University of Nottingham

First Year PhD Report

Higher Dimensional Type Theory

and other Aspects of Mathematics in Computer Science

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Abstract

We give an introduction to *higher dimensional* or *homotopy type theory*, trying to provide a comprehensive overview on this research direction that combines type theory, algebraic topology and higher category theory. Further, we describe the content and results of own research projects and outline our future work plan.

This report presents some of the topics the author has worked on during the first year as a PhD student.

Preface

During my first year, I have spent a lot of time learning the concepts of my PhD topic, *Higher Dimensional Type Theory* or *Homotopy Type Theory*. The first part of my report is therefore an introduction to this fairly new field in between of mathematics and theoretical computer science. The foundations for this subject were, in some way, laid by an article of Hofmann and Streicher [22] [1] by showing that in Intentional Type Theory, it is reasonable to consider different proofs of the same identity. Their strategy was to use groupoids for an interpretation of type theory. Pushing this idea forward, Lumsdaine [28] and van den Berg & Garner [12] noticed independently that a type bears the structure of a *weak omega groupoid*, a structure that is well-known in algebraic topology.

In recent years, Voevodsky proposed his *Univalence axiom*, basically aiming to ensure nice properties that traditional mathemticians use regulary, such as the ability to treat isomorphic structures as equal. Claiming that set theory has inherent disadvantages, he started to develop his *Univalent Foundations of Mathematics*, drawing a notable amount of attention from researchers in many different fields: homotopy theory, constructive logic, type theory and higher dimensional category theory, to mention the most important.

Note that there exists an extended version of this report[2]. While the extended version is fairly comprehensive, the version at hand is rather short and does not provide an introduction to the basic intuition and to a couple of concepts that are either well-known or too advanced and therefore not strictly necessary for this report. Moreover, the extended version contains descriptions of other, rather unrelated research projects I have worked on and results I have obtained, as well as a couple of other research ideas for the future.

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[1] this is not the original one, but an improved version

[2] it can be found on my homepage, `http://red.cs.nott.ac.uk/~ngk/`

Contents

1 Overview

The sections 2 to 7 of this report are a significantly abridged version of an exposition I am writing[3]. Originally, my motivation for writing up these contents has been to teach them to myself. At the same time, I had to notice that no detailed written introduction seems to exist, maybe due to the fact that the research branch is fairly new.

Concretely, in section 2, I introduce weak factorisation systems. I discuss in which way they can, refining the model described by Seely [38], be used to model dependently typed theories. This is based on work of many recent publications, including [7], [8], [10], [11] and [24]. An introduction to model categories is given in section 3. One example of a model category is the category of small groupoids. It was used by Hofmann & Streicher [22] to show that *uniqueness of identity proofs* is not implied by J (Section 4.2). Another model category is the one of simplicial sets, which plays a very central role in Voevodsky's model of type theory. We introduce it in Section 8. Section 9 deals with the notion of contractibility, homotopy levels, weak equivalences, univalence and concludes with a proof that univalence implies function extensionality. In Section 10, a proof of Hedberg's theorem is presented. Finally, in section 8, my own work on what I call *Yoneda Groupoids* is described and finally (section 9), some future research ideas are outlined.

2 Homotopic Models

The main sources for this section are van den Berg & Garner [11], Arndt & Kapulkin [7] and Awodey & Warren [10]. For some concepts, the $n$lab [41] is very useful.

2.1 Weak Factorization Systems

Weak factorization systems provide a useful setting for models of identity types. As described in the section about semantics, the critical point is to find a suitable subclass of morphisms that have the properties of types, and weak factorization system address exactly at this issue. It also works the other way round: As shown by Gambino & Garner [18], the classifying category of a type theory with identity types admits always a (nontrivial) weak factorization system.

We use the definitions stated in [32]. The concepts are well-known and wildly accepted, but to the best of my own knowledge, they are originally due to Bousfield [13].

[3] it can be found on my homepage, `http://red.cs.nott.ac.uk/~ngk/`

It will turn out later that a basic requirement for identity types is the following *lifting property*.

Definition 2.1 (Lifting Property)**.** Let c, f be morphisms in a category. c has the *left lifting property* with respect to f (equivalently, f has the *right lifting property* with respect to c) if, for any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "c"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

a *diagonal filler*, i.e. a morphism $j : B \to X$, exists, making the whole diagram commutative. This filler does not have to be unique (though this would be a useful property later).

Having this concept in hand, we are able to define the mentioned systems:

Definition 2.2 (weak factorization system)**.** Given a category C, a *weak factorization system* on C is a pair $(\mathfrak{L}, \mathfrak{R})$ of sets of morphisms of C such that

(W_1) every morphism in C can be written as $p \circ i$, where $p \in \mathfrak{R}, i \in \mathfrak{L}$ (not necessarily unique)

(W_2) every morphism in $\mathfrak{L}$ has the left lifting property with respect to every morphism in $\mathfrak{R}$

(W_3) $\mathfrak{L}$ and $\mathfrak{R}$ are maximal with this property (i.e. we cannot add any morphisms without violating the above requirements)

Example 2.3. A basic (but nonconstructive) example of a weak factorization system on the category of sets is (*monos*, *epis*), i.e. the set of injective (monomorphisms) and surjective maps (epimorphisms). The factorization of $f : A \to B$ is given by

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, dashed, hook, "a \mapsto (in_1 a, f(a))"'] & & B \\ & (1+A) \times B \arrow[ur, dashed, two heads, "(a,b) \mapsto b"'] & \end{tikzcd}$$

Note that the somewhat nasty $1+$ is necessary as there would be a problem if $A = \emptyset \neq B$ otherwise. For the lifting j, given an injective c and a surjective f in

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "c"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] & Y \end{tikzcd}$$

define $j(b) = u(a)$ if $a = c(a)$ and, if such an a does not exist, $j(b) = x$ for some x that satisfies $f(x) = v(b)$. Note that this is possible if and only if the Axiom of Choice holds true. Finally, the maximality condition is clearly satisfied.

This can also be done in a constructive way by requiring that all the monos and epis are split (and carry information about the corresponding retraction resp. section), thus essentially making a (deterministic) construction of the diagonal filler possible.

As an easy exercise and because it is important for further explanations, we prove the following:

Lemma 2.4. *Given a weak factorization system* $(\mathfrak{L}, \mathfrak{R})$*, the class* $\mathfrak{R}$ *is closed under pullbacks (whenever they exist).*

Proof. Let $f : B \to A \in \mathfrak{R}$ and $\sigma : X \to A$ any morphism. Then, for $Y = X \times_A B$, we have to show that in the pullback square

$$\begin{tikzcd}
Y \arrow[r, "\tau"] \arrow[d, "g"'] & B \arrow[d, "f"] \\
X \arrow[r, "\sigma"'] & A
\end{tikzcd}$$

the morphism g is in $\mathfrak{R}$. Therefore, let $c : S \to T \in \mathfrak{L}$ be any morphism and $s : S \to Y$, $t : T \to X$ be morphisms that make the left square and therefore the whole diagram commute:

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "c"'] & Y \arrow[r, "\tau"] \arrow[d, "g"'] & B \arrow[d, "f"] \\
T \arrow[r, "t"'] & X \arrow[r, "\sigma"'] & A
\end{tikzcd}$$

As $f \in \mathfrak{R}$, there exists a diagonal filler:

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "c"'] & Y \arrow[r, "\tau"] & B \arrow[d, "f"] \\
T \arrow[r, "t"'] \arrow[urr, dashed, "i"] & X \arrow[r, "\sigma"'] & A
\end{tikzcd}$$

Together with the pair (i, t), the property of the pullback guarantees that there is a morphism j:

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "c"'] & Y \arrow[r, "\tau"] \arrow[d, "g"'] & B \arrow[d, "f"] \\
T \arrow[r, "t"'] \arrow[ur, dotted, "j"] & X \arrow[r, "\sigma"'] & A
\end{tikzcd}$$

By our construction, we have $g \circ j = t$, so the only thing to check is whether s equals $j \circ c$. We know that $g \circ s = g \circ j \circ c$ and that $\tau \circ s = i \circ c = \tau \circ j \circ c$, so the pairs $(\tau \circ s, g \circ s)$ and $(\tau \circ j \circ c, g \circ j \circ c)$ are equal. Because of the pullback property, each of the two pairs provides a unique morphism $S \to Y$ making everything commutative, but for the first pair, this is obviously s and for the second pair, this is $j \circ c$. Consequently, they are equal and we are done. $\square$

The above lemma enables us to interpret substitution of types as pullbacks, as described in the section about semantics of dependent type theory in the extended version of this report.

2.2 Identity types in Weak Factorization Systems

I have learnt the content of this subsection by reading the (highly recommended) survey article [8]. Unfortunately, it is not very detailed, so I try to give a slightly longer explanation.

From now on, let us write A^I as short-hand for $\Sigma_{a,b:A}.Id_A\, a\, b$. Clearly, if A is a type in context Γ, then so is A^I[4].

So, let us now discuss the interpretation of identity types. Given a bicartesian closed categoriy $\mathbb{C}$ with a weak factorization system $(\mathfrak{L}, \mathfrak{R})$, assume we have interpreted everything apart from identity types as described in the semantics section. If $\Gamma \vdash A$ *: type*, then there is the context morphism $\delta_A : \Gamma.A \to \Gamma.A.A$. In $\mathbb{C}$, the morphism $[\![\delta]\!]_A$ can, according to the laws of a weak factorization system, be written as

$$[\![\delta_A]\!] = [\![\Gamma.A]\!] \xrightarrow{c \in \mathfrak{L}} X \xrightarrow{f \in \mathfrak{R}} [\![\Gamma.A.A]\!].$$

Assume we have a possibility to choose one of the (possible multiple) factorizations in a coherent way. Then, we can choose to model $\Gamma.A^I$ by X, and c will be the interpretation of $\lambda a\,.\,(a, a, \mathit{refl}_a)$, while f is the interpretation of the equality type (that depends on $\Gamma.A.A$).

