Two études on modal logic in computer science

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joint work with

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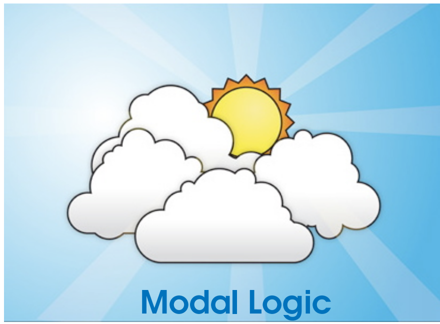
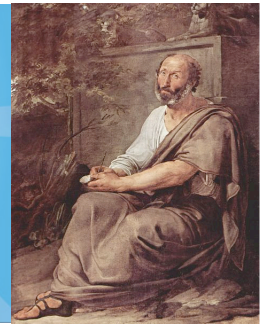
Modal Logic



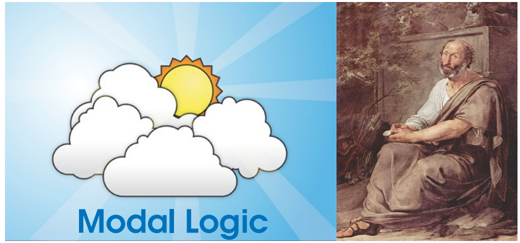


"Narrowly construed, modal logic studies reasoning that involves the use of the expressions **necessarily** and **possibly**.

However, the term **modal logic** is used more broadly to cover a family of logics with similar rules and a variety of different symbols."

(*Stanford Encyclopedia of Philosophy*)

Modal Logic and Computer Science



Dynamic logics

Description logics

Spatial logics

Temporal logics

Computer Science

Étude I: Spatial Logic

Spatial logics: formal languages interpreted over various classes of geometrical structures (topological, metric, Euclidean spaces, etc.)

- **Mathematics:** Hilbert's geometry, Tasrki's geometry, $\mathbf{Th}(\mathbb{R}^n, +, \times, \leq), \dots$
- **Philosophy:** Whitehead's & de Laguna's spatial ontologies (1920s)

  region-based theories of space
- **Theoretical physics:** Logics of space-time (part of Hilbert's 6th problem)

  e.g., Andréka&Németi (2007), Goldblatt (1980, 87), Shehtman&Shapirovsky (2003)
- **Computer science & AI:**

  – GIS, spatial databases and (first-order) query languages

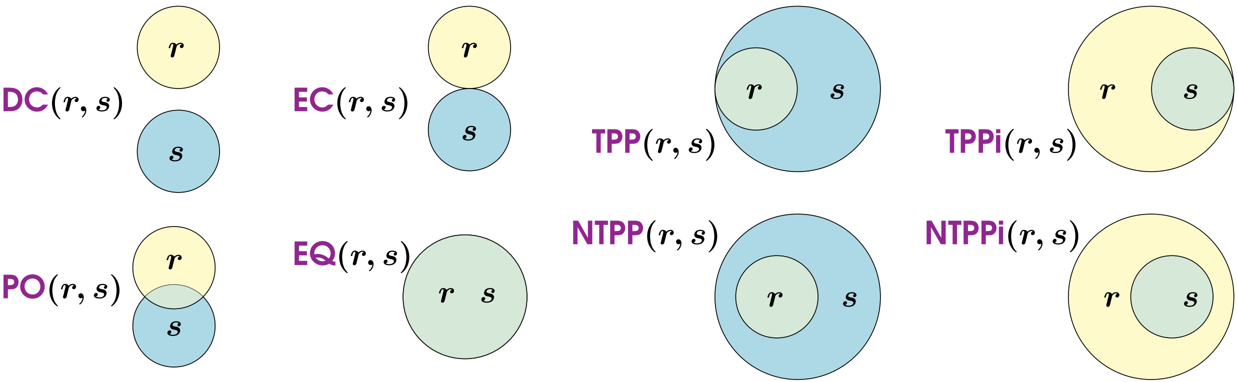
  Complexity of $\mathbf{Th}(\mathbb{R}, +, \times, \leq)$ *still open?*

  – qualitative spatial (region-based) KR&R; quantifier-free constraint systems and their algebraic counterparts

Intended models

Aim: (M. Egenhofer (GIS), A. Cohn, B. Nebel (KR&R), *et al.*, 1990s)

> Effective reasoning about basic topological relations between spatial regions over Euclidean plane $\mathbb{R}^2$



NB: First-order theories are undecidable

Models: topological spaces, metric spaces, $\mathbb{R}^2$, algebras, . . .

Regions: arbitrary, connected, 'regular,' semialgebraic/linear sets, . . .

‘Topological’ logic

Kuratowski’s axioms: A *topological space* is a pair $T = (U, \cdot^\circ)$ where

$$(X \cap Y)^\circ = X^\circ \cap Y^\circ, \quad X^\circ \subseteq X^{\circ\circ}, \quad X^\circ \subseteq X, \quad \text{and} \quad U^\circ = U$$

Similarity with axioms for **modal logic** $\mathcal{S}4$ spotted by McKinsey & Tarski (1944)
in their quest to ‘algebraise’ topology

$$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi, \quad \Box\varphi \rightarrow \Box\Box\varphi, \quad \Box\varphi \rightarrow \varphi, \quad \text{and} \quad \Box\top \leftrightarrow \top$$

Topological logic $\mathcal{S}4_u$ (interpreted over T)

