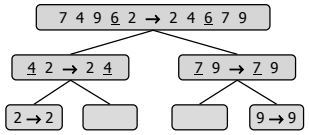
Plan of the lecture

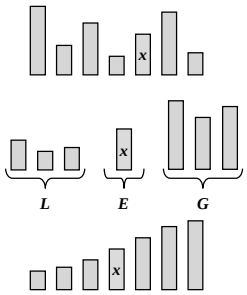
- Quick-sort
- Lower bounds on comparison sorting
- Correctness of programs (loop invariants)

Quick-Sort



Quick-Sort (§ 10.2)

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



Partition of lists (or using extra workspace)

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

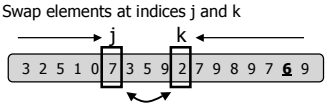
Partitioning arrays

- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

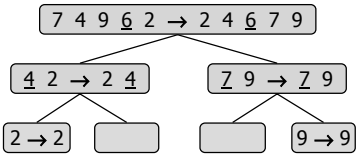
3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 (pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq$ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k



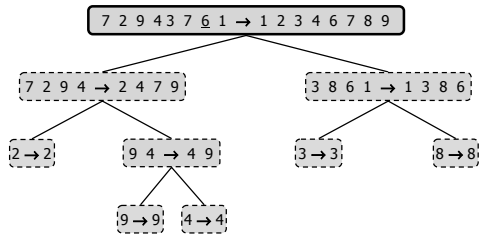
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



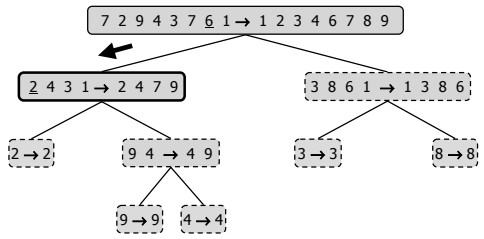
Execution Example

- Pivot selection



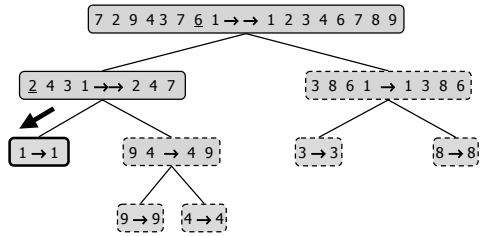
Execution Example (cont.)

- Partition, recursive call, pivot selection



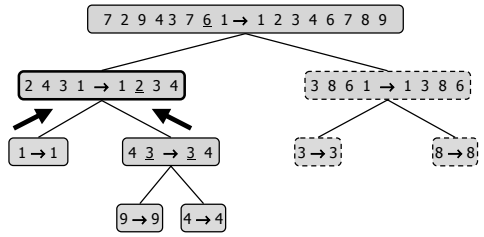
Execution Example (cont.)

- Partition, recursive call, base case



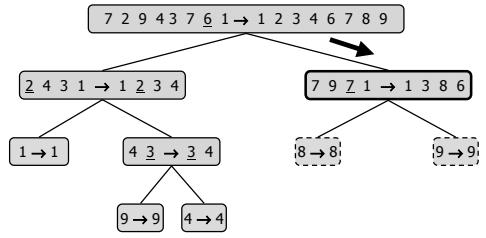
Execution Example (cont.)

- Recursive call, …, base case, join



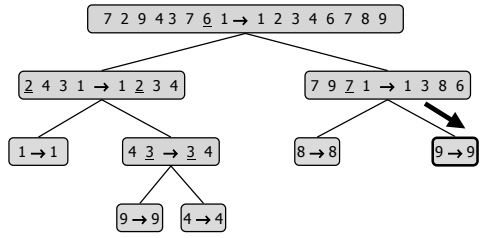
Execution Example (cont.)