For an easier notation, let us write r_A instead of $\lambda a : A\,.\,(a, a, \mathit{refl}_a)$.

The property of the weak factorization system makes sure that we can interpret the eliminator. Consider a type P that depends on A^I in context Γ and assume we have a term m of type $\forall a.P(a, a, \mathit{refl}_a)$ as in the commutative

[4] This is usually the notation for the *path object* in abstract homotopy theory of model categories; this coincidence is, of course, not random!

diagram

$$\begin{tikzcd}
\llbracket \Gamma.A \rrbracket \arrow[r, "\llbracket m \rrbracket"] \arrow[d, "\llbracket r_A \rrbracket"'] & \llbracket \Gamma.A^I.P \rrbracket \arrow[d, "\llbracket P \rrbracket"] \\
\llbracket \Gamma.A^I \rrbracket \arrow[r, "1"] & \llbracket \Gamma.A^I \rrbracket
\end{tikzcd}$$

This diagram represents exactly the assumptions of the J rule. We know that the left and the right morphism are in $\mathfrak{L}$ resp. in $\mathfrak{R}$. Therefore, the properties of the weak factorization system guarantee the existence of a morphism j making the diagram commutative:

$$\begin{tikzcd}
\llbracket \Gamma.A \rrbracket \arrow[r, "\llbracket m \rrbracket"] \arrow[d, "\llbracket r_A \rrbracket"'] & \llbracket \Gamma.A^I.P \rrbracket \arrow[d, "\llbracket P \rrbracket"] \\
\llbracket \Gamma.A^I \rrbracket \arrow[r, "1"] \arrow[ur, "j"'] & \llbracket \Gamma.A^I \rrbracket
\end{tikzcd}$$

If we have a coherent possibility to choose the filler j, we can use it as the interpretation of the elimination rule. Note that the upper triangle represents the computation rule, stating $j \circ \llbracket r_A \rrbracket = \llbracket m \rrbracket$.

2.3 Homotopy Theoretic Models of Identity Types

Let us state the above discussion in form of a theorem. Is is due to Awodey & Warren [10] and makes use of some abstract homotopy theory (in particular, *path objects*, which we do not repeat here; a good source is [15]):

Theorem 2.5. *Let $\mathbb{C}$ be a finitely complete category with a weak factorization system and a functorial choice $(\cdot)^I$ of path objects in C, and all of its slices, which is stable under substitution. I.e., given any $A \to \Gamma$ and $\sigma : \Gamma' \to \Gamma$,*

$$\sigma^*(A^I) \cong (\sigma^* A)^I.$$

Then $\mathbb{C}$ is a model of a form of Martin-Löf type theory with identity types.

Note that A^I is now defined in both the cases that A is a type and that A is an object in a category, which will hopefully not lead to confusion. The intuition is that the former should be modelled by the latter.

Here is a diagram that illustrates the theorem. Given an $f \in \mathfrak{R}$ and a morphism σ, which are painted as solid arrows, we can construct the pullbacks (dashed arrows) and factorizations of the diagonals (dotted arrows):

$$
\begin{tikzcd}
\sigma^* A \times_{\Gamma'} \sigma^* A & & & \\
& (\sigma^* A)^I \arrow[ul, dotted] & & \\
& & \sigma^* A \arrow[ul, dotted, "r_{\sigma * A}"'] \arrow[r, dashed] \arrow[dd, dashed] & \Gamma' \arrow[dd, "\sigma"] \\
\sigma^*(A^I) \arrow[uur, no head, squiggly] \arrow[urrr, dashed] \arrow[dddr, dashed] & & & \\
& & A \arrow[r, "f"] & \Gamma \\
& & A \times_\Gamma A \arrow[u, dashed] \arrow[r, dashed] & A \arrow[u, "f"'] \\
& A^I \arrow[ur, dotted] & & \\
A \arrow[ur, dotted, "r_A"] & & &
\end{tikzcd}
$$

As shown in the diagram, $\sigma^*(A^I)$ and $(\sigma^* A)^I$ have to be isomorphic in order to fulfil the assumptions of the theorem.

The proof given in [10] is basically a summary of our explanations in the previous sections.

Remark 2.6. At first sight, one might wonder if the choices of j in Awodey & Warren's theorem have to fulfil some coherence conditions. The authors do not mention any, but personally, I am still not completely sure about this.

3 Model Categories

In this section, we want to provide some background on *model categories*, a structure that can be found in many mathematical constructions and has two weak factorization systems built-in. They were first mentioned by Quillen

[36]. Many authors ([7], [8], [10], [11], [24], [27]) make use of model categories, but in fact, one of the two weak factorization systems is never used. However, if something has the structure of a weak factorization system, it nearly always is a model category as well, so not much harm is done if the stronger notion is assumed. The advantage is that model categories are a well-established concept in mathematics.

We first state the definition given in the $n$lab [31]:

Definition 3.1 (Model Category)**.** A *model structure* on a category C consists of three distinguished classes of morphisms of C, namely the cofibrations $\mathfrak{C}$, the fibrations $\mathfrak{F}$ and the weak equivalences $\mathfrak{W}$, satisfying:

(M_1) *2-out-of-3*: If, for any two composable morphisms f, g, two of the three morphisms $f, g, g \circ f$ are weak equivalences, then so is the third.

(M_2) there are two weak factorization systems, $(\mathfrak{C}, \mathfrak{W} \cap \mathfrak{F})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$.

A *model category* is a complete (all small limits exist) and cocomplete (all small colimits exist) category that carries a model structure.

Like most authors, we call a map that is a fibration and a weak equivalence at the same time an *acyclic fibration* (not that *trivial fibration* is also common), similarly, a cofibration that is a weak equivalence is called an *acyclic cofibration.* If $A \to 1$ is a fibration, then the object A is called *fibrant.* If $0 \to B$ is a cofibration, B is called *cofibrant.*

We allow us to make three comments here, all of which are well-known and are, for example, stated in [29].

Example 3.2. For any category C, let two of the three classes $\mathfrak{F}, \mathfrak{C}, \mathfrak{W}$ be the class of all morphisms and let the third be the class of all isomorphisms. This gives us three different model structures on C. If C has all small limits and colimits, each of these constructions forms a model category.

Example 3.3. The product of model categories is, in the obvious way, a model category.

Remark 3.4. The concept of a model category is self-dual.

Nontrivial examples include the categories of groupoids, simplicial sets and topological spaces, which will be discussed later.

As we will need it soon, we want to state the the following proposition ([15] Proposition 3.14):

Proposition 3.5. *Let C be a model category. Then the following statements are true:*

(i) $\mathfrak{C}$ *is closed under* cobase change*, i.e. if c is a cofibration in the following pushout diagram, then so is c':*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "c"'] & C \arrow[d, dashed, "c'"] \\
B \arrow[r, dashed] & D
\end{tikzcd}$$

(ii) $\mathfrak{W} \cap \mathfrak{C}$ *is closed under* cobase change*, i.e. if c is an acyclic cofibration in the diagram above, then so is c'.*

(iii) $\mathfrak{F}$ *is closed under* base change*, which is the dual statement of (i), i.e. the pushout is replaced by a pullback.*

(iv) $f \cap \mathfrak{W}$ *is closed under* base change*.*

Proof. The statement (ii) can be proved analogously to (i) and $(iii), (iv)$ are clearly dual to $(i), (ii)$, so we only prove (i). To prove that c' in the given diagram is a cofibration, let f be an acyclic fibration.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "c"'] & C \arrow[r, "u"] \arrow[d, dashed, "c'"] & E \arrow[d, "f"] \\
B \arrow[r, dashed, "s"'] & D \arrow[r, "t"'] & F
\end{tikzcd}$$

As c is a cofibration and f an acyclic fibration, there exists a lifting $j : B \to E$. Together with u and the pushout property, j implies that there is a morphism $j' : D \to E$. While [15] concludes the proof by stating that j' is the required lifting, the authors do not find it obvious that both triangles commute.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "c"'] & C \arrow[r, "u"] \arrow[d, dashed] & E \arrow[d, "f"] \\
B \arrow[r, dashed, "s"'] \arrow[urr, dotted, "j"] & D \arrow[r, "t"'] \arrow[ur, dotted, "j'"'] & F
\end{tikzcd}$$

While the commutativity of the upper triangle, i.e. $u = j' \circ c'$ is indeed a direct consequence of the pullback property, it is less clear that $t = f \circ j'$. However, note that $f \circ j' \circ c' = f \circ u = t \circ c'$ and $f \circ j' \circ s = f \circ j = t \circ s$ do directly follow from the commutativity of the diagram, where commutativity of the lower triangle is not assumed. We have therefore shown that the pairs $(C \xrightarrow{f \circ j' \circ c'} F, B \xrightarrow{f \circ j' \circ s} F)$ and $(C \xrightarrow{t \circ c'} F, B \xrightarrow{t \circ s} F)$ are identical, which means that they induce (by the pushout property) the same unique morphism $D \to F$, but as those morphisms are $f \circ j'$ respective t, those are equal. $\square$

We want to state another often used definition, used, for example, in [15] and [29] (however, both times slightly differently). One thing should be mentioned before:

Definition 3.6 (Retract)**.** If X, Y are two objects in a category C, we say that Y is a *retract* of X if there exists maps $Y \xrightarrow{s} X \xrightarrow{r} Y$ with $r \circ s = id_Y$. If f, g are maps, we call g a *retract* of f if this holds true in the arrow category $C^{\rightarrow}$, i.e. if there is a commuting diagram

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "g"'] & W \arrow[r, "p_1"] \arrow[d, "f"'] & A \arrow[d, "g"'] \\ B \arrow[r, "i_2"'] & Z \arrow[r, "p_2"'] & B \end{tikzcd}$$

satisfying $p_1 \circ i_1 = id_A$ and $p_2 \circ i_2 = id_B$.

