Non-Recursive Truncations

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Truncation

$$\exists(x : A), P(x) \quad \text{is not the same as} \quad \Sigma(x : A), P(x) \quad !$$

$:= \; \|\Sigma(x : A), P(x)\|$

Explanation: The *[propositional] truncation* $\|-\|$ makes a type *propositional* (all elements equal).

HoTT 2013; see also NuPRL 1986, Awodey-Bauer 2004.

Rules for $\|-\|$

$\|A\|$ is propositional

$A \to \|A\|$

$$\frac{\|A\| \to B}{A \to B}$$

if B is propositional

But then, what is $\|A\| \to B$?

$\|A\| \to B$ **is equivalent to . . .**

$$
\begin{array}{ll}
\Sigma(f : A \to B), & \text{if } B \text{ is } (-1)\text{-type} \\
\Sigma(c : \mathsf{wconst}_f), & \text{if } B \text{ is } 0\text{-type} \\
\Sigma(d : \mathsf{coh}_{f,c}), & \text{if } B \text{ is } 1\text{-type} \\
\quad \ldots & \quad \ldots
\end{array}
$$

general B: **infinitely many components!**

note:

$$\mathsf{wconst}_f \;:\equiv\; \Pi_{x,y:A}\; f\,x = f\,y$$

$$\mathsf{coh}_{f,c} \;:\equiv\; \Pi_{x,y,z:A}\; c(x,y) \cdot c(y,z) = c(x,z)$$

$$\ldots$$

$\|A\| \to B$ for general B

Theorem [K., TYPES 2014 proceedings]

We can define Reedy fibrant $\mathcal{T}A$ and $\mathcal{E}B : \Delta_+^{\mathrm{op}} \to \mathbf{Type}$ such that:

$$(\|A\| \to B) \quad \simeq \quad \text{nat. trans. from } \mathcal{T}A \text{ to } \mathcal{E}B$$

in any type theory with $\mathbf{1}, \Sigma, \Pi, \mathsf{Id}$, fun.ext., $\|-\|$, Reedy ω^{op}-limits.

This (directly or indirectly) generalises

- ⋆ Lurie, *Higher Topos Theory*, Prop. 6.2.3.4: ∞-semitopos instead of **Type**
- ⋆ Rezk, *Toposes and Homotopy Toposes*, Prop. 7.8: model topos instead of **Type**

Truncation as a Higher Inductive Type

$\|A\|$ as HIT (standard construction)

$|-| : A \to \|A\|$

$\mathsf{t} \ : \Pi_{x,y:\|A\|} \ x =_{\|A\|} y$

Can we give an equivalent definition of $\|A\|$ with a nicer elimination principle?

1st approximation: A_1

$f : A \to A_1$

2nd approximation: A_2

$f : A \to A_2$

$c : \mathsf{wconst}_f$

3rd approximation: A_3

$f : A \to A_3$

$c : \mathsf{wconst}_f$

$d : \mathsf{coh}_{f,c}$

$\|A\| \simeq \|A_1\|_0 \simeq \|A_2\|_1 \simeq \ldots$

Easier elimination principle into 0-, or 1-, or ...-types!

Purely non-recursive representations, I

We could try to consider the homotopy colimit of

$$A_1 \to A_2 \to A_3 \to \dots$$

which should be $\|A\|$.

Problem: for any n, we can write down A_n. However, we **cannot** write down $A : \mathbb{N} \to \mathcal{U}$.

("Semisimplicial Types Phenomenon")

Purely non-recursive representations, II

Solution: Make the sequence $A_1 \to A_2 \to A_3 \to \ldots$ "coarser".

- ⋆ van Doorn (CPP'16), independent of my analysis: do the first approximation in every step
  (easy to prove correct, but no finite special cases).

- ⋆ K. (LiCS'16): construct A_{n+1} by taking A_n and adding fillers for $S^{n-1} \to A_n$
  (harder to prove correct, but useful finite special cases);

  Any sequence of weakly constant functions has a propositional colimit!

- ⋆ Rijke - van Doorn / Buchholtz - Rijke, wip: localizations and related constructions

Thank you! Any comments or questions?