ABSTRACT: ***Betting on Fuzzy and Many-valued Propositions***

In a 1968 article, Lotfi Zadeh proposed accounts of absolute and conditional probability for fuzzy sets [Zadeh 1968]. What I offer here, *via* Dutch Book Arguments, is a vindication of those accounts when probability is assigned to propositions rather than sets.

I apply what Jeff Paris calls "the Dutch Book method" to fuzzy and many-valued logics that are, as I shall say, additive. That is, conjunction and disjunction satisfy this condition, that for any valuation v and for any sentences A and B

$$v(A \wedge B) + v(A \vee B) = v(A) + v(B).$$

Additivity is common: the Gödel, Łukasiewicz, and product fuzzy logics are all additive, as are Gödel and Łukasiewicz n-valued logics.

We need a betting scheme suitably sensitive to truth-values intermediate between the extreme values 0 and 1. Setting out the classical case the right way makes one generalization obvious. Using it we obtain Dutch Book arguments for certain seemingly familiar principles of probability, seemingly familiar in that formally they recapitulate classical principles.

- $0 \leq Pr(A) \leq 1$
- $\mathrm{Pr}(A) = 1$ when $\vDash A$
- $\mathrm{Pr}(A) = 0$ when $A \vDash$
- $\mathrm{Pr}(A \wedge B) + \mathrm{Pr}(A \vee B) = \mathrm{Pr}(A) + \mathrm{Pr}(B)$.

Here $\wedge$ and $\vee$ are the conjunction and disjunction, respectively, of an additive fuzzy or many-valued logic.

The connection with the classical case is tighter than that, though, because the mathematical details of quite a few of our Dutch Book arguments piggy-back on those of Dutch Book arguments applied to the two-valued case.

As an initial vindication of Zadeh's account, we find that in the context of a finitely-many-valued Łukasiewicz logic, all probabilities are *classical expectations*. That is, the probability of a many-valued proposition is the expectation of its truth-value *and* that a proposition has a particular truth-value is expressible using a two-valued proposition. So in this setting, in analogy with Zadeh's assignment of absolute probabilities to fuzzy sets, all probabilities are expectations defined over a classical domain.

Introducing the propositional constant t which has this interpretation in the semantics of the $(n + 1)$-valued Łukasiewicz logic, in which all formulas are assigned values in the set $\{0, 1/(n+1), 2/(n+1), \ldots, n/(n+1), 1\}$:

$$\text{under all valuations } v,\ v(t) = n/(n+1),$$

we can in fact derive this expectation account from the principles obtained by Dutch Book Arguments that are listed above; in the case of the fourth, we take the conjunction and disjunction to be those of $(n + 1)$-valued Łukasiewicz logic.

A little more messily, the representation of probabilities of all propositions as expectations defined over a classical domain to fuzzy logics. The increase in 'mess' comes in that we may have to introduce a countable infinity of primitive two-valued propositions of the sort 'the truth-value of A is less than or equal to x'. Also, niceties concerning countable additivity intrude. We need to take a stand on just what incoherence—laying oneself open to a Dutch Book—really means. Does

incoherence mean facing certain loss, no matter how small? Or is it, rather, as de Finetti maintained, facing at least a positive infimum loss, and possibly greater loss?

If conditional probabilities are to satisfy the same axioms as absolute probabilities, then the orthodox format for Dutch Book arguments with conditional bets *requires* that we opt for the weaker, de Finetti, reading.

We obtain the representations as classical expectations in two steps. First we use Dutch Book arguments to show that probabilities must be representable as expectations taken with respect to a restriction of the probability distribution to a family of two-valued propositions whose logic is classical. Next we derive the representation from other principles governing probability. There is a point to this two-step procedure because the derivations—unlike, for the most part, the Dutch Book arguments—raise issues relating to expressiblity in Łukasiewicz logics.

Turning attention to conditional probabilities, we obtain a betting scheme that generalises the classical/boolean case and we find that we vindicate the definition of conditional probability proposed by Zadeh. This definition gives prominence to the product fuzzy logic. To be sure, we can define notions of conditional bet appropriate to pretty much any means of evaluating conjunctions, but only in the case of product logic do the coherent betting quotients for such bets merit the name "probabilities"—or so it is argued.

In the case of Łukasiewicz logics with finitely many values, Jeff Paris has provided a converse Dutch Book Argument to show that a certain set of principles, closely analogous to classical principles, is sufficient for the avoidance of betting arrangements entailing certain loss [Paris 2001, §3]. Daniele Mundici has recently supplied an extension to infinitely many valued Łukasiewicz logics and to infinite families of bets [Mundici 2006]. Perhaps it is little surprise that principles closely analogous to the classical suffice for, as stated, against the Łukasiewicz background all probabilities are, effectively, either classical probabilities or expectations defined with respect to classical probabilities.

With that representation in hand, we are, thankfully, in a position to present alternative, *easy* proofs, modelled on Howson and Urbach's proof of the classical case [Howson and Urbach 1993], of the converse Dutch Book Argument for both finitely-many-valued and continuum-valued Łukasiewicz logics. The converse Dutch Book Argument for Łukasiewicz logics with finitely many values is extended to incorporate conditional probabilities.

[Howson and Urbach 1993] Colin Howson and Peter Urbach: *Scientific Reasoning: The Bayesian Approach*, second edition, La Salle: Open Court, 1993.

[Mundici 2006] Daniele Mundici: 'Bookmaking over infinite-valued events', *International Journal of Approximate Reasoning*, **43** (2006), 223-240.

[Paris 2001] Jeff Paris: 'A note on the Dutch Book method', in Gert De Cooman, Terrence Fine and Teddy Seidenfeld (eds.), *ISIPTA '01, Proceedings of the Second International Symposium on Imprecise Probabilities and Their Applications, Ithaca, NY, USA*, Maastricht: Shaker Publishing, 2001, pp. 301-306. A slightly revised version is available on-line at http://www.maths.manchester.ac.uk/_jeff/papers/15.ps.

[Zadeh 1968] Lotfi A. Zadeh: 'Probability Measures of Fuzzy Events', *Journal of Mathematical Analysis and Applications*, **23** (1968), 421-427.

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