Non-Recursive Higher Inductive Types

Fields Institute, Toronto

Nicolai Kraus

University of Nottingham

20 May 2016

Some reasons why HITs are difficult

type-parametrized, e.g. $\mathsf{Susp}(A)$

recursive path constructors, e.g. $\|A\|_n$

higher path constructors, e.g. the torus $\mathbf{T}^2$

inductive-inductive (inductive-recursive), e.g. syntax of type theory, Cauchy-Reals

Question: Reduction theorems?
(General theories of HITs: Lumsdaine-Shulman, Sojakova, Dijkstra [see next talk]/Altenkirch-Capriotti-Dijkstra, ...)

Recursive versus non-recursive HITs

example: Susp(A)

$$\mathrm{north} : \mathrm{Susp}(A)$$
$$\mathrm{south} : \mathrm{Susp}(A)$$
$$\mathrm{merid} : A \to \mathrm{north} = \mathrm{south}$$

example: $\|A\|$

$$|{-}| : A \to \|A\|$$
$$\mathrm{t} : \Pi_{x,y:\|A\|}\; x =_{\|A\|} y$$

universal property Susp(A)

$$\frac{\mathrm{Susp}(A) \to B}{\Sigma(x_n, x_s : B), (A \to x_n = x_s)}$$

for any B

universal property $\|A\|$

$$\frac{\|A\| \to B}{A \to B}$$

if B is propositional

Recursive path constructors make elimination principles difficult to use!

This talk: my view on the propositional truncation

$\|A\| \to B$ is equivalent to . . .

$\Sigma(f : A \to B)$,	if B is (-1)-type
$\Sigma(c : \mathsf{wconst}_f)$,	if B is 0-type
$\Sigma(d : \mathsf{coh}_{f,c})$	if B is 1-type
. . .	. . .

How to do this in general?

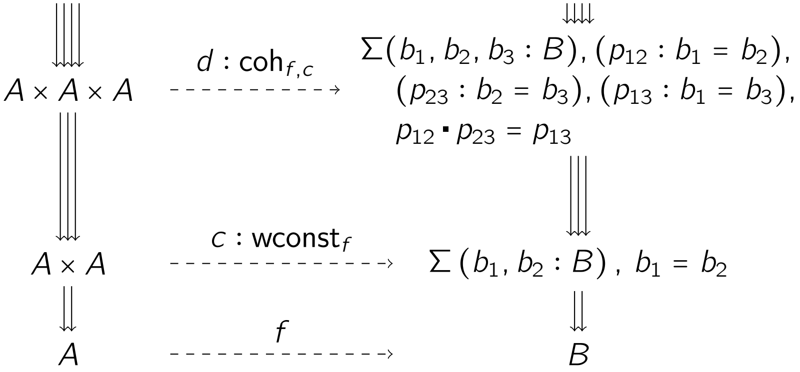
note:

$$\mathsf{wconst}_f \;:\equiv\; \prod_{x,y:A} f\,x = f\,y$$

$$\mathsf{coh}_{f,c} \;:\equiv\; \prod_{x,y,z:A} c(x,y) \cdot c(y,z) = c(x,z)$$

. . .

Coherently constant functions



$\mathcal{T}A :\ \Delta_+^{\mathrm{op}} \to \mathsf{Type}$

[0]-coskeleton of A

$\mathcal{E}B :\ \Delta_+^{\mathrm{op}} \to \mathsf{Type}$

Fibrant replacement of B

Theorem [K., TYPES 2014 proceedings]

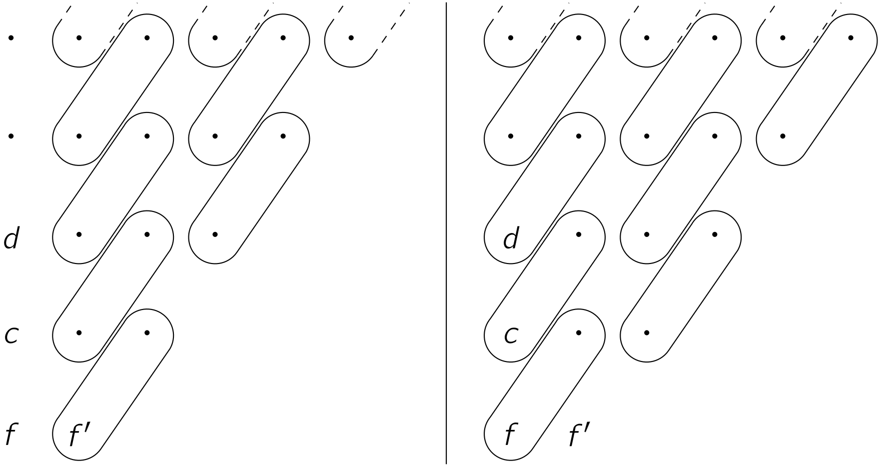
$$(\|A\| \to B) \quad \simeq \quad \text{nat. trans. from } \mathcal{T}A \text{ to } \mathcal{E}B$$

in any type theory with $\mathbf{1}, \Sigma, \Pi, \mathsf{Id}$, fun.ext., $\|{-}\|$, Reedy ω^{op}-limits.