Definition 3.7 (Model Category, alternative definition)**.** As above, a *model structure* on C consists of three classes $\mathfrak{C}, \mathfrak{F}, \mathfrak{W}$, so that:

(N_1) $\mathfrak{C}, \mathfrak{F}, \mathfrak{W}$ are all closed under composition and include all identity maps

(N_2) *2-out-of-three* is satisfied as above: If two of f, g, $g \circ f$ are in $\mathfrak{W}$, then so is the third

(N_3) $\mathfrak{C}, \mathfrak{F}, \mathfrak{W}$ are also closed under retracts, i.e. if g is a retract of f and f is in one of the classes, then g is in the same class

(N_4) if $c \in \mathfrak{C}$ and $f \in \mathfrak{F}$, then c has the left lifting property with respect to f if at least one of them is also in $\mathfrak{W}$

(N_5) any morphism m can be factored in two ways:

$$A \xrightarrow{m} B = A \xrightarrow{c \in \mathfrak{C} \cap \mathfrak{W}} X \xrightarrow{f \in \mathfrak{F}} B = A \xrightarrow{c' \in \mathfrak{C}} X' \xrightarrow{f' \in \mathfrak{F} \cap \mathfrak{W}} B$$

A *model category* is a category with all small limits and colimits together with a model structure.

Remark 3.8. There are a couple of variants to these definitions. For example, Dwyer & Spalinski [15] only requires the existence of finite limits and colimits. Also, Hovey [29] wants the factorizations to be functorial. The $n$lab [31] states: "Quillens original definition required only finite limits and colimits, which are enough for the basic constructions. Colimits of larger cardinality are sometimes required for the small object argument, however."

As an exercise, we have proved that each of the above definitions can be replaced by the other. This fairly involved proof can be found in the extended version of our report; here, we just state the claim as a proposition:

Proposition 3.9. *The definitions 3.1 and 3.7 are equivalent.*

4 Groupoids

The Groupoid model is the easiest one of those we want to present here. At the same time, it is the oldest one ([22]) and has inspired the whole research field of homotopy type theory.

The basic setting is the category **Grp** with (small) groupoids as objects and functors as morphisms. More precisely, we are talking about the "evil" version of **Grp**, which does *not* arise as the truncation of the natural 2-category of groupoids, functors and natural transformations. Instead, the equality on the hom-sets, i. e. the functors, is *strict* functor equality: Two functors $F, G : A \to B$ are only equal if $F(a) = G(a)$ and $F(m) = G(m)$ for every object a and morphism m of the groupoid A, while the "non-evil" version of the category of groupoids would identify F and G if they are natural isomorphic.

In category theory, there exist the so-called *Grothendieck fibrations* [30] and these are the fibrations we need, as described in [8], though it might be more appropriate to talk about *isofibrations*. As both concepts are the same in the simple case of groupoids, we do not make the distinction. We only give the definition for groupoids, not the more general one for categories.

Definition 4.1 (Fibrations in Grp)**.** A functor $p : E \to B$ between groupoids is a *fibration* if for any $e \in E$ and $f : b \to p(e)$, there is a morphism $g : e' \to e$ with $p(g) = f$. This immediately implies that every groupoid is fibrant.

The definition implies that any connected component of B is either disjoint from the image of p or completely objectwise contained in this image.

Groupoids carry the structure of a weak factorization system in the following way:

- $\mathfrak{L}$ is the class of functors which are injective and equivalences in the categorical meaning (i. e. embeddings which are injective on objects).
- $\mathfrak{R}$ are the fibrations defined above.

Theorem 4.2. *The structure* $(\mathfrak{C}, \mathfrak{F})$ *is a weak factorization system on the category of small groupoids.*

Proof. In my original composition, I had a very long and unnecessarily complicated proof here. I have decided to skip it, as there are much more elegant proofs available, for example at the $n$lab [16]. $\square$

We may define A^I to be the arrow groupoid $A^{\to}$. Then, *refl* maps objects $a \in |A|$ canonically on $id_a \in |A^{\to}|$ and morphisms $m : a \to b$ on (m, m). It is easy to check that this is indeed an injective equivalence between A and $A^{\to}$, i.e. a trivial cofibration. Moreover, there is an obvious fibration $A^{\to} \to A \times_{\Gamma} A$ that represents the Identity Type. Summarised, our decomposition of the diagonal is

$$A \xrightarrow{a \mapsto id_a} A^{\to} \xrightarrow{p \mapsto (dom(p), codom(p))} A \times A$$

This model has in some way marked the beginning of the whole development. It is due to Martin Hofmann and Thomas Streicher's work [22], who used it to answer an important question about the *uniqueness of identity proofs*:

Corollary 4.3 (*UIP* is not provable from J, M. Hofmann and T. Streicher)**.** *Given two terms $a, b : A$ and two proofs $p, q : Id_A\, a\, b$, it is not possible to construct an inhabitant of $Id_{Id_A\, a\, b}\, p\, q$ without using axioms beyond J.*

Proof. We use the weak factorization system given above, together with theorem 2.5. We do not work out the details here. A subtle point is the question if it is possible to model dependent function spaces, as the category of small groupoids is not locally cartesian closed. In fact, it is, as shown by Hofmann & Streicher [22], and Palmgren [34] explains this by discussing that pullbacks along fibrations have "semi-strict pseudo-adjunctions".

Remark. It causes regularly some confusion that two objects in A^I such as $a \xrightarrow{f} b$ and $a \xrightarrow{g} b$, are always propositionally equal, since $A^I = A^{\to}$ and the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, dashed, "id_a"'] & b \arrow[d, dashed, "g \circ f^{-1}"] \\ a \arrow[r, "g"] & b \end{tikzcd}$$

is obviously commutative. This should, however, not be too surprising, as [22] already mentioned that *UIP_tuple* is provable: Any (a, b, p) is equal to (a, a, refl_a), the crucial point is that p is *not* equal to refl_a.

Remark 4.4. While we have only used that small groupoids form a weak factorization system, they are even an example of a model category. To get this structure, take all the functors which are injective on objects as cofibrations and the usual categorical equivalences as weak equivalences.

5 Simplicial Sets

A *simplicial set* is a presheaf over the category of (isomorphy classes of) finite totally ordered nonempty sets and monotone functions. We want to introduce the most important concepts and provide some intuition.

5.1 General introduction to simplicial sets

Our main source for this subsection is Greg Friedman's article[17]. Another valuable article is [37] which is somewhat more direct but also discusses fewer concepts. As simplicial sets play an important role in homology and related topics, books such as [35], [19] and [29] can also serve as references. Here, we only give a brief summary. For everything beyond that, we highly recommend Friedman's article.

Definition 5.1 (category Δ)**.** For every natural number $n > 0$, we define $[n]$ to be the set $\{0, 1, \ldots, n-1\}$ (equivalently, the finite ordinal). Δ is the category that has the $[n]$ as objects and all monotone maps ($l \leq k$ implies $f(l) \leq f(k)$) as morphisms.

Remark 5.2. Caveat: The literature does not completely agree on the definition of Δ, but the different definitions are equivalent. It is still necessary to pay attention to avoid confusion. In particular, $[n]$ is sometimes defined to be $\{0, \ldots, n\}$ and the condition $n > 0$ is dropped, thereby shifting everything by 1. However, the empty set is usually never seen as an object of the category, i. e. Δ does not have an initial object.

Definition 5.3 (category **sSet**)**.** **sSet** is the functor category $\mathbf{Set}^{\Delta^{op}}$.

Although the above definition is short and precise, it is sometimes helpful to use a picture:

Given a simplicial set, i.e. a functor $X : \Delta^{op} \to \mathbf{Set}$, one can (to some extend) visualise $X[1]$ as a discrete set of points in the space. $X[2]$ is a set as well. We can visualise it as a set of directed connections, or lines, between pairs of points in $F[1]$. To see how this can be justified, notice that in Δ, there are two maps from the one-point set [1] to the two-point set [2]. Consequently, in Δ^{op}, there are two maps from [2] to [1]. Applying X on them, these are exactly the two maps that map every directed connection on its source respectively its target. Further, in Δ, there is exactly one map from [2] to [1]. After applying X, this map maps every point $x \in X[1]$ on the trivial connection from x to x. Therefore, the visualisation of only $X[1]$ and $X[2]$ looks like a directed multigraph with loops. Similarly, an object of the set $X[3]$ will be the shape of a triangle whose border is already given in the graph, $X[4]$ can be visualised as a tetrahedron, and so on. In fact, the image we have described can be seen as the *Grothendieck construction* $\int X$ (see)

In general, an element of $X[n]$ is called an *n-simplex of* X. We also call $X[n]$ the set of n-simplices. By the Yoneda lemma, this set can also be classified as $\mathbf{Set}^{\Delta^{op}}(\Delta^n, X)$.

It is handy to introduce a set of generators of the morphisms in Δ:

Definition 5.4. For i, n with $i \leq n$, we write D_i for the map $[n] \to [n+1]$ that is defined by

$$D_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{otherwise.} \end{cases}$$

Further, again for $i < n$, there is the map $S_i : [n+1] \to [n]$, defined by

$$S_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{otherwise.} \end{cases}$$

It is easy to see that those two classes of maps generate all the morphisms in Δ. They are central in the theory of simplicial sets:

Definition 5.5 (face and degeneracy maps)**.** Given a simplicial set X, the morphism $d_i := X(D_i) : X[n+1] \to X[n]$ is called *face map*, while $s_i := X(S_i) : [n] \to [n+1]$ is called *degeneracy map*.

