Handout 1
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Introduction to Model Theory

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Notation and definitions.

Structure A structure A consists of

- a set called the domain of A, $dom(A)$.
- a set of relations on A: an n-ary relation on A is a subset of $dom(A)^n$ ($n > 0$). Relations are *named* by relation symbols.
- a set of functions on A: an n-ary function on A is a map from $dom(A)^n$ to $dom(A)$ ($n > 0$). Functions are *named* by function symbols.
- a set of constant elements of A (distinguished elements which are going to have a name). Constants are *named* by constant symbols.

Relational structure is a structure where the last two sets are empty (no functions and constants).

Signature of a structure is a set of symbols for relations, functions and constants of the structure. We will refer to the relation named by a symbol R in the structure A by R^A, function named by f in A by f^A, and the element of A named by a constant c by c^A.

For example, a graph $\langle V, E\rangle$ is a relational structure; its signature consists of a single binary relational symbol E.

Homomorphism from a structure A to a structure B (of the same signature L) is a function h from $dom(A)$ to $dom(B)$ such that:

- for each constant c of L, $h(c^A) = c^B$ (where c^A is the element named by c in A, similarly for c^B).
- for each n-ary relation symbol R in L and every n-tuple of elements of $dom(A)$, if $\langle a_1, \ldots, a_n\rangle \in R^A$ then $\langle h(a_1), \ldots, h(a_n)\rangle \in R^B$.
- for each n-ary function symbol f in L and every n-tuple of elements of $dom(A)$, $h(f^A(a_1, \ldots, a_n)) = f^B(h(a_1), \ldots, h(a_n))$.

Isomorphism is a homomorphism which is bijective (one-to-one) and in addition satisfies

- $\langle a_1, \ldots, a_n\rangle \in R^A$ iff $\langle h(a_1), \ldots, h(a_n)\rangle \in R^B$.

Embedding is a homomorphism which is injective (no two elements of A mapped to the same element in B) and satisfies

- $\langle a_1, \ldots, a_n\rangle \in R^A$ iff $\langle h(a_1), \ldots, h(a_n)\rangle \in R^B$.

Substructures A is a substructure of B if they have the same signature, $dom(A) \subseteq dom(B)$ and the function which maps every element of A to itself is an embedding.

First order logic

A first order language: logical symbols:

- variables $x, y, z, \ldots$
- boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$
- quantifiers $\forall$, $\exists$

plus some signature L (non-logical symbols):

- constants $c_1, c_2, \ldots$;
- relation symbols;
- function symbols;

First order logic with equality also contains a special relation symbol $=$ (logical symbol).

Terms

- variables are terms;
- constants are terms;
- if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term.

Atomic formulas

- if t_1 and t_2 are terms, then $t_1 = t_2$ is an atomic formula;
- if $t_1, \ldots, t_n$ are terms and R is an n-ary relation symbol, then $R(t_1, \ldots, t_n)$ is an atomic formula.

Formulas

- An atomic formula is a formula;
- If φ, ψ are formulas, then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$.
- If φ is a formula, and x a variable, then $\forall x\varphi$ and $\exists x\varphi$ are formulas. Occurrences of x in the scope of the quantifier are bound by this quantifier (unless they are already bound by another quantifier. Variables which are not bound are called free.

A formula with no free variables is called a sentence.

Meaning of formulas First, we need some way to find the value of an L-term t in a given L-structure A: $v_A(t)$. It depends on the way the variables are interpreted. Let s be a function from the set of variables to the domain of A.

- $v^s_A(x) = s(x)$;
- $v^s_A(c) = c^A$;
- $v^s_A(f(t_1, \ldots, t_n) = f^A(v^s_A(t_1), \ldots, v^s_A(t_n))$.

Let φ be a first order formula in some signature L and A a structure of signature L. **We only consider structures with non-empty domain as the models of first order logic.**

Then the relation φ is satisfied by s on A ($A, s \models \varphi$) is defined inductively as follows:

- $A, s \models R(t_1, \ldots, t_n)$ iff $\langle v_A^s(t_1), \ldots, v_A^s(t_n)\rangle \in R^A$.
- $A, s \models \neg\varphi$ iff $A, s \not\models \varphi$
- $A, s \models \varphi \wedge \psi$ iff $A, s \models \varphi$ and $A, s \models \psi$
- $A, s \models \exists x\varphi$ iff for some assignment s' which differs from s at most in the value assigned to x, $A, s' \models \varphi$.

A is a model of φ (φ is true on A, $A \models \varphi$) if φ is satisfied on A by all assignments.

φ is logically valid if it is true on all L-structures.

φ and ψ are logically equivalent iff for all A and s, $A, s \models \varphi$ if, and only if, $A, s \models \psi$.

φ logically follows from a set of formulas Γ iff for all A and s, if $A, s \models \Gamma$, then $A, s \models \varphi$.

Exercises

Exercises marked by (*) are moderately difficult, (**) are difficult or impossible.

1. Prove that if h_1 is a homomorphism from A to B and h_2 is a homomorphism from B to C, then their composition is a homomorphism from A to C.
2. Give a truth definition for $\forall x\varphi$ given that $\forall x\varphi$ should be logically equivalent to $\neg\exists x\neg\varphi$.
3. Write a first order formula which is only true
   (a) on structures with more than three elements (assume there are no constants; (*) assume that there is no equality, but two unary relation symbols available);
   (b) (*) on structures with infinitely many elements (assume that you only have one binary relation symbol);
   (c) (**) on structures with even number of elements.
4. (*) Prove that only the values assigned by s to the free variables of φ matter for deciding whether $A, s \models \varphi$. Namely, if there are two assignments s and z which agree on the values assigned to the free variables of φ, then $A, s \models \varphi$ iff $A, z \models \varphi$.
5. Prove that isomorphic structures satisfy the same first order formulas.
6. (*) Can you say anything similar about homomorphic structures: e.g. if there is a homomorphism from A to B and ψ is a certain kind of formula, then if A satisfies ψ, so does B?