Betting on Fuzzy and Many-valued Propositions

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1. Introduction

In a 1968 article, ‘Probability Measures of Fuzzy Events’, Lotfi Zadeh proposed accounts of absolute and conditional probability for fuzzy sets [Zadeh 1968]. Where P is an ordinary (“classical”) probability measure defined on a σ-field of Borel subsets of a space X, and μ_A is a fuzzy membership function defined on X, *i.e.* a function taking values in the interval [0,1], the probability of the fuzzy set A is given by

$$P(A) = \int_X \mu_A(x)dP.$$

The thing to notice about this expression is that, in a way, there’s nothing “fuzzy” about it. To be well defined, we must assume that the “level sets”

$$\{x \in X: \mu_A(x) \leq \alpha\},\ \alpha \in [0,1],$$

are P-measurable. These are ordinary, “crisp”, subsets of X. And then $P(A)$ is just the expectation of the random variable μ_A. — This is entirely classical. Of course, you may *interpret* μ_A as a fuzzy membership function but really we have, if you’ll pardon the pun, in large measure lost sight of the fuzziness.

So you might ask:

- is this the only way to define fuzzy probabilities?

The answer, I shall argue, is yes.

Defining conditional probability Zadeh offered

$$P(A|B) = P(AB) / P(B), \text{ when } P(B) > 0,$$

where

$$\forall x \in X,\ \ \mu_{AB}(x) = \mu_A(x) \times \mu_B(x).$$

One might wonder

- is this the only way to define conditional probabilities?

The answer is:

> no, it is not the *only* way but it is the only *sensible* way.

Zadeh assigns probabilities to sets. What I offer here, using Dutch Book Arguments, is a vindication of Zadeh’s specifications when probability is assigned to propositions rather than sets. (But translation between proposition talk and set and event talk is straightforward. It’s just that proposition talk fits better with betting talk.)

2. Bets and many-valued logics

I apply "the Dutch Book method", as Jeff Paris calls it [Paris 2001], to fuzzy and many-valued logics that meet a simple linearity condition. I shall call such logics additive.

Additivity

For any valuation v and for any sentences A and B

$$v(A \wedge B) + v(A \vee B) = v(A) + v(B)$$

where '$\wedge$' and '$\vee$' the conjunction and disjunction of the logic in question. Additivity is common: the Gödel, Łukasiewicz, and product fuzzy logics are all additive, as are Gödel and Łukasiewicz n-valued logics.

In order to employ Dutch Book arguments, we need a betting scheme suitably sensitive to truth-values intermediate between the extreme values 0 and 1. Setting out the classical case the right way makes one generalization obvious.

Rather than betting odds, which are algebraically less tractable, we use, as is standard, a "normalized" betting scheme with fair betting quotients. Classically, with a bet on A at betting quotient p and stake S:

- the bettor gains $(1 – p)S$ if A;
- the bettor loses pS if not-A.

Taking 1 for truth, 0 for falsity, and $v(A)$ to be the truth-value of A, we can summarise this scheme like this:

the pay-off to the bettor is $(v(A) – p)S$.

And now we see how to extend bets to the many valued case: we adopt the same scheme but allow $v(A)$ to have more than two values. The slogan is: the pay-off is the larger, the more true A is.

Using this betting scheme, we obtain Dutch Book arguments for certain seemingly familiar principles of probability, seemingly familiar in that formally they recapitulate classical principles.

- $0 \leq Pr(A) \leq 1$
- $Pr(A) = 1$ when $\vdash A$
- $Pr(A) = 0$ when $A \vdash$
- $Pr(A \wedge B) + Pr(A \vee B) = Pr(A) + Pr(B)$.

Here $\wedge$ and $\vee$ are the conjunction and disjunction, respectively, of an additive fuzzy or many-valued logic.

Other principles that may or may not be independent, depending on the logic:

- $Pr(A) + Pr(\neg A) = 1$ when $v(\neg A) = 1 – v(A)$;
- $Pr(A) \geq x$ when, under all valuations, $v(A) \geq x$;
- $Pr(A) \leq x$ when, under all valuations, $v(A) \leq x$;
- $Pr(A) \leq Pr(B)$ when $A \vdash B$.

