Eliminating out of Truncations

HoTT/UF Workshop, Warsaw (mostly based on arXiv:1411.2682, to appear in TYPES'14)

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General Question

What is $\|A\|_n \rightarrow B$?

I mainly talk about:

What is $\|\mathbf{A}\| \to \mathbf{B}$?

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(where $\|-\|$ is the propositional truncation, i.e. $n \equiv -1$.)

Prop. Truncation ||-||

What is a function $\mathbf{g} : \|\mathbf{A}\| \to \mathbf{B}$?

A function $f : A \rightarrow B$ that cannot look at its input?

wconst_f :=
$$\prod_{a_1,a_2:A} f(a_1) = f(a_2)$$
.

 $\begin{array}{rcl} & \text{Theorem} \\ (\|A\| \to B) & \simeq & \Sigma \left(f : A \to B \right) . \text{ wconst}_f \\ & \text{if } B \text{ is a 0-type (h-set).} \end{array}$

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wconst_f :=
$$\prod_{a_1,a_2:A} f(a_1) = f(a_2)$$

Coherence condition on $c : wconst_f$

$$\operatorname{coh}_{f,c} :\equiv \prod_{a^1 a^2 a^3:A} c(a^1, a^2) \cdot c(a^2, a^3) = c(a^1, a^3).$$

		Theorem	1
$(\ A\ \to B)$	\simeq	$\Sigma(f: A \rightarrow B)$. $\Sigma(c: \operatorname{wconst}_f)$. $\operatorname{coh}_{f,c}$	I
		if <i>B</i> is a 1-type.	

Assume $\mathfrak{a}_{\mathfrak{o}}$: *A* is given.

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Assume $\mathfrak{a}_{\mathfrak{o}} : A$ is given. $\Sigma(f_1 : B)$. **1**

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$$\Sigma (f : A \to B) \cdot \Sigma (c_1 : \prod_{a:A} f(a) = f_1) .$$
1

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$$\Sigma (f_1 : B) .$$

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$$\Sigma (c : \operatorname{wconst}_f) . \Sigma (d_1 : \prod_{a^1 a^2 : A} c(a^1, a^2) \cdot c_1(a^2) = c_1(a^1)) .$$

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$$\Sigma (c_2 : f(\mathfrak{a}_o) = f_1) . \Sigma (d_3 : c(\mathfrak{a}_o, \mathfrak{a}_o) \cdot c_1(\mathfrak{a}_o) = c_2) .$$

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$$\Sigma (d : \operatorname{coh}_{f,c}) .$$
1

Assume $a_o : A$ is given.

$$\begin{split} &\Sigma\left(f_{1}:B\right).\\ &\Sigma\left(f:A \rightarrow B\right).\Sigma\left(c_{1}:\prod_{a:A} f(a) = f_{1}\right).\\ &\Sigma\left(c:\operatorname{wconst}_{f}\right).\Sigma\left(d_{1}:\prod_{a^{1}a^{2}:A} c(a^{1},a^{2})\cdot c_{1}(a^{2}) = c_{1}(a^{1})\right).\\ &\Sigma\left(c_{2}:f(\mathfrak{a}_{o}) = f_{1}\right).\Sigma\left(d_{3}:c(\mathfrak{a}_{o},\mathfrak{a}_{o})\cdot c_{1}(\mathfrak{a}_{o}) = c_{2}\right).\\ &\Sigma\left(d:\operatorname{coh}_{f,c}\right).\\ &\Sigma\left(d_{2}:\prod_{a:A} c(\mathfrak{a}_{o},a)\cdot c_{1}(a) = c_{2}\right).\\ &\mathbf{1} \end{split}$$

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Assume $\mathfrak{a}_{\mathfrak{o}}$: *A* is given.

$$\begin{split} & \Sigma\left(f_{1}:B\right) \cdot \\ & \Sigma\left(f:A \rightarrow B\right) \cdot \overline{\Sigma\left(c_{1}:\prod_{a:A} f(a) = f_{1}\right)} \cdot \\ & \Sigma\left(c:\operatorname{wconst}_{f}\right) \cdot \overline{\Sigma\left(d_{1}:\prod_{a^{1}a^{2}:A} c\left(a^{1},a^{2}\right) \cdot c_{1}\left(a^{2}\right) = c_{1}\left(a^{1}\right)\right)} \cdot \\ & \Sigma\left(c_{2}:f\left(\mathfrak{a}_{o}\right) = f_{1}\right) \cdot \overline{\Sigma\left(d_{3}:c\left(\mathfrak{a}_{o},\mathfrak{a}_{o}\right) \cdot c_{1}\left(\mathfrak{a}_{o}\right) = c_{2}\right)} \cdot \\ & \Sigma\left(d:\operatorname{coh}_{f,c}\right) \cdot \\ & \Sigma\left(d:\operatorname{coh}_{f,c}\right) \cdot \\ & \mathbf{1} \end{split}$$

Assuming $a_o : A$, we have constructed an equivalence

$$g: B \rightarrow \Sigma(f: A \rightarrow B) \cdot \Sigma(c: \operatorname{wconst}_f) \cdot \operatorname{coh}_{f,c}$$
.

