Handout 4
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Introduction to Model Theory

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Modal Logic

Basic modal logic

The language of basic modal logic: logical symbols:

- boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$
- modal operators $\Box$, $\Diamond$

plus some non-logical symbols:

- propositional variables $p_1, p_2, \ldots$

Formulas

- A propositional variable is a formula;
- If φ, ψ are formulas, then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$.
- If φ is a formula, then $\Diamond\varphi$ and $\Box\varphi$ are formulas.

Models of basic modal logic: structures in the signature containing one binary relation R and arbitrary many unary relations:

$$A = \langle W, R, P_1, \ldots, P_n, \ldots \rangle$$

The elements of W are referred to as: possible worlds; moments of time; states of a program; sets of beliefs, or theories ....

Meaning of formulas The relation φ is true in a world w in A ($A, w \models \varphi$) is defined inductively as follows:

- $A, w \models p_i$ iff $w \in P_i^A$.
- $A, w \models \neg\varphi$ iff $A, w \not\models \varphi$
- $A, w \models \varphi \wedge \psi$ iff $A, w \models \varphi$ and $A, w \models \psi$
- $A, w \models \Diamond\varphi$ iff for some $v \in W$ such that $\langle w, v\rangle \in R^A$, $A, v \models \varphi$.
- $A, w \models \Box\varphi$ iff for all $v \in W$ such that $\langle w, v\rangle \in R^A$, $A, v \models \varphi$.

A is a model of φ (φ is true on A, $A \models \varphi$) if φ is satisfied on A by all elements.

φ is logically valid if it is true on all structures.

φ and ψ are logically equivalent iff for all A and w, $A, w \models \varphi$ if, and only if, $A, w \models \psi$.

φ logically follows from a set of formulas Γ iff for all A and w, if $A, w \models \Gamma$, then $A, w \models \varphi$.

Multi-modal modal logic Instead of $\Box$ and $\Diamond$, multi-modal modal logic has modalities $[a]$, $[b]$, ..., $\langle a\rangle$, $\langle b\rangle$, ... for some set of labels $a, b, \ldots$. Correspondingly, the models have a set of binary relations R_a, R_b, ..., for every label. The truth definition for $[a]$ is the same as for $\Box$, but uses R_a instead of R. The truth definition for $\langle a\rangle$ is the same as for $\Diamond$, but uses R_a instead of R.

Further extensions: use regular expressions made up of labels, sequential composition ;, union | and Kleene star * inside modal operators: $\langle (a;b)*\rangle p$: a state where p holds is reachable by a path of alternating a and b labeles.

n-ary modal operators $\Box$ and $\Diamond$ are unary (monadic) modal operators. It is possible to define polyadic modalities Δ such that: if $\phi_1, \ldots, \phi_n$ are formulas, then so is $\Delta(\phi_1, \ldots, \phi_n)$. Example: $Until(\phi_1, \phi_2)$ - ϕ_1 is true until (the moment of time when) ϕ_2 is true. Another example (intuitionistic implication): $\phi_1 \Rightarrow \phi_2$ if for every possible expansion of our current knowledge, if ϕ_1 is true there, then ϕ_2 is true there. Both operators can be interpreted on the same structures as $\Box$ and $\Diamond$ using a binary relation.

The following generalisation of monadic modalities is sometimes given, but there are not many interesting modalities which have such a definition. If Δ is an n-ary modality, the relation used to give meaning to Δ is an $(n+1)$-ary relation R_Δ. The general truth definition:
$A, w \models \Delta(\phi_1, \ldots, \phi_n)$ iff there exist $v_1, \ldots, v_n$ such that $R_\Delta(w, v_1, \ldots, v_n)$ and $A, v_i \models \phi_i$.

Exercises

Exercises marked by (*) are moderately difficult, (**) are difficult or impossible.

1. Are $\Box$ and $\Diamond$ interdefinable?
2. (*) Define a fragment of first order logic which can express exactly the same properties of structures in the signature $\langle R, P_1, \ldots, P_n\rangle$ as basic modal logic.