These second four principles follow from the first four in the case of Łukasiewicz *n*-valued logics

I'll show you how two of the arguments go as there's a very interesting connection with the standard Dutch Book arguments used in the classical, two-valued case.

Here's the easy one. We let x range over the possible truth-values (which all lie in the interval [0, 1]). Clearly, for given p, we can choose a value for the stake S that makes

$$G_x = (x - p)S$$

negative, for *all* values of x in the interval [0, 1], if, and only if, p is less than 0 or greater than 1. Hence

$$0 \leq Pr(A) \leq 1.$$

So far so good, but here's the cute bit:

$$G_x = xG_1 + (1 - x)G_0,$$

so G_x is negative for all values of $x \in [0, 1]$ *if, and only if*, G_1 and G_0 are both negative. From the classical case, we know that the necessary and sufficient condition for the latter is that p lie outside the interval [0, 1]. It suffices to look at the classical extremes to fix what holds good for all truth-values in the interval [0, 1].

Next, a harder case. We consider four bets:

1. a bet on A, at betting quotient p with stake S_1;
2. a bet on B, at betting quotient q with stake S_2;
3. a bet on $A \wedge B$, at betting quotient r with stake S_3;
4. a bet on $A \vee B$, at betting quotient s with stake S_4.

We assume that for all allowed values of $v(A)$ and $v(B)$,

$$v(A \wedge B) + v(A \vee B) = v(A) + v(B) \text{ and } v(A \wedge B) \leq min\{v(A), v(B)\}.$$

Then, where x, y, and z are the truth-values of A, B and $A \wedge B$ respectively, the pay-off is

$$G_{x,y} = (x - p)S_1 + (y - q)S_2 + (z - r)S_3 + ((x + y - z) - s)S_4.$$

This can be rewritten as

$$G_{x,y} = zG_{1,1} + (x - z)G_{1,0} + (y - z)G_{0,1} + (1 - x - y + z)G_{0,0}.$$

The co-efficients are all non-negative and cannot all be zero. Thus $G_{x,y}$ is negative, for all allowable x, y, and z, *just in case* $G_{1,1}$, $G_{1,0}$, $G_{0,1}$, and $G_{0,0}$ are all negative. From the standard Dutch Book argument for the two-valued, classical case, we know this to be possible if, and only if, $p + q \neq r + s$. Hence

$$Pr(A \wedge B) + Pr(A \vee B) = Pr(A) + Pr(B).$$

3. The classical expectation thesis for finitely-many-valued Łukasiewicz logics

As an initial vindication of Zadeh's account, we find that in the context of a finitely-many-valued Łukasiewicz logic, all probabilities are *classical expectations*. That is, the probability of a many-valued proposition is the expectation of its truth-value *and* that a proposition has a particular truth-value is expressible using a *two-valued* proposition. So in this setting, in analogy with Zadeh's assignment of absolute probabilities to fuzzy sets, *all* probabilities are expectations defined over a classical domain.

In all Łukasiewicz logics, conjunction and disjuction are evaluated by the functions max{0, x + y – 1} and min{1, x + y}, respectively.

Employing Łukasiewicz negation and one or more of Łukasiewicz conjunction, disjunction, and implication, one can define a sequence of n +1 formulas of a single variable, $J_{n,0}(p)$, $J_{n,1}(p)$, . . ., $J_{n,n}(p)$, which have this property: in the semantic framework of (n + 1)-valued Łukasiewicz logic it is the case that for every formula A, for all k, $0 \leq k \leq n$, and for every valuation v,

- $v(J_{n,k}(A)) = 1$, if $v(A) = k/n$;
- $v(J_{n,k}(A)) = 0$, if $v(A) \neq k/n$ ([Rosser and Turquette 1945]).

In the semantic framework of (n + 1)-valued Łukasiewicz logic, for all sentences A,

$$\vdash J_{n,0}(A) \vee_Ł J_{n,1}(A) \vee_Ł \ldots \vee_Ł J_{n,n}(A) \text{ and } J_{n,i}(A) \wedge_Ł J_{n,j}(A) \vdash \text{ for } 0 \leq i < j \leq n. \quad (*)$$

From the probability axioms, we have, for all sentences A, that

$$\textstyle\sum_{0 \leq i \leq n} Pr(J_{n,i}(A)) = 1.$$

The propositions of the form $J_{n,i}(A)$ are two-valued, so, (n + 1)-valued Łukasiewicz logic reducing to classical logic on the values 0 and 1, the logic of these propositions is classical. So, when restricted to these propositions and their logical compounds, the probability axioms give us *a classical, finitely additive, probability distribution*. What we show next is that this classical probability distribution determines the probabilities of all propositions in the language.

