Free Higher Groups in Homotopy Type Theory

Nicolai Kraus

jww Thorsten Altenkirch, thanks to Paolo Capriotti

University of Nottingham

Oxford, LICS'18, 9 July

Equalities in intensional type theory

If x, y are two terms of the same type:

$$\mathbf{x} \equiv \mathbf{y}$$

- *definitional/judgmental* equality;
- meta-theoretic;
- used for type-checking.

$$\mathbf{x} = \mathbf{y}$$

- a.k.a. $\mathsf{Id}(x, y)$;
- equality *type*;
- can be proved internally.

Hofmann-Streicher: We **cannot** show UIP, which says $\Pi(p, q : x = y), (p = q)$.

extensions

→ add UIP as an axiom

→ add univalence $\rightsquigarrow$ HoTT (homotopy type theory)

Free groups (set-based)

The HoTT book[1] defines the *free group* over a type/set A as a higher inductive type FA_0 with constructors:

$$\eta : A \to \mathsf{FA}_0$$
$$e : \mathsf{FA}_0 \qquad \textit{(neutral element)}$$
$$m : \mathsf{FA}_0 \times \mathsf{FA}_0 \to \mathsf{FA}_0 \qquad \textit{(multiplication)}$$
$$\alpha : m(x, m(y, z)) = m(m(x, y), z) \qquad \textit{(associativity)}$$
$$\vdots$$
$$h : (p, q : x = y) \to p = q \qquad \textit{(set truncation)}$$

This is purely based on sets (h-sets).
Can we do *free ∞-groups*?

[1] The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, 2013.

What is an ∞-group in homotopy type theory?

Simple observation:

Assume A is a type, $x : A$. Then:

- $\mathsf{refl}_x : x = x$
- if $p, q : x = x$, then $p \cdot q : x = x$
- $p \cdot (q \cdot r) = (p \cdot q) \cdot r$
- ...

Note : One often writes ΩA or $\Omega(A, x)$ for $(x = x)$.

Define: ∞-group $\overset{\text{def}}{=}$ a type of the form ΩA

(for pointed connected A). See next talk.

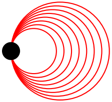
. . . and what's a free ∞-group?

Wedge of A-many circles

HIT *WA* where

b : *WA*

l : $A \to b = b$



$A \rightrightarrows \mathbf{Unit} \dashrightarrow WA$

Potential definition:
Free higher group is $\Omega(WA)$.

Directly as a HIT

Intuition: "lists where elements can be negative"

HIT FA where

nil : FA

cons : $A \to$ FA $\to$ FA

i : $(a : A) \to \text{isequiv}(\text{cons}_a)$

Potential definition:
Free higher group is FA.

These two definitions are equivalent!

Have we generalised the set-based free group?

For a type A, we now have:

(1) the set-based free group FA_0

(2) the free ∞-group FA (equivalently, $\Omega(\mathit{WA})$).

Question: Does (2) generalise (1)?
That is: if A is a set, do we have $\mathsf{FA}_0 \simeq \mathsf{FA}$?

This boils down to:
If A is a set, is FA (equivalently, $\Omega(\mathit{WA})$) also a set?
(Because the rest is easy.)

This is a known open problem in homotopy type theory. Our result:

Thm: All fundamental groups of FA are trivial.

Idea of the proof

Thm: All fundamental groups of FA are trivial.

- There is a canonical map $\eta : \mathsf{List}(A \times \mathbf{2}) \to \mathsf{FA}$.
  $N_{-1} :\equiv \mathsf{List}(A \times \mathbf{2})$ is a (very bad) approximation of FA.
  More precisely: $\|\mathsf{List}(A \times \mathbf{2})\|_{-1} \simeq \|\mathsf{FA}\|_{-1}$.
- Next step: Define relation $\sim$ on lists, by
  $[\ldots, x, a^+, a^-, y, \ldots] \sim [\ldots, x, y, \ldots]$
  (or $+/-$ exchanged).
  Define HIT N_0 with points given by lists, paths by $\sim$
  ("quotient without coherences").
  Easy to show: $\|N_0\|_0 \simeq \|\mathsf{FA}\|_0$.
- Next step: add one level of coherences to define N_1. We show (a weakened but sufficient variant of)
  $\|N_1\|_1 \simeq \|\mathsf{FA}\|_1$.

Idea of the proof (2)

- "Rewriting combinatorics" plus "weak constancy" argument shows: N_1 has trivial fundamental group at nil.
- This implies that all fundamental groups of N_1 are trivial.
- Since $\|N_1\|_1 \simeq \|\mathsf{FA}\|_1$, all fundamental groups of FA are trivial. $\square$

Conjecture: In HTS/2LTT (allowing semisimplicial types), we can define a canonical sequence

$$N_{-1} \to N_0 \to N_1 \to N_2 \to \ldots$$

(no truncations), show that it is weakly constant on path spaces, and show that FA is a retract of its colimit.
This would solve the open problem (for HTS/2LTT).

Thank you for your attention!