The reason for these names can again be explained using the visualisation: d_i maps an element of $X[n+1]$, i.e. an $n+1$-simplex, on its ith face. This can also be expressed by saying that the ith corner is deleted, which collapses the rest. For $n = 1$, we have already seen that this morphism maps a directed line on one of its endpoints. For $n = 2$, it maps a triangle on one of its three faces, for $n = 3$, a tetrahedron is mapped on one of its four faces, and so on. Dually, s_i maps an n-simplex on an $n+1$-simplex by just using the ith corner twice. In the case of $n = 1$, we have described s_0 above as the map that maps a 1-simplex on the trivial connection to itself, i.e. a point is mapped on the degenerated line which has the point as both endpoints.

An n-simplex is therefore called *degenerated* if it can be written as $s_i(x)$ for some $n-1$-simplex x, else it is called *non-degenerated*.

A morphism in **sSet** is, of course, just a natural transformation between functors $F, G : \Delta^{op} \to \mathbf{Set}$. It maps points on points, lines on lines, triangles on triangles and so on and is therefore easy visualise as well. If we specify $\mu_{[n]}$ for such a natural transformation, all $\mu_{[m]}$ with $m < n$ are determined.

There is one type of simplicial sets that can, in some sense, be seen as the most basic type, often referred to as the *standard simplices*:

Definition 5.6 (standard simplex Δ^n)**.** For any positive integer n, we define $\Delta^n := y[n]$, where y is the *Yoneda embedding* $y : C \to \mathbf{Set}^{C^{op}}$ for a locally small category C. If we spell this out, Δ^n is the simplicial set given by the functor $\Delta(\cdot, [n]) : \Delta^{op} \to \mathbf{Set}$. Its visualised version looks like a (regular) triangle of dimension $n-1$; i.e., Δ^1 looks like a single point, Δ^2 like a line, Δ^3 like a triangle (with its body), Δ^4 like a tetrahedron, and so on.

Note that Δ^n always has exactly one non-degenerated n-simplex. More general, for each m, Δ^n contains exactly $\binom{n}{m}$ non-degenerated elements as any set of m distinct points (elements of $\Delta^n[1]$) form the boundary of exactly one non-degenerated m-simplex. Also note that Δ^1 is the terminal object of **sSet**. Caveat, again: With the alternative formalization of remark 5.2, Δ^0 is the terminal object. Another possible way of dealing with this is defining Δ^n to be $y[n+1]$.

5.2 Kan fibrations

Basically, a *Kan fibration* is a simplicial map satisfying a certain lifting property, which should not be surprising. Again, we recommend [17] and the articles mentioned above as our main references.

Definition 5.7 (k^{th} *horn* Λ^n_k)**.** For $k < n$, the k^{th} *horn* (denoted by Λ^n_k) of the simplex Δ^n can be defined by the full subcategory that is given by

$$\Lambda^n_k[j] := \{f : [j] \to [n] | \exists i \in [n]. i \neq k \wedge i \notin f[j]\}.$$

Here, we make use of the Yoneda embedding again. Λ^n_k is obtained by removing the "interior" and the $n-1$-dimensional boundary piece at position k. There is therefore an obvious inclusion $\Lambda^n_k \hookrightarrow \Delta^n$.

Definition 5.8 (Kan fibration)**.** Finally, a morphism $f : E \to B$ in **sSet** is a *Kan fibration* if, for any (k, n), any commutative diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & E \arrow[d, "f"] \\ \Delta^n \arrow[r] & B \end{tikzcd}$$

has a diagonal filler $j : \Delta^n \to E$.

The idea is the same as for all fibrations: "If we can complete something in B, then we can also complete it in E". The fibrant objects, i.e. those objects E such that $E \to \Delta^1$ is a Kan fibration, are called *Kan complexes.*

Note that even such simple examples as Δ^n fail to be Kan complexes (see [17]):

$$\begin{tikzcd} \Lambda^3_0 \arrow[rr, "0\mapsto 0, 1\mapsto 1, 2\mapsto 0"] \arrow[d, hook] & & \Delta^2 \arrow[d, "1"] \\ \Delta^n \arrow[rr] & & \Delta^1 \end{tikzcd}$$

Λ^3_0 has the three constant functions as 1-simplices, which we have just called $0, 1, 2$ above. It is a good idea to think about the triangle with vertices labelled $0, 1, 2$ and, as it is the 0^{th} horn, without the body and without the edge between 1 and 2. The given mapping for these constants determines the whole simplicial map. Δ^2 is, thinking this way, the line with two endpoints, labelled 0 and 1. The upper morphism works well; however, there is *no* diagonal filler because we would have to map the edge between 1 and 2 to the edge between 1 and 0, but in the wrong way ($1 \mapsto 1, 2 \mapsto 0$), which is not a morphism in **sSet**.

5.3 Simplicial sets and spaces

Simplicial sets are used as a completely algebraic model of "nice" topological spaces. To make the connection clear, we first need to become more serious of what we have called "visualisation so far:

Definition 5.9 (Realisation of standard simplices)**.** For all $n > 0$, we denote by $|\Delta^n|$ the realisation of the standard simplex, that is the topological space given by

$$|\Delta^n| := \{(x_0, x_1, \ldots, x_n \in \mathbb{R}^{n+1} | 0 \leq x_i \leq 1, \sum x_i = 1\}$$

This definition is straightforward and well-known. We can use it to define a functor $\mathbf{Top} \to \mathbf{sSet}$:

Definition 5.10 (singular set functor)**.** The *singular set functor* $\mathfrak{S} : \mathbf{Top} \to \mathbf{sSet}$ is given by

$$\mathfrak{S}Y := \mathbf{Top}(|\Delta^n|, Y)$$

This means, if Y is a space, then $\mathfrak{S}Y$ is the set of "ways how Δ^n can be embedded in Y", i.e. the set of all "pictures" of Δ^n. Of course, $\mathfrak{S}Y$ is very large unless Y is discrete.

We now want to be more precise about the notion of realisation, or "visualisation". While intuitively easy, it is surprisingly hard to define a functor that builds a space out of a simplicial set X in a reasonable way. The definition we state is given in [37]. We choose it as it is compact and at the same time not (much) more confusing than the "more down-to-earth" definition given in [17]. Clearly, any set can be viewed as a discrete space, in particular, $X[n]$ is one. Consider the product of spaces $|\Delta^m| \times X[n]$. Given $f : [n] \to [m]$ in Δ, there is a canonical continuous map $f_* : |\Delta^m| \times X[n] \to |\Delta^n| \times X[n]$ doing nothing on $X[n]$ and sending the standard simplex of dimension m to the one of dimension n. Similarly, there is the map $f^* : |\Delta^m| \times X[n] \to |\Delta^m| \times X[m]$. We now define (where we write $\coprod$ instead of $+$ for the coproduct):

Definition 5.11 (geometric realisation functor)**.** The *geometric realisation* *functor* $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ is given by

$$|X| := \mathit{colimit}\left(\coprod_{f:[n]\to[m]} |\Delta^m| \times X[n] \overset{f_*,f^*}{\rightrightarrows} \coprod_{[n]} |\Delta^n| \times X[n]\right)$$

Note that this is often always written as $\int^n |\Delta^n| \times X[n]$.[5]

Intuitively, the realisation functor just does what it is supposed to do: For every standard simplex occurring in the simplicial set, it constructs its geometric version. This gives us quite a lot of "pyramids" in every dimension and we have to make sure that all the face and degeneracy maps are respected. This is done by taking the colimit.

Theorem 5.12 ($|\cdot| \dashv \mathfrak{S}$)**.** *The geometric realisation functor is left adjoint to the singular set functor.*

Proof. For an even more general statement see [37]. □

Especially interesting is that Kan complexes are actually in some way the same as CW-complexes:

Theorem 5.13. *The category of Kan complexes and homotopy classes of maps between them is equivalent to the category of CW complexes and homotopy classes of continuous maps, where the equivalences are given by* $|\cdot|$ *and* $\mathfrak{S}$*.*

Proof. See [35], theorem I.11.4. □

While **Top** is quite nice for the intuition (as, for example, described in the extended version of this report), it is not very well-behaved, which raises a lot of problems when it comes to the details of an interpretation of type theory. On the other hand, **sSet** is a purely combinatorial formulation with much better properties. The above theorem demonstrates how one should think, in summary: Work in **sSet**, but get intuition from **Top**!

5.4 The model in sSet

Multiple sources (including [8], [10]) explain that there is the following model structure on **sSet**:

1. cofibrations are the monomorphisms

2. weak equivalences are the weak homotopy equivalences (see below)

[5] This is a generalized version of the Grothendieck construction (if I am not mistaken)

3. fibrations are the Kan fibrations

Therefore, the weak factorization system we should use is

- $\mathfrak{L} = \mathit{monos} \cap \mathit{weak\ equivalences}$
- $\mathfrak{R} = \mathit{Kan\ fibrations}$

A morphism $f : X \to Y$ in **sSet** is a *weak homotopy equivalence* iff it induces isomorphisms on all homotopy groups. For the homotopy groups of a simplicial set X, there are several equivalent definitions (see [17], section 9), one of them saying that the n^{th} homotopy group is defined to be the n^{th} homotopy group of the topological space that is obtained by applying the realisation functor on X. Applying Whitehead's theorem [45], we should be able to conclude that a *weak homotopy equivalence* is just a map that becomes, after realization, a homotopy equivalence.

