Higher Categorical Structures, Type-Theoretically

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Cambridge, Logic and Semantics Seminar
12 May 2017

Longterm goals, all related

Some of our goals in (homotopy) type theory:

- Find definition + properties of higher categories
- For $\mathcal{C}$ a category, find definition of diagrams $\mathcal{C} \to \text{Type}$
- (general specification of higher inductive types)
- (Directed HoTT)
- (. . .)

Categories, naively

- $\mathsf{Ob} : \mathsf{Type}$
- $\mathsf{Hom} : \mathsf{Ob} \times \mathsf{Ob} \to \mathsf{Type}$
- $\_ \circ \_ : \mathsf{Hom}(y,z) \times \mathsf{Hom}(x,y) \to \mathsf{Hom}(x,z)$
- $h \circ (g \circ f) = (h \circ g) \circ f$
- $\mathsf{id} : \mathsf{Hom}(x,x)$ and equations
- pentagon and id-triangles
- associahedron and . . .
- ???

Similar: What is $F : \mathcal{C} \to \mathsf{Type}$?
(E.g. for $\mathcal{C}$ externally fixed category.)

Diagrams, special cases

Have notions of diagrams $F : \mathcal{C} \to \mathsf{Type}$ for special $\mathcal{C}$ (where $\mathcal{C}$ finite or represented internally), e.g.:

- $\mathcal{C}$ discrete
- More generally: $\mathcal{C}$ generated by simpler structure, e.g. by a graph
- $\mathcal{C}$ groupoidal
- $\mathcal{C}$ inverse (+ finite, externally given)

Reedy fibrant diagrams

E.g. take $\mathcal{C}$ to be the following category, where $u \circ w \equiv v \circ w$:

$$z \xrightarrow{\ w\ } y \underset{v}{\overset{u}{\rightrightarrows}} x.$$

A (Reedy fibrant) diagram $\mathcal{C} \to \mathsf{Type}$ is given by:

$$\begin{aligned} &R_x : \mathsf{Type} \\ &R_y : R_x \times R_x \to \mathsf{Type} \\ &R_z : (\Sigma(a : R_x).R_y(x, x)) \to \mathsf{Type}. \end{aligned}$$

Prominent inverse category: Δ_+^{op} – “semi-simplicial types”

Univalent categories via Reedy fibrant diagrams

Univalent category (Ahrens - Kapulkin - Shulman):

(1) Ob : Type

(2) Hom : Ob × Ob → Type

(3) $\_ \circ \_$: Hom(y, z) × Hom(x, y) → Hom(x, z)

(4) $h \circ (g \circ f) = (h \circ g) \circ f$

(5) Hom(x, y) always a *set*

(6) id, and id-equations, and univalence

Observations:

- (1), (2) are diagram over $(\Delta_+^{\leq 1})^{\mathsf{op}}$
- (1), (2), (3) are diagram over $(\Delta_+^{\leq 2})^{\mathsf{op}}$ with contractible inner horn-filler
- (1), (2), (3), (4) are diagram over $(\Delta_+^{\leq 3})^{\mathsf{op}}$ with contractible inner horn-fillers

Univalent categories via Reedy, cont.

(5) $\mathsf{Hom}(x, y)$ always a *set*

(6) id, and id-equations, and univalence

Observations, cont.:

- (5) is direct translation
- (6) can be neatly expressed as:

$$\Pi(x : \mathsf{Ob}), \mathsf{isContr}(\Sigma(y : \mathsf{Ob}), (f : \mathsf{Hom}(x, y)), \mathsf{isEquiv}(f))$$

where $\mathsf{isEquiv}(f)$ means that
$(\_ \circ f) : \mathsf{Hom}(y, z) \to \mathsf{Hom}(x, z)$ and
$(f \circ \_) : \mathsf{Hom}(w, x) \to \mathsf{Hom}(w, y)$ are equivalences of types: **completeness condition**.

$(\infty, 1)$-categories

Definition: complete semi-Segal type (Capriotti)

A Reedy fibrant $A : \Delta_+^{\mathsf{op}} \to \mathsf{Type}$ such that

- Segal condition: each map $A_n \twoheadrightarrow A_1 \times_{A_0} A_1 \times_{A_0} \ldots \times_{A_0} A_1$ is an equivalence
- Completeness: for $a : A_0$, the type $\Sigma(b : A_0), (f : A_1(a, b)), \mathsf{isEquiv}(f)$ is contractible.