Theorem (Classical Expectation Thesis)**.** In the framework of (n + 1)-valued Łukasiewicz logic,

$$\mathrm{Pr}(A) = 1/n \textstyle\sum_{0 \leq i \leq n} i Pr(J_{n,i}(A)).$$

Proof. From (*) and the two-valuedness of the $J_{n,i}(A)$'s we have

$$A \dashv\vdash (A \wedge_Ł J_{n,0}(A)) \vee_Ł (A \wedge_Ł J_{n,1}(A)) \vee_Ł \ldots \vee_Ł (A \wedge_Ł J_{n,n}(A)).$$

From our probability axioms it follows that logically equivalent propositions must receive the same probability, so

$$Pr(A) = \textstyle\sum_{0 \leq i \leq n} Pr(A \wedge_Ł J_{n,i}(A))). \qquad (\dagger)$$

We consider two bets, one on $A \wedge_Ł J_{n,k}(A)$ at betting quotient p and stake S_1, the other on $J_{n,k}(A)$ at betting quotient q with stake S_2. The pay-offs are:

$$G_{=k/n} = (k/n - p)S_1 + ((1 - q)S_2) \text{ when } A \text{ has truth-value } k/n,$$

and

$$G_{\neq k/n} = - pS_1 - qS_2 \text{ when } A \text{ has truth-value other than } k/n.$$

Setting $S_2 = -(k/n)S_1$ gives a pay-off, independent of the truth-value of A, of $(qk/n - p)S_1$, which can be made negative by choice of S_1 provided $p \neq qk/n$. On the other hand, for arbitrary S_1 and S_2, when $p = qk/n$ the two pay-offs are

$$G_{=k/n} = (1 - q)[(k/n)S_1 + S_2] \text{ when } A \text{ has truth-value } k/n,$$

and

$$G_{\neq k/n} = - q[(k/n)S_1 + S_2] \text{ when } A \text{ has truth-value other than } k/n.$$

These cannot both be negative. Hence

$$Pr(A \wedge_Ł J_{n,k}(A)) = (k/n)Pr(J_{n,k}(A)).$$

Substituting in (†), we obtain:

$$Pr(A) = 1/n \textstyle\sum_{0 \le i \le n} iPr(J_{n,i}(A)). \qquad \textit{Q.e.d.}$$

Two comments

Having been obtained by an independent Dutch Book argument, the Classical Expectation Thesis may seem to be an additional principle. In fact it is not; it is derivable from our axioms for probability. To show this we have introduce a propositional constant, introduced into Łukasiewicz logic by Słupecki in order to obtain expressive completeness ([Słupecki 1936]).

In the semantics of $(n + 1)$-valued Łukasiewicz logic, in which all formulas are assigned values in the set $\{0, 1/(n+1), 2/(n+1), \ldots, n/(n+1), 1\}$, the propositional constant t has this interpretation:

$$\text{under all valuations } v,\ v(t) = n/(n+1).$$

Let t_1 be the $(n - 1)$-fold $\wedge_Ł$-conjunction of t with itself. For $1 \le k \le n$, let t_k be the k-fold $\vee_Ł$-disjunction of t_1 with itself. $v(t_1) = 1/n$ and $v(t_k) = k/n$. Since we have

$$t_k \wedge_Ł t_1 \vdash\ ,\ 1 \le k < n, \text{ and } \vDash t_n,$$

from our probability axioms we obtain:

$$Pr(t_k) = kPr(t_1),\ 1 \le k \le n, \text{ and } Pr(t_n) = 1,$$

hence

$$Pr(t_k) = k/n\ ,\ 1 \le k \le n.$$

Using the t_i's we can then derive the Classical Expectation Thesis, for

$$A \wedge_Ł J_{n,k}(A) \dashv\vdash t_k \wedge_Ł J_{n,k}(A),$$

$$J_{n,k}(A) \dashv\vdash t_n \wedge_Ł J_{n,k}(A) \dashv\vdash (t_1 \wedge_Ł J_{n,k}(A)) \vee_Ł (t_1 \wedge_Ł J_{n,k}(A)) \vee_Ł \ldots \vee_Ł (t_1 \wedge_Ł J_{n,k}(A))$$