Simplicial sets are used by Voevodsky to model univalent type theory ([42], [43], [44]). For a good explanation of the construction (which is unfortunately quite involved), we would like to recommend Kapulkin & Lumsdaine & Voevodsky's [23] or Streicher's [40].

6 Univalence

We now switch to a different aspect: Instead of discussing model constructions, we examine interesting (possible) properties of identity types.

6.1 Contractibility

Contractibility is, in topology, a well-known property of topological spaces: A space is called contractible if (and only if) it is homotopically equivalent to the point. This means, a space X is contractible iff there exists a continuous map $H : X \times [0, 1] \to X$ and a point $a \in X$ so that, for all $x \in X$, we have $F(x, 0) = x$ and $F(x, 1) = a$; in other words, F is at "time" 0 the identity and at "time" 1 constant.

For a type A, the notion is defined analogously:

Definition 6.1 (Contractibility)**.** A type A is called *contractible* if the type $\mathit{Contractible}(A) := \Sigma_{a:A}.\Pi_{a':A}.\mathit{Id}_A\, a\, a'$ is inhabited.

Unsurprisingly, this definition requires A to be inhabited by a distinguished element a. Furthermore, every other element has to be equal to a. At first, this property might look a bit weak: The corresponding ω-groupoid of A has, obviously, "up to propositional equality" exactly one 0-cell, but what about higher cells? There is no need to worry, as we will soon understand that this definition does indeed imply the same thing for all levels.

In the homotopy interpretation, the above definition looks like the definition of path-connectedness. However, if we have a closer look, we notice that it gets interpreted as *There exists a point $a \in A$ and a continuous function f which maps every point $x \in A$ on a path between a and x.* The continuity of f is the important detail. For example, consider the space S^1 It can, for example, be defined as the set of all points in the euclidean plane that have distance 1 from the origin. Another possible definition is to define it as the CW-complex with one 0-cell, where we attach one 1-cell in the obvious way; and this meets the type theoretic definition of the circle as a *higher inductive type* (see [26], [39]) quite well. For the moment, let us identify the circle with $[0, 2\pi]$, divided by the relation that unifies 0 and 2π. Assume that there is a continuous map f mapping a point x of this interval to a path from 0 to this point. $f(0)$ is a path from 0 to 0. Now, increase the argument x continuously; this makes the path $f(x)$ change continuously. Therefore, $f(2\pi)$ is homotopic to the path from 0 to 2π, composed with $f(0)$; but of course, $f(2\pi)$ is just $f(0)$, which shows that any path from 0 to itself is null-homotopic, contradicting the properties of the circle.

6.2 Homotopy Levels

The notion of *homotopy levels* corresponds (roughly) to the question which homotopy groups of a space are nontrivial. Clearly, for a contractible space, they are all trivial; and in fact, we define *H-level*$_0$ just to be the same as contractible. A space is still "relatively simple" if it becomes contractible after replacing it by it's path space (or iterating this step several times). For types, we define analogously:

Definition 6.2 (homotopy level)**.** A type A is said to be of *homotopy level 0* just if it is contractible:

$$\textit{H-level}_0\,(A) := \textit{Contractible}\,(A)$$

Moreover, if all the identity types are of *homotopy level n*, then A is of *homotopy level $n + 1$*.

$$\textit{H-level}_{n+1}\,(A) := \Pi_{ab:A}.\textit{H-level}_n\,(Id_A\,a\,b)$$

Remark 6.3. For small homotopy levels, the following notions are commonly used:

- A type of *homotopy level 0* is a singleton type, (isomorphic to) the unit type, or just contractible.

- A proposition, i.e. a type with at most one inhabitant, has *homotopy level 1*.
- The types of *homotopy level 2* are called *sets*.
- Less frequently used, but reasonable, is writing *groupoids* for types of *homotopy level 3*.
- Similarly, the *homotopy level 4* types are 2-groupoids, those of level 5 are 3-groupoids, and so on. In general, types do not need to have a finite homotopy level or the homotopy level might just not be provable. Univalence ensures that there is a type of homotopy level 3, the universe type. The hierarchy of universes determines for which levels a type exists, such that the type is provable not of the corresponding level.

6.3 Weak Equivalences

To understand the notion of a *weak equivalence*, homotopical intuition is, again, quite helpful. First of all, if $f : A \to B$ is a function and $b \in B$, we define the preimage of b:

Definition 6.4 (Preimage of $f : A \to B$)**.** The preimage of a function at b is defined as the set of pairs of a point, together with a proof that this point is indeed mapped to b:

$$f^{-1}b := \Sigma_{a:A}.\mathit{Id}_B\, b\, f(a)$$

Definition 6.5 (Weak equivalence property)**.** A function $f : A \to B$ is called a *weak equivalence* if all preimages are contractible:

$$\mathit{isWeq}\, f := \Pi_{b:B}.\mathit{Contractible}\left(f^{-1}b\right)$$

Definition 6.6 (Weak equivalence)**.** We define a *weak equivalence* between types A, B to be a map, together with a proof that this map is indeed a weak equivalence:

$$\mathit{Weq}\, A\, B := \Sigma_{f:A\to B}.\mathit{isWeq}\, f$$

A second where natural definition is the one of an *isomorphism*:

Definition 6.7 (Isomorphism)**.** An *isomorphism* between types A, B is a tuple consisting of a map in each direction, together with a proof that each composition is (extensionally equal to) the identity:

$$\mathit{Isos}\, A\, B := \Sigma_{\phi:A\to B,\psi:B\to A}.\Pi_{a:A}.\mathit{Id}_A\, (\psi \circ \phi(a))\, a \times \Pi_{b:B}.\mathit{Id}_B\, (\phi \circ \psi(b))\, b$$

Remark 6.8. The notions of a weak equivalence and an isomorphism are logically equivalent, but not isomorphic: a weak equivalence always is an isomorphism and vice versa, but the two types $\mathit{Weq}\,A\,B$ and $\mathit{Isos}\,A\,B$ are in general not isomorphic. However, it is the case that we can make them isomorphic by adding a concrete coherence proof to the second one, thereby making the 4-tuple a 5-tuple. This coherence condition does follow from the other terms, but the crucial point is that no *unique* proof follows.

Lemma 6.9 (Composition with weak equivalences is weak equivalence)**.** *Assume* A, B, C *are types. If* $w : \mathit{Weq}\,B\,C$ *is a weak equivalence, then composition with* w *is a weak equivalence, i.e* $\mathit{Weq}\,(A \to B)\,(A \to C)$ *is inhabited.*

Proof. If u is the inverse of w in the alternative definition of a weak equivalence, it is enough to show that $\lambda f.w \circ f$ and $\lambda f.u \circ f$ are inverse. More precisely, it is sufficient to prove that their composition is extensionally equal to the identity on $A \to B$ respective $A \to C$, which is straightforward. □

6.4 The Univalence Axiom

From the previous section, it is clear that the identity function on any type A is always a weak equivalence (more precisely, can be completed canonically to a weak equivalence); by $\mathit{idIsWeq}$, we denote the canonical map of type $\forall A\,.\,\mathit{Weq}\;\,A\,A$. Assume A, B are types. Furthermore, assume p is an inhabitant of $\mathit{Id}\,A\,B$. From p, we can construct a weak equivalence between A and B: Using the J eliminator, we only have to give this construction if p is the reflexivity proof; but in that case, $\mathit{idIsWeq}$ is just what we need. The *Univalence Axiom* states that this map is again a weak equivalence.

The contents of this sections are, by the best of my knowledge, originally by Voevodsky; they are nicely presented in Bauer & Lumsdaine's notes [25].

We first define the "canonical map" precisely:

Definition 6.10. There is map of type $\forall A\,B\,.\,\mathit{Id}\,A\,B \to \mathit{Weq}\;\,A\,B$, constructed as

$$\mathit{eqToWeq} = J\,(\lambda\,A\,B\,.\,(p : \mathit{Id}\,A\,B) \to \mathit{Weq}\;\,A\,B)\;\mathit{idIsWeq}$$

Definition 6.11 (Univalence Axiom)**.** The map $\mathit{eqToWeq}$ is a weak equivalence. In other words, the Univalence axiom postulates a term of type

$$\forall A\,B\,.\,\mathit{isWeq}\;\mathit{eqToWeq}$$

The Univalence Axiom provides us with the possibility to treat weak equivalences similarly as propositional equalities. To make this clear, we prove the following:

Theorem 6.12 (Induction on weak equivalences)**.** *Given some*

$$P : \forall U, V \,.\, \mathit{Weq}\, U\, V \to \mathit{Type},$$

assume we can construct a term for "canonical weak equivalences". More precisely, assume we can construct a term of the type

$$m : \forall T \,.\, P\, T\, T\, (\mathit{idIsWeq}\ T)$$

Then we can also construct an inhabitant of P.

Proof. Define

$$P' : \forall U\, V \,.\, \mathit{Id} U\, V \to \mathit{Type}$$

by

$$P' = \lambda U\, V\, q \,.\, P\, U\, V\, (\mathit{eqToWeq}\ q)$$

Now, $\forall U \,.\, P'\, U\, U\, \mathit{refl}_U$ is inhabited by m (this uses the β rule of identity types). By J, P' is inhabited. Given any U, V as well as $w : \mathit{Weq}\ \ U\, V$ and univalence, we get a proof $p : \mathit{Id} U\, V$. But $\mathit{eqToWeq}\, p$ is equal to w, so using the constructed inhabitant of $P'\, U\, V\, p$ and J (or just a substitution rule that follows from J), we get an inhabitant of $P\, U\, V\, w$. □

We want to conclude with a proof that Univalence implies Extensionality of functions, i.e., if two functions are pointwise equal, we can prove that they are equal. We summarise the main argument of [25].

