Handout 3
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Introduction to Model Theory
Natasha Alechina

Fraisse-Ehrenfeucht Games.

EF games are used to establish a more fine-grained relation between two structures than isomorphism. The relation which EF game (of length n) establishes is a back-and-forth equivalence (of depth n). The corresponding logical property on finite structures (or for finite signatures) is that two structures satisfy the same formulas of quantifier rank n.

Quantifier rank is basically the number of nested quantifiers. A formal definition is given on the next page.

How can we show that some property P is not first order definable? Show that for every n we can find two structures, one of which has P, the other does not, and there is a back-and-forth equivalence of depth n between them. Which means that there is no first order formula of quantifier rank n which distinguishes them. Since we show this for every n, this means that there is no formula which defines P on all finite structures.

Back-and-forth equivalence First we need to define partial, or local, isomorphism of size n between two structures. Let L be a signature and A and B L-structures. Assume that L does not contain constants and function symbols.

A partial isomorphism of size n between A and B is a bijective function h defined on a subset of $dom(A)$ of size n, such that

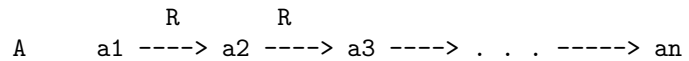
- if h is defined for $a_1, \ldots, a_k$, then $\langle a_1, \ldots, a_k \rangle \in R^A$ iff $\langle h(a_1), \ldots, h(a_k) \rangle \in R^B$.

The idea of back-and-forth equivalence (of size n) between A and B is that we can always construct a partial isomorphism of size n between them, no matter where we start and which elements we choose to include in the domain and range of h - we can find a match for any element (so long as there are not more than n elements matched in total). To make this idea precise, the following notion of a game has been introduced.

Two players, a *Spoiler* and a *Duplicator*, play a game on A and B. The game has n steps. The Spoiler tries to show that A and B are different, and the Duplicator tries to show that they are really the same. At the ith step in the game the Spoiler chooses an element in A or in B. The Duplicator answers by choosing the element in the other structure. At every step they generate a pair (a_i, b_i) where a_i is an element of A and b_i is an element of B. The Duplicator wins if the sequence $(a_1, b_1), \ldots, (a_n, b_n)$ forms a partial isomorphism between A and B (i.e. the function h which sends a_i to b_i is a partial isomorphism). Otherwise the Spoiler wins.

If A and B are isomorphic, the Duplicator can always win (there is a winning strategy, following which the Duplicator wins regardless of how cleverly the Spoiler plays). But sometimes the Duplicator has a winning strategy on non-isomorphic structures, provided the number of steps is less than the size of structures.

Exercise. Consider two structures in the signature containing a single binary relation R:



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B     b1 ----> b2 ----> b3 ----> . . . -----> bn-1
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Show that the Duplicator has a winning strategy in the game which has less than $log_2(n-1)$ steps.

Back-and-forth equivalence of depth n holds between A and B iff the Duplicator has a winning strategy in EF game of length n between A and B. We denote this by $A \approx_n B$.

Quantifier rank of a formula φ, $qr(\varphi)$, is defined inductively:

- if φ is an atomic formula, $qr(\varphi) = 0$;
- $qr(\neg\psi) = qr(\psi)$;
- $qr(\psi_1 \wedge \psi_2) = qr(\psi_1 \vee \psi_2) = qr(\psi_1 \to \psi_2) = max(qr(\psi_1), qr(\psi_2))$;
- $qr(\exists x\psi) = qr(\forall x\psi) = qr(\psi) + 1$.

To make the proofs easier, we need the following convention. Let A be a structure and $a \in dom(A)$. Then (A, a) is a structure in a signature extended by a (called a simple expansion of A) where a is a constant interpreting itself. So we can write formulas like $\varphi(a)$ which is true in (A, a) if $A \models \varphi[a]$. We will also assume that if A and B are structures with the same signature, then so are (A, a) and (B, b) (i.e. a and b are really the same constant).

We say that $(A, a) \approx_n (B, b)$ if the Duplicator has a winning strategy in a game of length n after a is matched to b. Clearly, $A \approx_n B$ iff $\forall a \in A \exists b \in B(A \approx_{n-1} B)$ ('forth' property) and $\forall b \in B \exists a \in A(A \approx_{n-1} B)$ ('back' property).

Theorem 1 *The Duplicator has a winning strategy in EF game of length n between A and B iff A and B satisfy the same sentences of quantifier rank $\leq n$ (denoted $A \equiv_n B$).*

Proof. By induction:

- $n = 0$: there are no sentences of quantifier rank 0 in purely relational signatures, so A and B trivially satisfy the same sentences; also a function with an empty domain is a trivial isomorphism.
- Suppose the theorem holds for n. We need to show that $A \approx_{n+1} B$ iff $A \equiv_n B$.

  Assume $A \approx_{n+1} B$. Let φ be a sentence with $qr(\varphi) \leq n+1$. We need to show that $A \models \varphi$ iff $B \models \varphi$. φ can only be $\neg\psi$, where ψ has the same quantifier rank as φ; $\psi_1 \wedge \psi_2$, where ψ_1, ψ_2 have the same quantifiers rank (or less) as φ, or $\exists x\psi$, where $qr(\psi) \leq n$. We can reduce all cases to the last case. So, assume $\varphi = \exists x\psi$.

  Let $A \models \exists x\psi$. Then for some a $A \models \psi[a]$, or $(A, a) \models \psi(a)$. Since $A \approx_{n+1} B$, there exists a $b \in B$ such that $(A, a) \approx_n (B, b)$. By the inductive hypothesis, $(A, a) \equiv_n (B, b)$. Since $qr(\psi) \leq n$, $(B, b) \models \psi(b)$, so $B \models \psi[b]$ and $B \models \exists x\psi$.

  The other direction ($B \models \exists x\psi$ implies $A \models \exists x\psi$) uses the 'back' property.

  Assume $A \equiv_{n+1} B$. We need to show that $\forall a \in A \exists b \in B((A, a) \equiv_n (B, b))$ and vice versa: $\forall b \in B \exists a \in A((A, a) \equiv_n (B, b))$. Then by the inductive hypothesis we get $\forall a \in A \exists b \in B((A, a) \approx_n (B, b))$ and $\forall b \in B \exists a \in A((A, a) \approx_n (B, b))$, which gives $A \approx_{n+1} B$.

Take any $a \in A$. Suppose by contradiction that there is no $b \in B$ such that $(A, a) \equiv_n (B, b)$. Then for every $b \in B$ there is a formula ψ_b of quantifier rank $\leq n$ such that $A \models \psi_b(a)$ and $B \not\models \psi_b(b)$. *If B is infinite, but its signature is finite, there are finitely many non-equivalent formulas of quantifier rank at most n.* $\exists x \bigwedge_{b \in B} \psi_b(x)$ is a formula of quantifier rank at most $n + 1$ which distinguishes A and B, which contradicts our assumption. So $\forall a \in A \exists b \in B((A, a) \equiv_n (B, b))$. The other statement has a similar proof.