- Recursive call, pivot selection



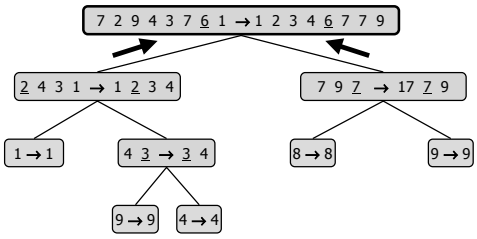
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



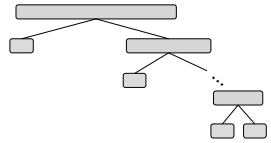
Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

- Thus, the worst-case running time of quick-sort is $O(n^2)$

depth	time
0	n
1	$n - 1$
…	…
$n - 1$	1



Best-case Running Time

- The best case for quick-sort occurs when the pivot is the median element
- The L and G parts are equal – the sequence is split in halfs, like in merge sort
- Thus, the best-case running time of quick-sort is $O(n \log n)$

Average-case Running Time

- The average case for quick-sort: half of the times, the pivot is roughly in the middle
- Thus, the average-case running time of quick-sort is $O(n \log n)$ again
- Detailed proof in the textbook

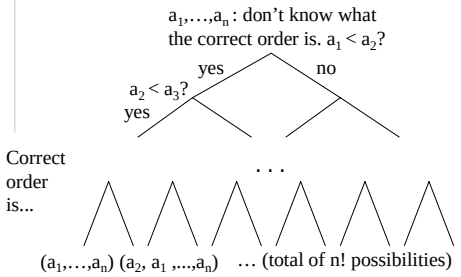
Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	in-place (no extra memory) slow (good for small inputs)
insertion-sort	$O(n^2)$	in-place (no extra memory) slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	in-place (+stack), randomized fast (good for large inputs)
heap-sort	$O(n \log n)$	in-place (if heapify the sequence to be sorted) fast (good for large inputs)
merge-sort	$O(n \log n)$	sequential data access fast (good for huge inputs)

Lower bound for comparison sorting

- Can model sorting which depends on comparisons between elements as a binary decision tree.
- At each node, a comparison between two elements is made; there are two possible outcomes and we find out a bit more about the correct order of items in the array.
- Finally arrive at full information about the correct order of the items in the array.

Comparison sorting



Comparison sorting

- If a binary tree has n! leaves than the minimal number of levels (assuming the tree is perfect) is log n! +1.
- This shows that O(n log n) sorting algorithms are essentially optimal ($\log_2 n!$ is not equal to n $\log_2 n$ but has the same growth rate modulo some hairy constants).

Correctness of algorithms

Two issues:

- Given an algorithm, prove that it is correct (always achieves the intended result, e.g. a sorted array).
- Design an algorithm with intended properties from specification (even more difficult)

Additional reading

- Goodrich and Tamassia, Chapter 3.5
- Frank M. Carrano, Janet J. Prichard. *Data abstraction and problem solving with Java.* Addison Wesley Longman, 2001. Chapter 1, Problem solving and software engineering (on verification).
- Duane A. Bailey. *Data structures in Java for the principled programmer*. McGraw-Hill 1999. Chapter 2, Comments, conditions and assertions (pre- and postconditions).
- Roland Backhouse. *Program Construction : Calculating Implementations from Specifications*. John Wiley & Sons 2003.

Disproving correctness

- Just one counterexample is enough
- Testing may fail to discover a bug

Proving correctness

- Formulate precisely the property which has to hold
- If necessary, formulate relevant properties for smaller parts of an algorithm : program assertions

Assertions

- Assertion: claim about values of program variables before or after a statement or a group of statements is executed

  Typical assertions:
- Precondition (usually of a method): what we expect to hold before the method is executed.
- Postcondition: what holds after the method is executed.

Hoare triples

- {P} S {Q}: P precondition, S statements, Q postcondition.
- Meaning: provided P holds before S is executed, then after S is executed, Q holds.
- for example:

{x < 10} **`x = x + 20`** {x < 30}

{x < y} **`while (x < y) x++`** {x=y}

- For a small programming language, can provide axioms for every construct in the language and derive postconditions using axioms

Assignment axiom

- If the only programming construct was assignment, here is an axiom to verify all programs:

{Q(e substituted for x)} **`x = e`** {Q}

- For example, if want to prove

{x < 10} **`x = x + 20`** {x < 30} then the assignment axiom gives

{x+20 < 30} **`x = x + 20`** {x < 30} and from extra knowledge about math etc we derive that {x+20 < 30} is equivalent to {x < 10}.

Example

- In the programs we usually write there are lots of constructs and they also use other people's code.
- Less formal approach (but good practice): write pre- and postconditions for significant chunks of code/methods.
- Example: code in Bailey's book.

Proving correctness

To prove that an algorithm is correct:

- Determine preconditions and postconditions for the whole algorithm.
- Cascade statement assertions together, so that postconditions for one provide preconditions for the next.
- Prove correctness of individual statements.
- Hence show that executing algorithm with stated preconditions terminates and leads to stated post-conditions.