Notes:

1. Segal condition is equivalent to saying that all inner horns have contractible fillers.
2. This can *probably* not be internalised in pure "standard HoTT", but it is possible in HTS / 2-level type theory with some assumption.
3. Fix n; then, "univalent $(n, 1)$-categories" can always be internalised.

Type universe

Example of a complete semi-Segal type: universe $\mathbf{T}$

$$
\begin{array}{lcl}
\mathbf{T}_0 & \cong & \mathsf{Type} \\
\mathbf{T}_1(X, Y) & \cong & X \to Y \\
\mathbf{T}_2(X, Y, Z, f, g, h) & \cong & g \circ f = h \\
\ldots & \ldots & \ldots
\end{array}
$$

Notes:

- Can be constructed as Reedy fibrant replacement of the semi-simplicial nerve of Type
- Univalence axiom $\simeq$ completeness for $\mathbf{T}$

Homotopy coherent diagrams[1]

For $\mathcal{C}$ a finite inverse category, define

$$N_+(\mathcal{C}) : \Delta_+^{\mathsf{op}} \to \mathsf{Set}$$

to be the "positive nerve" (chains of non-identity morphisms).

Define a *homotopy coherent diagram* $\mathcal{C} \to$ Type to be a "natural transformation" $N_+(\mathcal{C}) \to \mathbf{T}$; formally:

Definition: homotopy coherent diagram

The type of homotopy coherent diagrams is the Reedy limit of the composition $\left(\int N_+(\mathcal{C})\right) \xrightarrow{\mathsf{shape}} \Delta_+^{\mathsf{op}} \xrightarrow{\mathbf{T}}$ Type.

Example: do it for

$$z \xrightarrow{w} y \underset{v}{\overset{u}{\rightrightarrows}} x.$$

[1] The work on diagrams is jww Sattler

Homotopy coherent diagrams II

Theorem

For $\mathcal{C}$ a (finite) inverse category, homotopy coherent diagrams over $\mathcal{C}$ are equivalent to Reedy fibrant ones.

Notes:

- Homotopy coherent diagrams make precise the idea to "add all coherences explicitly"
- Construction works for any $(\infty, 1)$-category, not only $\mathbf{T}$
- Completely finite (since $\mathcal{C}$ is finite) $\Rightarrow$ can be internalised
- But: only works for inverse category (or semicategory, but finiteness not guaranteed).

Homotopy coherent diagrams with identities

Now: $\mathcal{C}$ *any* category, write

$$N(\mathcal{C}) : \Delta_+^{\mathsf{op}} \to \mathsf{Set}$$

for the nerve (chains of morphisms).

Definition: general homotopy coherent diagram

A *general homotopy coherent diagram* is an

$$h : \lim_{\int N(\mathcal{C})} (\mathbf{T} \circ \mathsf{shape})$$

such that for each object x of $\mathcal{C}$, the function $h(\mathsf{id}_x) : h(x) \to h(x)$ is an equivalence.

Note: $\int N(\mathcal{C})$ is infinite.

Homotopy coherent diagrams, comparison

Theorem

For $\mathcal{C}$ an inverse category, homotopy coherent diagrams are equivalent to general homotopy coherent diagrams: adding identities makes no difference up to homotopy.

Corollary

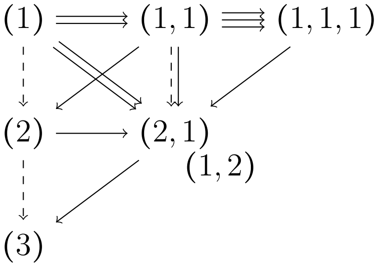
Let A be a complete semi-Segal type. We can construct all the degeneracy maps $s_i : A_n \to A_{n+1}$ such that the equalities

$$
\begin{aligned}
d_i \circ s_j &\equiv s_{j-1} \circ d_i && \text{if } i < j \\
d_i \circ s_j &\equiv s_j \circ d_{i-1} && \text{if } i > j+1 \\
d_i \circ s_j &\equiv \mathsf{id} && \text{if } i = j \text{ or } i = j+1
\end{aligned}
$$

hold judgmentally.

Simplicial types

We can construct a direct category D with "marked" arrows such that Reedy fibrant diagrams over D^{op}, which send marked arrows to equivalences, are "simplicial types".
Sketch of the the beginning of D:



(End of talk. Thanks for your attention!)