and

$$t_k \wedge_Ł J_{n,k}(A) \dashv\vdash (t_1 \wedge_Ł J_{n,k}(A)) \vee_Ł (t_1 \wedge_Ł J_{n,k}(A)) \vee_Ł \ \ldots \vee_Ł (t_1 \wedge_Ł J_{n,k}(A))$$

where there are n disjuncts in the first of these and k in the second, so

$$Pr(A \wedge_Ł J_{n,k}(A)) = Pr(t_k \wedge_Ł J_{n,k}(A)) = k\, Pr(t_1 \wedge_Ł J_{n,k}(A)) = (k/n)Pr(J_{n,k}(A)).$$

The Dutch Book argument for the Classical Expectation Thesis goes through with *any* notion of conjunction for which $v(A \,\&\, B) = v(A)$ when $v(B) = 1$ and $v(A \,\&\, B) = 0$ when $v(B) = 0$. Also, the $J_{n,i}(A)$'s being truth-functional, the Classical Expectation Thesis holds good of every proposition in the semantic framework, not just those expressible using the Łukasiewicz connectives.

4. The extension to infinitely many truth-values (a sketch)

For any rational number x in the interval [0,1], there is a formula $\varphi(p)$ of a single propositional-variable p, constructed using Łukasiewicz negation and any one or more of Łukasiewicz conjunction, disjunction, or implication, such that, under *any* valuation taking values in [0,1], $v(\varphi(A/p)) = 0$ if $v(A) \leq x$ and $v(\varphi(A/p)) > 0$ otherwise [McNaughton 1951].

Employing the Gödel negation ~, then, we have,

- for each interval $[0, x]$ with x rational, a formula $J_{[0,x]}(A)$ that takes the value 1 under any valuation v for which $v(A) \leq x$ and otherwise takes the value 0;
- for each half-open interval $(x, y]$ with rational endpoints x and y, $x < y$, a formula $J_{(x,y]}(A)$ that takes the value 1 under a valuation v when $v(A) \in (x, y]$ and otherwise takes the value 0.

Given a strictly increasing, finite sequence $x_0, x_1, \ldots, x_{n-1}$ of rational numbers in the open interval (0, 1), consider the family of $n + 1$ bets:

- a bet on A at betting quotient q with stake S;
- a bet on $J_{[0,x_1]}(A)$ at betting quotient p_1 with stake S_1;
- a bet on $J_{(x_{i-1},x_i]}(A)$ at betting quotient p_i with stake S_i, $1 < i < n$;
- a bet on $J_{(x_{n-1},1]}(A)$ at betting quotient p_n with stake S_n.

A Dutch Book argument then shows that

$$\textstyle\sum_{2 \leq i \leq n} x_{i-1}\, Pr(J_{(x_{i-1},x_i]}(A)) \leq Pr(A) \leq x_1 Pr(J_{[0,x_1]}(A)) + \sum_{2 \leq i \leq n} x_i\, Pr(J_{(x_{i-1},x_i]}(A))$$

where $x_n = 1$.

So by taking finer and finer partitions we can more closely approximate the probability of A from above and below. This may not quite do to fix $Pr(A)$ exactly. For that we *may* also need the probabilities of at most a countable infinity of (two-valued) statements of the form

$$v(A) \leq x$$

where x is an irrational number.

With these in hand, we then find that

$$Pr(A) = \textstyle\int^1_0 x dF_A(x),$$

Where F is the ordinary, "classical" distribution function determined by the probabilities of the $J_{[0,x]}(A)$'s, $J_{(x,y]}(A)$'s and however many $v(A) \leq x$'s with x irrational we have used.

By introducing a countably infinite family of logical constants, we can *derive* this classical representation from the previously given principles of probability together with the principle

- for any proposition A logically constrained to take only the values 0 and 1 and for rational values of x in the interval [0, 1], $Pr(t_x \wedge A) = xPr(A)$

where t_x takes the value x under all valuations v.