Loop Invariants

- Assertions for loops are difficult, because loops may be executed many times over, with slightly different assertions holding before and after each iteration. Focus on those assertions that remain constant between iterations.
- Known as l*oop invariants:* true before and after each iteration through a loop.

Example

```
pos_greatest = 0;
for (int j = 0; j < =i; j++) {
   if( arr[j] > arr[pos_greatest]) {
      pos_greatest = j;
   }
}
```

Invariant: **`pos_greatest`** is the index of the largest array element between 0 and j.

(More formally, for all k such that **`0 ≤ k < j`**, **`arr[k] ≤ arr[pos_greatest].)`**

Correctness of loops

To prove correctness of a while loop (or: that assertion A holds after the loop terminates):

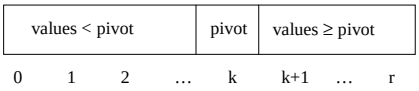
- Prove that the loop eventually terminates (by finding the *bound function* for the loop)
- Find a suitable invariant (there are infinitely many invariants for each loop, most of them useless)
- Prove that A is true after last iteration (usually by substituting the state in which the loop terminates in the invariant)

Partition algorithm

```
small = l;  // set at the left border of the
          // range
large = r; // set at the right border where the
          // pivot sits
while(small < large) {
   if (arr[small] < pivot) small++;
   else {
         large--;
         temp = arr[small];
         arr[small] = arr[large];
         arr[large] = temp; }
}
temp = arr[r];
arr[r] = arr[large];
arr[large] = temp;
return large;
```


Postcondition for the partition

```
public int partition(int[] arr, int l, int r)
// post: returns and integer k such that
//       for all indices i such that l<=i<k,
//       arr[i]<arr[k] and
//       for all indices i such that k<=i<r
//       arr[i]>=arr[k]
```

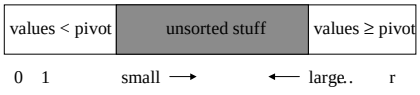


Loop invariant for partition

for all indices i,

if **`l ≤ i < small`**, then **`arr[i] < pivot`**

if **`large ≤ i < r`** then **`arr[i] ≥ pivot`**



Bound function for partition

- bound function = **`large - small`**
- decreases by 1 at every step; the loop terminates when it is equal to 0 (**`small = large`**).

After the loop…

- When **`large = small`**, the invariant:
  - if **`l ≤ i < small`**, then **`arr[i] < pivot`**
  - if **`large ≤ i < r`** then **`arr[i] ≥ pivot`**

  becomes (substitute **`large`** for **`small`**):
  - if **`l ≤ i < large`**, then **`arr[i] < pivot`**
  - if **`large ≤ i < r`** then **`arr[i] ≥ pivot`**
- After the pivot is swapped, **`pivot = arr[large].`**

Partition algorithm

```
Assertion 1: l < r; pivot is arr[r]
while(small < large) {
   if (arr[small] < pivot) small++;
   else {
         large--;
         temp = arr[small];
         arr[small] = arr[large];
         arr[large] = temp;}}
Assertion 2: pivot is arr[r];
if l ≤ i < large, then arr[i] < pivot;
if large ≤ i < r then arr[i] ≥ pivot
temp = arr[r];
arr[r] = arr[large];
arr[large] = temp;
Assertion 3: arr[large] is the pivot
return large;
```


Informal coursework (tutorials from the 6th of December)

Given that l < r in the partition method, which of the following are loop invariants of the while loop:

1) small < r
2) small < large
3) small ≤ large
4) for all i such that l ≤ i < small, arr[i] < pivot
5) for all j such that large ≤ j ≤ r, arr[j] ≥ pivot
6) for all j such that large < j ≤ r, arr[j] ≥ pivot

Informal coursework 2

Prove correctness of selection sort:

```
void selectionSort(int arr[], int len){
int i,j,temp,pos_greatest;
for( i = len - 1; i > 0; i--){
   pos_greatest = 0;
   for(j = 0; j <= i; j++){
      if(arr[j] > arr[pos_greatest]) pos_greatest = j;
   }//end inner for loop
   temp = arr[i];
   arr[i] = arr[pos_greatest];
   arr[pos_greatest] = temp; }}
```