Compare:

- ⋆ Lurie, *Higher Topos Theory*, Prop. 6.2.3.4: ∞-semitopos
- ⋆ Rezk, *Toposes and Homotopy Toposes*, Prop. 7.8: model topos

Proof sketch: Expanding and Contracting



About the theorem:

(1) Does Book-HoTT have the required limits? Guess: exactly iff semi-simplicial types are definable!

(2) Does this allow us to construct the propositional truncation with a nice elimination principle?
It would be an "infinite HIT" A_∞ with constructors

$$f : A \to A_\infty$$

$$c : \mathsf{wconst}_f$$

$$d : \mathsf{coh}_{f,c}$$

$$\ldots$$

Can we construct finite approximations of A_∞?

HIT A_1

$f : A \to A_1$

HIT A_2

$f : A \to A_2$

$c : \mathsf{wconst}_f$

HIT A_3

$f : A \to A_3$

$c : \mathsf{wconst}_f$

$d : \mathsf{coh}_{f,c}$

Finally: take the colimit of $A_1 \to A_2 \to \ldots$.

Feature: A_n is already correct with respect to $(n-2)$-types. Put differently, $\|A_n\|_{n-1} \simeq \|A\|$.

Problem: we can write down every A_n, but **not** a family $A : \mathbb{N} \to \mathcal{U}$ of types.

Analysis:

- ⋆ for any two points, f gives two points; c connects them
- ⋆ for any three points, f and c give an empty triangle; d fills it
- ⋆ in general, in step $n+1$: for any $(n+1)$ points in A, the previous n constructors give an "empty n-dimensional tetrahedron"; the next constructor fills it

My alternative sequence, based on this analysis:

- ⋆ In step $n+1$: fill **every** boundary of an n-dimensional tetrahedron.

"filling every n-boundary"

=

"take the n-pseudo-truncation" ! – write $\{-\}^n$

$A_1 :\equiv A$

$A_2 :\equiv \{A_1\}^{-1}$

$A_3 :\equiv \{A_2\}^0$

$A_{n+1} :\equiv \{A_n\}^{n-2}$

This works! Additional features:

- ⋆ It is a sequence of approximations – $\|A_n\|_{n-2} \simeq \|A\|$.
- ⋆ Side-results for free (characterisation of maps $\|A\|_n \to B$).

Comparison: the van Doorn sequence (see previous talk) *always* uses $\{-\}^{-1}$:

$A_1 :\equiv A$

$A_2 :\equiv \{A_1\}^{-1}$

$A_3 :\equiv \{A_2\}^{-1}$

$A_{n+1} :\equiv \{A_n\}^{-1}$

This is much coarser.

Advantage: much simpler to prove correct.

Disadvantage: the finite parts are not well-behaved.

For both sequences, the proof that their colimits are propositional factors through:

Lemma

Given a sequence $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots$.
If every f_i is weakly constant, then the colimit is propositional.

Clearly fulfilled for the van Doorn sequence; much harder for our sequence (note: $X \to \{X\}^n$ is not weakly constant!)

Intuition:

- ⋆ the van Doorn sequence is the coarsest sequence that works;
- ⋆ the sequence I wanted original is the finest sequence;
- ⋆ my sequence with n-pseudo-truncation is the finest sequence that is definable in Book-HoTT.

Final remarks

- $\star$ Can probably find all sorts of constructions of $\|A\|$ with this lemma.
- $\star$ One more construction (Rijke): $A_{n+1} :\equiv A \star A_n$.
- $\star$ Obvious conjecture: get n-truncation if we skip $\{-\}^i$ for $i < n$.
- $\star$ Less obvious conjecture (Rijke): can use my strategy to construct localizations with better properties of "finite initial segments".
- $\star$ **Open question: Can all HITs be represented non-recursively?** – probably it does not work for inductive-inductive ones (Cauchy Reals).