Lemma 6.13 (source and target are weak equivalences)**.** *Recall that we write A^I for $\Sigma_{a,b:A}.\mathit{Id}_A\, a\, b$. Given a type A, the canonical projection maps $\mathit{src}_A : A^I \to A$ and $\mathit{trg}_A : A^I \to A$ are weak equivalences.*

Proof. We only give a sketch of the proof for the *src* function. Here, it seems to be advantageous to use definition 6.7. We want to prove that the map $r_A : A \to A^I$ is an inverse of src_A (recall $r_A = \lambda a \,.\, (a, a, \mathit{refl}_a)$). It is clear that $\mathit{src}_A \circ r_A$ is extensionally equal to id_A. For the other direction, we have to show that every term $(a, b, p) : A^I$ equals (a, a, refl_a). But, using the J eliminator, it is enough to show this if (a, b, p) is (a, a, refl_a), and in this case, it follows by reflexivity. □

Theorem 6.14 (Univalence and η imply Extensionality)**.** *Assume we have, for types A, B and functions $f, g : A \to B$, a proof that f and g are pointwise equal; i.e. we have $p : \Pi_{a:A}.\mathit{Id}_A\, (f a)\, (g a)$. Using the Univalence axiom (and the usual η law for functions), we can construct an inhabitant of $\mathit{Id}_{A \to B}\, f\, g$.*

Proof. We sketch the proof that is given in [25]. Define

$$d := \lambda a \,.\, (f a, f a, \mathit{refl}_{f a})$$
$$e := \lambda a \,.\, (f a, g a, p a)$$

Now, $\mathit{src}_A \circ d = \lambda a \,.\, f a = \mathit{src}_A \circ e$. But for any weak equivalence s, $\mathit{Id}\, d\, e$ is inhabited iff $\mathit{Id}\,(s \circ d)(s \circ e)$ is, which is easily shown by induction on weak equivalences. We therefore just need to apply lemma 6.12 to see that $\mathit{Id}\, d\, e$ is inhabited and also $\mathit{Id}\,(\mathit{trg}_A \circ d)\,(\mathit{trg}_A \circ e)$, which is just $\mathit{Id}\,(\lambda a \,.\, f a)\,(\lambda a \,.\, g a)$, so the η law solves it. □

Remark 6.15. For simplicity, we have only stated the nondependend form of extensionality. The dependent version holds as well, but is more involved.

7 Hedberg's Theorem

In 1998, Michael Hedberg has published a proof that, for a given type, decidable equality implies uniqueness of identity proofs [20]. His original proof is quite lengthy, though it provides a couple of very interesting insights. Here, I want to present a much more direct proof, which I have also formalised in Coq (available on my homepage). There is also a post on the HoTT blog [1] on the topic.

Definition 7.1 (decidability)**.** A type A is said to be *decidable* if there is either a proof that it is inhabited or a proof that it is not:

$$\mathit{Decidable}_A = A + \neg A$$

where, of course, $\neg A$ is just short-hand for $A \to \bot$. *Decidable equality* means that we can, for each pair of terms, decide their equality type:

$$\mathit{DecEqu}_A = \forall a\, b \,.\, \mathit{Decidable}_{(\mathit{Id}_A\, a\, b)}$$

Uniqueness of identity proofs has already been introduced at the very beginning of this composition, we just repeat the definition in the form of a type:

Definition 7.2 (uip)**.**

$$\mathit{UIP}_A = \forall a\, b : A \,.\, \forall p\, q \;:\; \mathit{Id}_A\, p\, q \,.\, \mathit{Id}\, p\, q$$

Hedberg's theorem states that decidable equality implies UIP:

Theorem 7.3 (Hedberg)**.**

$$\mathit{DecEqu}_A \to \mathit{UIP}_A$$

Proof. Assume *dec: deceqA* (in some context Γ). Further, assume $(a, b, p) : A^I$ (in the context). We can now "ask" the "deciding function" *dec* what it "thinks" about a, b respectively a, a; it will either tell us that they are equal or unequal. The latter case would, however, immediately lead to a contradiction, as we already know that a and b are equal. Therefore,

$$\begin{aligned} dec\, a\, b &= \texttt{inl}\ \ q_1 \qquad \textit{for some } q_1 : Id_A\, a\, b \\ dec\, a\, a &= \texttt{inl}\ \ q_2 \qquad \textit{for some } q_2 : Id_A\, a\, a \end{aligned}$$

We claim that p equals $q_1 \circ q_2^{-1}$ propositionally (using the notation of remark 1.2.2 in the extended version: $\circ$ and $\cdot^{-1}$ are the transitivity and symmetry functions that arise from the equality eliminator). But applying J, we only need to prove it for $(a, b, p) = (a, a, \mathit{refl}_a)$, in which case q_1 and q_2 are the same, so that it suffices to observe that $q_2 \circ q_2^{-1}$ equals reflexivity. As every inhabitant of $Id_A\, a\, b$ equals $q_1 \circ q_2^{-1}$, there cannot be more than one. □

8 Yoneda Groupoids, Higher Dimensional Quotients and the Root of Equality

As the content of this section has originally been an independent note and it has not been completely adapted to this report, it might contain a couple of things that have been discussed before.

Ordinary quotients in type theory have already been examined ([4]). These are just types modulo an equivalence relation, where an equivalence relation can be seen as a setoid, i. e. as a type of homotopy level 2. With the possibility to speak about types of a higher level at hand, it is natural to ask what a *higher quotient* could be. Intuitively, we should be able to divide a type not only by a setoid, but by a type of an arbitrary homotopy level. However, this is quite involved: Already for the division by a groupoid, there is not really a canonical generalization of the setoid case. The problem is, as one might have expected, that the number of coherence conditions grows rapidly with the number of levels. A nice, convincing formulation has not been found yet.

We define the notion of a *Yoneda Groupoid* formally, the name of which is inspired by the relation to the Yoneda lemma, and show how a *weak ω groupoid* structure can be extracted. We also prove that, in the presence of bracket types (in the sense of Awodey & Bauer [9]), every Yoneda Groupoid gives rise to a higher quotient. All of this is done purely syntactically, thereby making Yoneda Groupoids a very powerful concept inside the theory itself and completely independent of the Meta theory.

The question whether and in which way a Yoneda Groupoid is a stronger structure than an ordinary weak ω groupoid leads to the notion of the *Root of*

Equality, giving rise to a problem in (∞, ∞)-category theory. This question seems to be fundamental but has, to the best of our knowledge, not been considered so far and is therefore an open problem.

8.1 Introduction to this topic

With the development of *Homotopy Type Theory*, the notion of a *weak ω groupoid* has gained central importance. As van den Berg & Garner [12] and Lumsdaine [28] have proved, every type, together with its equality, carries this structure.

A *weak ω groupoid* has, as the name suggests, three basic characterising features. First, it is a higher category with one level of cells for every natural number. While this might look like a complicated concept (which it certainly is!), an ω-category is more symmetric and therefore more natural than an ordinary or an n-category. Every level is an ω-category again, meaning that every level has the same fundamental structure. Second, as a *groupoid*, all the morphisms (which are usually referred to as *non-zero cells*) are isomorphisms. Finally, the structure is called *weak* as the usual laws, such as associativity, hold only "up to homotopy" (or "up to isomorphy"), meaning that $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are not strictly equal, but only isomorphic. This is natural as category theorists do not speak about equality on the object level, and here, every cell is, for some category, on the object level.

While varies informal definitions of this concept exist, a formalization is involved. One recent approach by Altenkirch & Rypacek [6] makes use of *globular sets*. Another attempt are Coquand & Huber's constructive Kan complexes [14].

As homotopy type theory gives up the principle of *uniqueness of identity proofs*, is is natural to ask how a type with a nontrivial higher structure can in general be constructed. One interesting approach are *higher inductive types* (pushed forward by many researchers in this area, including Peter LeFanu Lumsdaine and Michael Shulman). Another approach, which is apparently expressionwise less powerful, but also does not require the theory to provide as many additional features (the semantics and computational rules of which still have not completely been developed yet), are *higher quotients*. These are a straightforward generalization of ordinary quotients in type theory (at least as long as we only divide a set-like type). However, the problem is that clearly, we can only quotient a type by a *higher relation* if this relation has the structure of a weak ω groupoid, which, then again, has no reasonable formalization so far.

In this work, we introduce the notion of a *Yoneda Groupoid*. Those groupoids are weak ω groupoids and in fact, we can, purely syntactically, extract the groupoid structure. All coherence conditions hold, the proof of which can also be constructed without Meta-theoretic reasoning. Finally, and we consider this the high point of our work, in the presence of bracket types (in the sense of Awodey & Bauer [9]), we can purely syntactically construct the corresponding higher quotient from a Yoneda Groupoid. In the special case of an equivalence relation, our quotient is the *exact quotient* of Altenkirch, Anberrée & Li [4]. We can prove all of their conditions inside the theory. Further, their definition can be straightforwardly generalised to higher dimensions, and our Yoneda quotients still fulfil this definition.

We work in a univalent Intensional Type Theory, i. e. there are equality types and the *univalence axiom*, but not *axiom K* or *uniqueness of identity proofs*. Our theory has dependent sum and function types, (preferably) a hierarchy of universes and is, in total, exactly the kind of theory that is used for *homotopy type theory* and *univalent foundations*. For the type of propositional equality proofs that a equals b, we write $a \equiv b$. The convention is that $\equiv$ binds stronger than $\to$. We also use the symbol $\sim$ (for relations) and $\leftrightarrow$, where $A \leftrightarrow B$ is a shorthand for $(A \to B) \times (B \to A)$. The decreasing order of binding strength of the symbols is $\sim$, $\equiv$, $\to$, $\leftrightarrow$.

