Twisted Cubes via Graph Morphisms

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Short description. *Twisted cubes* are a variation of a cube category that has previously been used to model homotopy type theory [1]. Here, we describe how twisted cubes are constructed and what their properties are. The talk is based on our preprint [5].

General Motivation. Intensional Martin-Löf type theory admits models where types are interpreted as groupoids (categories where every morphism is invertible) and even higher groupoids. The latter view is particularly important to explain *homotopy* type theory. One way of approaching higher groupoids is via presheaves on cube categories with certain filling conditions, and this is how Bezem, Coquand, and Huber [1] have built an important model of homotopy type theory.

$$\begin{tikzcd} y \arrow[r, dashed] & w \\ x \arrow[u, "p"] \arrow[r, "q"] & z \arrow[u, "r"'] \end{tikzcd}$$

The basic idea is as follows: a type is modeled as a cubical set (think of a collection of cubes of various dimensions). Points are the elements of the type, lines are the equalities, squares are the equalities between equalities, and so on. The filling condition says that, whenever we have a partial cube satisfying some properties, it can be completed (this corresponds to the Kan filling condition of horns for simplicial sets). For example, a "partial square" could be given by x, y, z, w and p, q, r as shown on the left, where we think of x, y, z, w as elements and p, q, r as equalities. The filling then tells us that there is another line, namely the dashed one, as shown in the picture; we think of it as the composition of the other three (the inner part of the square would be the evidence that it is indeed the composition).

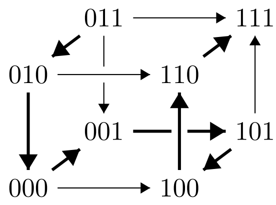
The direction of the arrows in the picture above is determined by the concrete cube category that is used. We see that the three arrows in the above diagram cannot really be composed directly, since p goes into the wrong direction. Since equality in (homotopy) type theory is invertible, this is not an issue.

However, the idea to create a type theory where equality is not necessarily invertible is subject to current research as well, and goes under the name *directed type theory* [4, 7, 3]. In this case, the above observation is problematic. If z and w are simply x, and q and r are identities (i.e. reflexivity on x), then the composition gives us an inverse of p which should not necessarily exist. To remedy this, in the case of a square, we swap the direction of the left vertical arrow p and call it a *twisted square*. This can be generalised and gives rise to *twisted n-cubes* by recursion: To construct a twisted $(n+1)$-cube, we multiply a twisted n-cube with a directed interval (in the same way as a standard $(n+1)$-cube can be constructed from a standard n-cube), and invert everything at the first endpoint of the interval. This ensures that every face of a twisted $(n+1)$-cube is a twisted n-cube.

Technical Details. To make the above idea precise, we represent a twisted n-cube as a directed graph without parallel edges, T_n. This graph is defined recursively using $\textsf{tw-iter}$, a function that takes any graph and "thickens and inverts at the source":

$$\begin{aligned}
(\textsf{tw-iter}\ (V,E))_{\textsf{nodes}} &:\equiv \{0,1\} \times V & T_0 &:\equiv (\{\epsilon\},\ \{(\epsilon,\epsilon)\}) \\
(\textsf{tw-iter}\ (V,E))_{\textsf{edges}} &:\equiv \{\ (\ (0,\ t),\ (0,s)\ )\ \mid\ (s,t) : E\} & T_{n+1} &:\equiv \textsf{tw-iter}\ (T_n) \\
&\quad \cup \{\ (\ (1,\ s),\ (1,t)\ )\ \mid\ (s,t) : E\} \\
&\quad \cup \{\ (\ (0,\ v),\ (1,v)\ )\ \mid\ v : V\}.
\end{aligned} \tag{1}$$

One interesting feature of T_n is a connection to the $(2^n - 1)$-simplex, witnessed by a unique Hamiltonian path through T_n. T_3 is shown on the right, with the Hamiltonian path drawn using thicker arrows. This also implies that the transitive closure of T_n is a total linear order, something that happens for simplices but not for cubes. The sketch of proof is that the third clause in the definition of edges of $\mathsf{tw\text{-}iter}$ links the biggest node in the first copy (which was the smallest node before it got inverted) to the smallest node in the second copy.



The category of twisted cubes that we consider, denoted by $\bowtie_{\mathsf{grp}}$, has natural numbers as objects, and morphisms from m to n are graph homomorphisms from T_m to T_n.

We further add the condition to $\bowtie_{\mathsf{grp}}$ that dimensions are preserved, i.e. we only consider graph homomorphisms $f : T_m \to T_n$ such that, if e_1, e_2 are edges of T_m that go into the same direction, $f(e_1)$ and $f(e_2)$ also go into the same direction. This defines a subcategory that we denote by $\bowtie_{\mathsf{dim}}$. This subcategory has the same objects as, but fewer morphisms than $\bowtie_{\mathsf{grp}}$; and this restriction essentially excludes connections. To show why the definition of $\bowtie_{\mathsf{dim}}$ makes sense, we define a category for the standard cubes counterpart called $\square_{\mathsf{dim}}$ and prove in our preprint [5] that $\square_{\mathsf{dim}}$ is isomorphic to the opposite of the category of cubes used by Bezem, Coquand, and Huber [1]. $\square_{\mathsf{dim}}$ is essentially the same as $\bowtie_{\mathsf{dim}}$ but skips the "twisting" step. In other words, $\bowtie_{\mathsf{dim}}$ is the "twisted analogue" of the BCH cube category.

Other interesting features of $\bowtie_{\mathsf{dim}}$ include the observation that there is exactly one surjective morphism in $\bowtie_{\mathsf{dim}}(m, n)$ for all $m \geqslant n$ (and clearly none if $m < n$). As a consequence, the degeneracies are unique, i.e. one can only degenerate an n-cube to an $(n+1)$-cube in exactly one way, something that happens for globes but not for simplices or cubes.

Status of this work. We have defined the category of twisted cubes and proved several of its properties, details can be found in our arXiv preprint [5]. Our goal is to use it to model a version of "higher directed type theory", but we have not yet done this. Another goal is to analyse whether the setting allows for a development of higher categories in homotopy type theory: The construction given in (1) can easily be adapted to define the category of twisted semi-cubes simply by starting with $T_0 :\equiv (\{\epsilon\}, \emptyset)$. We can then consider Reedy-fibrant diagrams on this category into the universe of types, i.e. *twisted semi-cubical types*. The unique Hamiltonian path suggests that we can equip these types with an analogue of Rezk's *Segal condition* [6] similar to how it has been done in homotopy type theory (see e.g. [2]).

References

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