By examining the steps, we see that the function is

$$g(b) \equiv (\lambda_.b, \lambda_., _.refl_b, \lambda_., _, _.refl_{refl_b}).$$

It does not depend on $\mathfrak{a}_o!$

$$A \rightarrow \text{isequiv}(g)$$

thus $||A|| \rightarrow \text{isequiv}(g)$.

Therefore:

$$\|A\| \rightarrow (B \simeq \Sigma(f : A \rightarrow B) . \Sigma(c : wconst_f) . coh_{f,c})$$

$$(\|A\| \rightarrow B) \simeq \Sigma(f : A \rightarrow B) . \Sigma(c : wconst_f) . coh_{f,c}$$

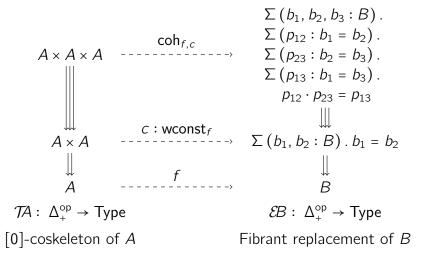
This strategy is so frugal that it can be done at any level, with minimalistic assumptions on the theory: we need $\mathbf{1}, \Sigma, \Pi, \text{Id}$ with function extensionality, $\|-\|$.

Main result: In a type theory with Reedy ω^{op} -limits (infinite Σ -types), the type $||A|| \rightarrow B$ corresponds to the type of *coherently constant* functions $A \rightarrow B$.

Setting: type-theoretic fibration category (Shulman, Univalence for inverse diagrams and homotopy canonicity)

Main part of this talk: a very, very rough outline of the proof.

Coherently constant functions are morphisms between semi-simplicial types ($\Delta^{op}_+ \rightarrow Type$)

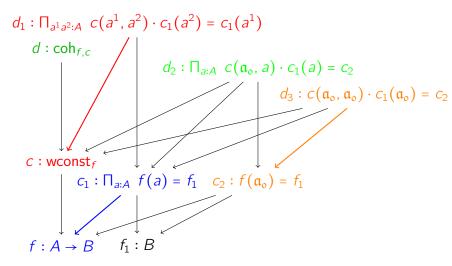


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 $\mathcal{E}B_{[n]}$ is the type of *n*-dimensional tetrahedra, built of the identity type (defined as a Shulman-kind diagram over the inverse category Δ^{op}_+). We can also define the type of *horns*.

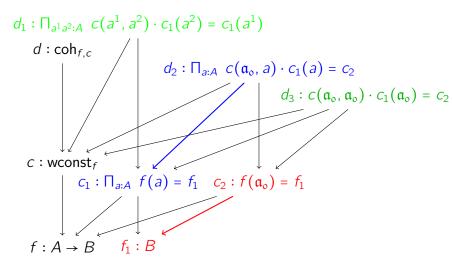
Important Kan-filling lemma: **The projection from full tetrahedra to the type of** (*k*-**)horns is an equivalence.**

(Side remark: This is a strong "Kan filling" property and gives a "simplicial" version of Lumsdaine's / van den Berg-Garner's "globular" result that types are weak ω-groupoids.) Nat. trans. between $\widehat{\mathcal{T}}A$ and $\widehat{\mathcal{E}}B$ (extended index cat. $\widehat{\Delta}^{op}_+$)



Kan-filling lemma \Rightarrow ... extensive calculation ... \Rightarrow Any two Σ -components connected by a "diagonal arrow" form a contractible pair!

Nat. trans. between $\widehat{\mathcal{T}}A$ and $\widehat{\mathcal{E}}B$ (extended index cat. $\widehat{\Delta}^{op}_+$)



Kan-filling lemma \Rightarrow ... extensive calculation ... \Rightarrow Any two Σ -components connected by a "diagonal arrow" form a contractible pair!

Rest as in the special case:

• Assuming \mathfrak{a}_{o} : *A*, we have shown that the can. map

 $B \rightarrow$ nat. trans. from $\mathcal{T}A$ to $\mathcal{E}B$

is an equivalence.

- This map is independent of \mathfrak{a}_{o} .
- Thus, ||A|| implies that this map is an equivalence.
- Therefore:

Theorem				
$(A \rightarrow B) \simeq$ nat. trans. from $\mathcal{T}A$ to $\mathcal{E}B$				
in any theory with $1, \Sigma, \Pi, Id, fun.ext., \ -\ , \mathbb{R}$ Reedy ω^{op} -limits.				

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If you don't like Reedy ω -limits, you still get all the cases where *B* is *n*-truncated.

Higher Truncations What is $\|\mathbf{A}\|_{n} \rightarrow \mathbf{B}$?

Conjecture: Natural Transformations from the [n + 1]-coskeleton of $\mathcal{E}A$ to $\mathcal{E}B$.

This talk: Case $n \equiv -1$.

Paolo Capriotti, N.K., Andrea Vezzosi: Proof for the case that B is (n + 1)-truncated (to appear at CSL'15).

Caveat, wild speculation following.

Case $n \equiv 0$ can be used to solve the open problem "univalent type theory eats itself" with n univalent universes, but without HITs; trick: interpret U_i as universe of i-types.