Given a type A , we can talk about a *higher relation*

$$\sim : A \to A \to \mathbf{U},$$

where $\mathbf{U}$ is some universe. At the moment, we restrict ourselves to the smallest universe which we call **Type**. We are interested in the question whether this relation can be given the structure of a weak ω groupoid, where the 0-cells are just the terms of A, the 1-cells between a and b are just the terms of $a \sim b$, the 2-cells between $s, t : a \sim b$ are just the proofs that s equals t, and so on.

This question is very closely related of a problem stated by Thorsten Altenkirch:

Question 8.1 (Altenkirch)**.** Given $\sim : A \to A \to \mathbf{Type}$ with terms

- $\mathit{refl}^{\sim} : \forall a \,.\, a \sim a$,
- $\mathit{sym}^{\sim} : \forall a \,,\, b \,.\, a \sim b \to b \sim a$,
- $\mathit{trans}^{\sim} : \forall a \,,\, b \,,\, c \,.\, a \sim b \to b \sim c \to a \sim c$,

how can we formalise the proposition that it is a weak ω groupoid?

A straightforward idea of approaching this question is stating all the coherence conditions. For example,

$$\lambda : \forall p \,.\, (\mathit{trans}^{\sim}\ \mathit{refl}^{\sim} p) \equiv p$$
$$\rho : \forall p \,.\, p \equiv (\mathit{trans}^{\sim}\ \mathit{refl}^{\sim} p)$$

(where we hide arguments that can be inferred for readability) are necessary coherence conditions. But now, we get a new coherence condition,

$$\lambda\, \mathit{refl}^{\sim} \equiv \rho\, \mathit{refl}^{\sim},$$

and in general, every new condition gives rise to even more new conditions. Nevertheless, a similar approach was taken by Altenkirch & Rypacek [6].

8.2 Yoneda Groupoids

Compared to the strategy described above, we chose a different approach. If a relation $\sim: A \to A \to \mathbf{Type}$ is well-behaved, it should satisfy some kind of Yoneda property. The condition we state is quite strong and, in particular, sufficient, but not necessary. For the rest of the section, assume $\sim: A \to A \to \mathbf{Type}$ is a given higher relation.

Definition 8.2 (Yoneda Groupoid)**.** A relation $\sim$ is a *Yoneda Groupoid* if there is a function mapping every $a : A$ to a pair (n, X), where n is the "label" of its equivalence class and X represents this class's structure (we discuss the latter point later in detail).

$$\mathbf{isGrp}(\sim) = \Sigma F : A \to \mathbb{N} \times \mathbf{U} \,.\, \forall a\,,\, b : A \,.\, a \sim b \equiv (F\, a \equiv F\, b)$$

$\mathbf{U}$ could be any available universe or type. However, if $\mathbf{U}$ is just some type $Q : \mathbf{Type}$, then Q would already have to be some sort of super-quotient (meaning that a subtype of Q is the quotient), and therefore, we consider this case rather uninteresting. Our focus shall lie on the possibility that $\mathbf{U}$ is a universe, as univalence provides then additional equality proofs. For our discussion, we find it convenient to choose $\mathbf{U} = \mathbf{Type}$, so let us assume that we are using the smallest universe.

If the cardinality of $\mathbb{N}$ is not sufficient, any other proper set could serve for the labelling. In fact, we could even make the indexing set part of the definition in the form of

$$\mathbf{isGrp}(\sim) = \Sigma I : \mathbf{Type}, \mathit{h\text{-}level}_2(I), F : A \to I \times \mathbf{U} \,.\, \forall a\,,\, b : A \,.\, a \sim b \equiv (F\, a \equiv F\, b).$$

Note that, for Yoneda Groupoids, we could make univalence unnecessary by replacing equality by weak equivalence. For the quotients discussed later, this will not be possible any more.

This definition is inspired by two different formalisations of equivalence relations in the proof-irrelevant case and can actually be understood as a combination of those. The first is, for equivalence relations $\sim: A \to A \to \mathbf{Prop}$, the "Yoneda"-characterisation

$$\forall a, b \,.\, a \sim b \leftrightarrow \forall x \,.\, a \sim x \leftrightarrow b \sim x$$

We have not seen this concrete characterisation before, but we assume that it is well-known as it looks completely natural. Unfortunately, it is not possible to generalise it in the straightforward way to higher relation as there is an unwanted "shift" of the level by 1 included. If we try to use the type

$$\forall a, b \,.\, a \sim b \equiv \forall x \,.\, a \sim x \equiv b \sim x,$$

we quickly realise that the right-hand side goes up "one equality level too much". For example, if we have a type with only one term a, and $a \sim a \equiv \mathbf{5}$, then the left-hand side is $\mathbf{5}$, while the right-hand side is $\mathbf{120}$ (there are 5! automorphisms on the set $\mathbf{5}$). We could try to fix this by stating

$$\forall a, b \,.\, a \sim b \leftrightarrow \forall x \,.\, a \sim x \equiv b \sim x,$$

but clearly, logical equivalence is not enough for a valid characterisation.

The second source of inspiration has been the definition of an equivalence class by Voevodsky:

$$\mathbf{isCl}(P : A \to \mathbf{Prop}) = [\Sigma(a : A).P\, a] \times ...$$

Here, it is already assumed that $\sim$ is an equivalence relation and the idea is that the quotient is just the collection of equivalence relations. Originally, our definition used equivalence classes, making it very similar to the one of Voevodsky. After realising that it is not necessary to have a whole type of equivalence classes indexed over A (which works, but it involved), we were able to simplify it by just using a single $F : A \to \mathbf{Type}$ which also adds indices from a proper set (such as the natural numbers) to distinguish classes that are isomorphic, but distinct.

Notation. Whenever a term is quantified universally by $\forall$, we consider it an *implicit argument*. This has no meaning for the theory but only for our representation: If a term is applied on an implicit argument, we use indices for better readability, i. e. if we have $i : \forall a \,,\, b : A \,.\, a \sim b \equiv (F\, a \equiv F\, b)$, we write $i_{a,b}(s)$ instead of $i\, a\, b\, s$.

Our main result, which we prove step-by-step in the next sections, is the following:

Theorem 8.3. *Given a proof term of* $\mathbf{isGrp}(\sim)$*, we can construct terms* $\mathit{refl}^{\sim}$*,* $\mathit{sym}^{\sim}$ *and* $\mathit{trans}^{\sim}$ *that turn* $(A, \sim)$ *into a weak* ω *groupoid. In particular, all the coherence conditions (as mentioned, e. g., in [6]) hold, the proof of which can also be constructed. Moreover, if bracket types [9] are available in the theory, we can directly construct the* quotient $A/\sim$*. If* A *is a* h-set *in the sense of homotopy type theory and* $\sim$ *is an equivalence relation, i. e.* $\sim : A \to A \to \mathbf{Prop}$*, our quotient is just an exact quotient in the sense of Altenkirch, Anberrée & Li [4], without the Meta-property that* *the β rule holds definitionally. Further, their definition of an exact quotient can be generalised to arbitrary higher relations, and our quotients satisfy this generalised definition.*

However, the constructed quotient will not be in $\mathbf{Type}(= \mathbf{Type}_0)$ anymore, but in the next universe $\mathbf{Type}_1$ instead. This is not at all surprising, as univalence and *uniqueness of identity proofs* are not inconsistent as long as only one universe is available. More general, in order to construct a type which has provably not h-level (n+1), a universe $\mathbf{Type}_n$ is required.

8.3 Groupoid Structure of a Yoneda Groupoid

We begin with the fairly simple proof that a Yoneda Groupoid is indeed a weak ω groupoid.

Lemma 8.4. *Given $p : \mathbf{isGrp}(\sim)$, the higher relation $\sim$ carries the structure of a weak ω groupoid (in the sense of [6]??) and this structure can be extracted purely syntactically from the proof p.*

Remark 8.5. Of course, the structure is not unique in general, as we have no way to distinguish between terms of $a \sim b$ (without looking at p). But, and this is more important, even up to isomorphism, there are fundamentally different structures. For example, for $A = \mathbf{1}$ and $\_ \sim \_ = \mathbf{6}$, $\sim$ could be either the equivalent of the group $\mathbb{Z}/(6)$ or the equivalent of the permutation group S_3. It really is the proof of **isGrp** that makes the choice.

Proof. The main ingredient of our construction is the groupoid property of equality itself. In particular, equality provides the usual terms $\mathit{refl} : \forall a . a \equiv a$ and $\mathit{sym} : \forall a\,,\, b\,.\, a \equiv b \to b \equiv a$ as well as $\mathit{trans} : \forall a, b, c . b \equiv c \to a \equiv b \to a \equiv c$.

The proof $p : \mathbf{isGrp}(\sim)$ is necessarily a pair (F, i).

We define $\mathit{refl}^{\sim}$, $\mathit{sym}^{\sim}$ and $\mathit{trans}^{\sim}$ in terms of p. For readability, we first omit all implicit arguments in the definitions (we give the precise definitions below):

$$\begin{aligned} \mathit{refl}^{\sim} &= \mathit{sym}\, i\, \mathit{refl} \\ \mathit{sym}^{\sim}\, s &= \mathit{sym}\, i\, (\mathit{sym}\, (i\, s)) \\ \mathit{trans}^{\sim}\, t\, s &= \mathit{sym}\, i\, (\mathit{trans}\, (i\, t)\, (i\, s)) \end{aligned}$$

The strategy is the same in each case: We use the isomorphism (or equality) i to translate the problem to the case where $\sim$ is replaced by $\equiv$. Now, in the case of equality, we know exactly how the required operation can be done, and we can just transport the result back using the inverse of i.