The really neat feature of infinitely many-valued Łukasiewicz logics is that this principle is derivable from the basic principles

- $0 \leq Pr(A) \leq 1$
- $Pr(A) = 1$ when $\vDash A$
- $Pr(A) = 0$ when $A \vDash$
- $Pr(A \wedge_Ł B) + Pr(A \vee_Ł B) = Pr(A) + Pr(B)$.

5. Conditional probabilities

In the classical setting, a bet on A conditional on B is a bet that goes ahead if, and only if, B is true and is then won or lost according as to whether A is true or not. The pay-offs for such a conditional bet with stake S at betting quotient p are:

- the bettor gains $(1 - p)S$ if A and B;
- the bettor loses pS if not-A and B;
- the bettor neither gains nor loses if not-B.

We can summarise this betting scheme like this:

$$v(B)(v(A) - p)S.$$

And so, as with ordinary bets, we now know one way to extend the scheme for conditional bets on classical, two-valued propositions to many-valued propositions.

A straightforward Dutch Book argument, which again piggy-backs on the proof in the two-valued case, then tells us that

$$Pr(A \wedge_{\times} B) = Pr(A|B) \times Pr(B)$$

where

$$v(A \wedge_{\times} B) = v(A) \times v(B).$$

— Just what Zadeh said.

You can, if you are so minded, generalize the classical scheme using *any* many-valued or fuzzy conjunction that is "classical at the extremes":

$$(v(A \wedge B) - v(B)p)S.$$

A Dutch Book argument — in all essentials, the *same* Dutch Book argument — will then deliver:

$$Pr(A \wedge B) = Pr(A|B) \times Pr(B).$$

However, $Pr(.|B)$ satisfies the axioms for an absolute probability measure *only* when product conjunction is used.

Observation The betting scheme

$$v(B)(v(A) - p)S$$

makes sense for *any* many-valued logic. But product conjunction requires infinitely many truth-values (when there are more than two). So when we work with a finitely-many-valued logic with more than two values, we can't properly *express* the relation between conditional and absolute probabilities.

6. Converse Dutch Book Arguments

In the case of a finitely-many-valued Łukasiewicz logic, relative to the *classical* probability distribution induced by Pr on the classical domain of propositions of the form $J_{n,k}(A)$ and their logical compounds, the expected value of a bet on A with betting quotient $Pr(A)$ and stake S is

$$\sum_{0 \leq i \leq n} ((i/n) - Pr(A))S \times Pr(J_{n,i}(A)) = 0.$$

But if a bettor faces a Dutch book, the expected net pay-off on a finite number of bets must be negative. So in the case of a finitely-many-valued Łukasiewicz logic these axioms are necessary and sufficient for avoidance of a Dutch book:

- $0 \leq Pr(A) \leq 1$
- $Pr(A) = 1$ when $\models A$
- $Pr(A) = 0$ when $A \models$
- $Pr(A \wedge_{Ł} B) + Pr(A \vee_{Ł} B) = Pr(A) + Pr(B)$.

One has to be a bit more careful when considering infinite families of bets and infinitely-many-valued logics. But we can get the same result—satisfaction of these axioms suffices for avoidance of a Dutch book—*if we give this reading to a Dutch book*:

> a bettor faces a Dutch book on a family of bets if, no matter what truth-values the propositions bet upon take, there is some positive sum that she must lose. (She may lose more than this for some combinations of truth-values.)

Put another way, she faces a Dutch book if she would be better off in all circumstances paying some flat fee to cancel the bets made.

This is de Finetti's weak notion of a Dutch book. It turns out to be *exactly* what we need in showing that the conditional probability distribution *Pr*(.|*B*) satisfies the axioms for an absolute probability measure.

Moral By first obtaining the classical expectation representation for probabilities assigned to fuzzy propositions, we can avoid the horrifying technicality of more direct Converse Dutch Book Arguments (e.g. in [Paris 2001] and [Mundici 2006]). Instead, as Howson and Urbach do in the classical case ([Howson and Urbach 1993]), we use purely classical reasoning about expectations to show the constraints on assignments of fair betting quotients to fuzzy or many-valued propositions sufficient for avoidance of a Dutch book.

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