With all implicit arguments, the definitions are:

$$\begin{aligned} \mathit{refl}^{\sim}{}_{a} &= \mathit{sym}_{a \sim a, F\,a \equiv F\,a}\, i_{a,a}\, (\mathit{refl}_{F\,a}) \\ \mathit{sym}^{\sim}_{a,b}\, (s : a \sim b) &= \mathit{sym}\, i_{b,a}\, (\mathit{sym}_{Fa,Fb}(i_{a,b}\, s)) \\ \mathit{trans}^{\sim}_{a,b,c}\, (t : b \sim c)\, (s : a \sim b) &= \mathit{sym}_{a \sim c, Fa \equiv Fc}\, i_{a,c}\, (\mathit{trans}_{Fa,Fb,Fc}\, (i_{b,c}\, t)\, (i_{a,b}\, s)) \end{aligned}$$

Every single coherence condition just holds because it holds for equality. For example,

$$\mathit{sym}^{\sim} \circ \mathit{sym}^{\sim}(s) = \mathit{sym}\, i\, (\mathit{sym}\, (i\, (\mathit{sym}\, i\, (\mathit{sym}\, (i\, s)))))$$

is propositionally equal to s. For a proof, we just need to use that $i \circ (\mathit{sym}\, i)$ is the identity, then the same for $\mathit{sym} \circ \mathit{sym}$, and finally, that $(\mathit{sym}\, i) \circ i$ is the identity as well. It becomes even clearer if we write $\cdot^{-1}$ for sym and $f \circ g(a)$ instead of $f(g(a))$:

$$\mathit{sym}^{\sim} \circ \mathit{sym}^{\sim}(s) = i^{-1}\, ((i \circ i^{-1}\, (i\, s)^{-1})^{-1}$$

□

8.4 Quotienting by a Yoneda Groupoid

Lemma 8.6. *If the type theory has bracket types as introduced by Awodey & Bauer [9], every Yoneda Groupoid gives rise to a higher quotient in the sense of Altenkirch, Amberr'ee & Li [4].*

Proof. As before, the proof is some tuple (F, i). Define the "carrier" of the quotient

$$Q = \Sigma[a : A]; x : \mathbb{N} \times \mathbf{Type}; [F(a) \equiv x]$$

and the projection into the carrier

$$q(a) = ([a], F(a), [\mathit{refl}_{F(a)}]).$$

For soundness and exactness, we need to prove

$$\forall a\,,\, b\,.\, a \sim b \equiv (q(a) \equiv q(b))$$

which is obvious from i.

Concerning the eliminator, whenever $B : Q \to \mathbf{Type}$ and $m : (a : A) \to B[a]$ are given (with coherences), we can a given $([a], x, [w])$ just map to $m\, a$. Here, we have to use the property that $m\, a$ does not depend on the concrete representant a. This property is one of the assumption of the eliminator.

Note that it is in general not possible to construct an embedding $Q \to A$ and the quotient is therefore, in the sense of [4], not *definable*. □

8.5 Examples

Some examples:

- $A = \mathbf{1}$, $\_ \sim \_ = \mathbf{6}$ is a Yoneda Groupoid, proved by $(\lambda\_ \to \mathbf{3}, \mathit{someproof})$. The quotient is the symmetric group on $\mathbf{3}$, which is not inside the universe **Type** anymore. This is exactly how it should have been expected, as a single universe with univalence is consistent with the assumption that equality proofs are always unique. One universe above **Type** does not allow this assumption anymore, and indeed, we have constructed the group $S_3 : \mathbf{Type}_1$. $(A, \sim)$ has another possible quotient which is the group $\mathbb{Z}/(6)$, but unfortunately, we cannot get it with our construction.

- $A = \mathbf{1}$, $\_ \sim \_ = S_3$ (where we already need $\sim$ to be of the type $A \to A \to \mathbf{Type}_1$) is a Yoneda Groupoid as $(\lambda\_ \to S_3, \mathit{someotherproof}) : \mathbf{isGrp}(\sim)$ (If I am not mistaken, but it should be true, the symmetric group over S_3 is S_3 again.) The quotient gives us (assuming that enough universes are available) a type of h-level 4, let us call it $1_{S_3} : \mathbf{Type}_2$. Obviously, we could carry on this example to get types with higher and higher structure, making more and more universes necessary.

- In the same way, we can construct the quotients for $A = \mathbf{1}$, $\_ \sim \_ = \mathbf{n}!$ for any natural number n. It is always a Yoneda Groupoid by $(\lambda\_ \to \mathbf{n}, \mathit{yetanotherproof})$ and the quotient will be the symmetric group S_n. However, our construction does not provide us with any other group structure on $\mathbf{n}!$. If we carry on as in the example before, the only thing we have to care about is that the automorphism groups of S_2 and S_6 is not, as in every other case, S_2 and S_6 again, thereby making these two cases special.

- We can now freely combine the groups on different levels constructed above, for example, we get $S_3 \times 1_{S_7} + S_5 + 3 : \mathbf{Type}_2$, which is a groupoid with 5 distinguishable cells on level zero, 11 on level one, and 29 on level two. There are also 29 n-cells for every $n > 2$.

8.6 The Root of Equality

Our definitions immediately give rise to the question: When does this function F exist? Or, formulated more basically: Given a type C, in which cases is $\Sigma(B : \mathit{Type}), C \equiv (B \equiv B)$ inhabited?

In the example $C = \mathbf{6}$, a solution exists, namely $B = \mathbf{3}$, leading to the symmetric group S_3 as discussed before. However, we cannot construct the group $\mathbb{Z}/(6)$. In the case of $C = \mathbf{1}$, we get two solutions, namely S_0 and S_1.

The first question is: Can we find an appropriate structure if C is not a discrete set where the number of terms equals a factorial? First, does this structure exist in the ω groupoid model (resp. the simplicial set model)? Such a group does exist indeed for $C = \mathbf{3}$ which can be generalised to other non-factorial sets (Christian Sattler) and a conjecture is that, given an $n - groupoid$, there always is an $n + 1 - groupoid$ with the required property (Christian Sattler).

There are several things left to do, see the section on the future work-plan.

9 Future Work-plan

There are quite a couple of projects and subjects that I find interesting and plan to work on. Here, I only outline the ones that are related to Homotopy Type Theory. Other ideas are explained in the extended version of this report.

9.1 Higher Dimensional Quotients and the Root of Equality

Our Yoneda Groupoids give rise to the following question in infinity category theory: Given an (ω, n)-category A (an interesting case being $n = \infty$), in which cases is there an ω-category B such that the ω-category formed by the automorphisms (we could also ask about endofunctors) on B is isomorphic (or weakly equivalent) to A? This is an obvious generalisation of the question which groups appear as automorphism groups. It seems to be fundamental (even independently of type theory) and is, to the best of my knowledge, an open problem and might not even have been considered yet; however, I might be completely wrong here.

The future work for this project is therefore kind of obvious. Further, if this root always exists, it would be interesting to check if it satisfies a naturality condition such that the simplicial sets model is still a model of the theory together with

$$\textit{postulate:}(A : \mathbf{Type}_n) \to \Sigma(B : \mathbf{Type}_{1+n}), A \equiv (B \equiv B),$$

The reason why I assume that this question is not answered yet is that infinity category theory is not very exhaustively explored so far.

9.2 Weak Canonicity

First, there is a question asked by Thorsten Altenkirch on what I call "weak canonicity". *Canonicity* is, together with strong normalization (decidability of typechecking) and subject reduction, a very feasible property that is usually given in intensional type theories. It states that every term of a (base) type in the empty context reduces to a canonical one, i. e. one that makes use of a constructor. For example, in "standard" intensional type theories, a natural number in the empty context always reduces (β) to either zero or the successor of some number.

Another feasible property is *function extensionality*, i. e. the property that two functions are propositionally equal whenever they are pointwise equal. Unlike canonicity, this is not necessarily the case in intensional theories. Of course, we could fix this by postulating the existence of a term *ext*, but that would destroy canonicity (see, e. g., [3]). The same problem arises for Voevodsky's univalence axiom (see part 2 of this report).

However, there is still hope that a weaker form of canonicity could hold. It could be the case that every term (of a base type) in the empty context is, if not definitionally, then at least *propositionally* equal to a canonical one. In fact, all the examples of irreducible natural number I have looked at so far have been propositionally equal to a canonical number.

The (general) question to ask is therefore: Given a type theory that satisfies (strong or weak) canonicity, which constants can be added without loosing weak canonicity (how can those constants be characterized)? In particular, is this true for the univalence axiom? If yes, can the existence of the equality proof be given constructively (which seems very likely), so that it is not only true that a term is equal to a canonical one, but this canonical one can also be computed and ("automatically") be proven equal?

This question seems to be quite relevant, not only from a theoretical point of view. If every term is (constructively) equal to a canonical one, it is natural to ask whether the system can be extended in a way that allows us to exchange the two terms (treat them as definitionally equal). This might lead to an alternative approach, or a supplement, of observational type theory ([2], [5]) and possibly computational rules for the univalence axiom.

So far, I have not made any mentionable progress on this question. A possibly helpful strategy is described in [21].

9.3 Topos Theory

From various vague statements of mathematicians (mostly without type theoretic background) at the Swansea mini-school [6], it became clear that topos theory is somewhat connected to univalence. Indeed, the $n$lab [33]

[6] *Modern Perspectives in Homotopy Theory: Infinity Categories, Infinity Operads and Homotopy Type Theory*, `http://maths.swan.ac.uk/staff/jhg/minischool2012/index.html`

states that, just as type theory formalizes the internal logic of type theory, the internal logic of an $(\infty, 1)$-topos is homotopy type theory. In particular, univalence is (apparently) naturally present in such topoi. Although most of this statements seems to be a reformulation of topics I have discussed in part 1 of this report, it appears to be very advisable to study a basic amount of topos theory.

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