- **terms:** **subsets of** T

$$\tau \quad ::= \quad v_i \quad | \quad \overline{\tau} \quad | \quad \tau_1 \cap \tau_2 \quad | \quad \tau^\circ \quad | \quad \tau^-$$

complement — interior — closure

= modal logic $\mathcal{S}4$

$\tau^\circ = \Box\tau, \quad \tau^- = \Diamond\tau$

- **formulas:** **true** or **false**

$$\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$$

+ ‘universal modality’

McKinsey & Tarski *et al.*: $\mathcal{S}4_u \vdash \varphi$ *iff* φ *is ‘valid’ in all topological spaces*

Kripke frames = Aleksandrov spaces

A space is called **Aleksandrov** if *arbitrary* intersections of open sets are open

Aleksandrov spaces $\equiv$ **Kripke frames** $\mathfrak{F} = (W, R)$, R is quasi-order on W,
where the R-closed sets are open

Theorem. (McKinsey & Tarski, Shehtman, Statman, Areces *et. al*, ...)

$$\textsf{Sat}(\mathcal{S}4_u, \textsc{All}) = \textsf{Sat}(\mathcal{S}4_u, \textsc{Alek}) = \textsf{Sat}(\mathcal{S}4_u, \textsc{FinAlek}) \neq \textsf{Sat}(\mathcal{S}4_u, \mathbb{R}^n), \; n \geq 1,$$

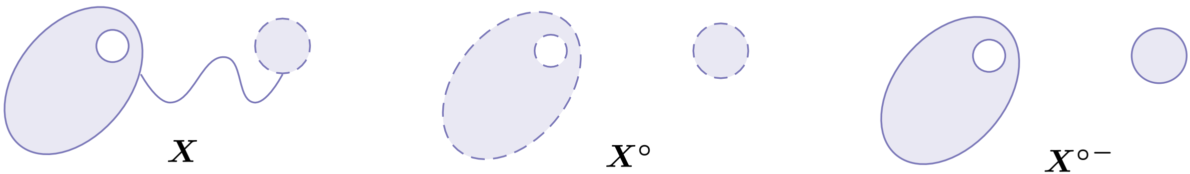
and these sets are **PSPACE**-complete

- Reasoning too **complex**?
- **Strange sets** are allowed as regions?
- The language is **not natural**?
- The language is too **weak**? ...

Regular closed sets and logic $\mathcal{B}$ (mereology)

$X \subseteq T$ is **regular closed** if $X = X^{\circ -}$

RC(T) regular closed subsets of T



(**RC**$(T), \cdot, -, \emptyset, T$) is a Boolean algebra

where $X \cdot Y = (X \cap Y)^{\circ -}$ and $-X = (\overline{X})^{-}$

$\mathcal{B}$-terms: $\tau \quad ::= \quad r_i \quad | \quad -\tau \quad | \quad \tau_1 \cdot \tau_2$ **regular closed sets!**

$\mathcal{B}$-formulas: $\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$

$\mathcal{B}$ is a **fragment** of $\mathcal{S}4_u$

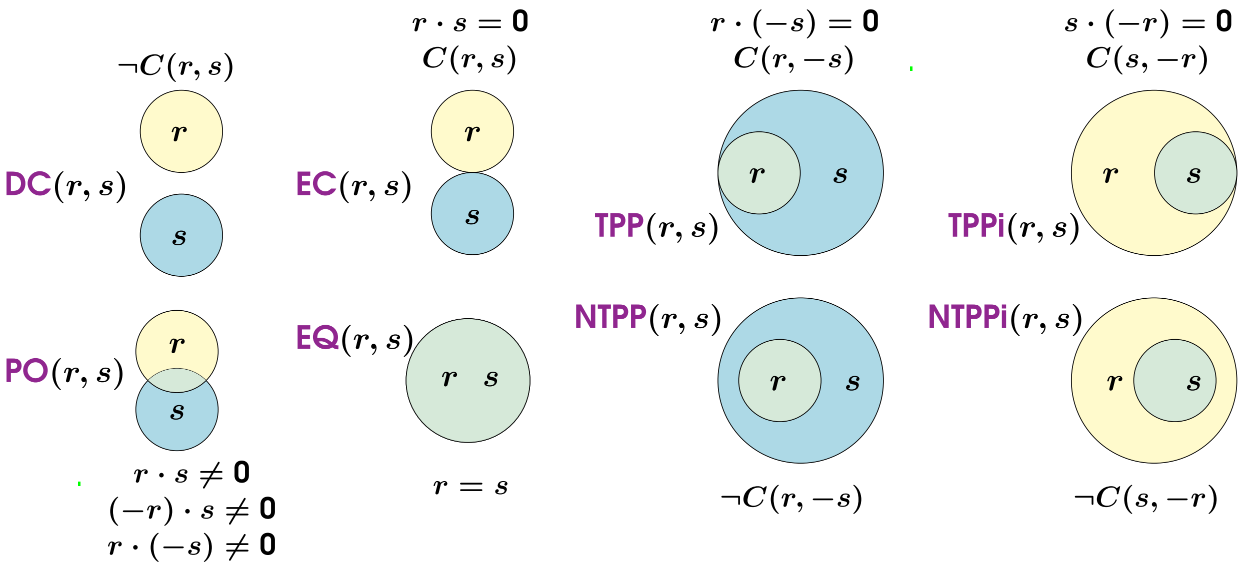
Sat$(\mathcal{B}$, Reg$)$ = **Sat**$(\mathcal{B}$, AlekReg$)$ = **Sat**$(\mathcal{B}$, **RC**$(\mathbb{R}^n))$, $n \geq 1$,
and this set is **NP**-complete

$\mathcal{C}$ = $\mathcal{B}$ + contact predicate (mereotopology)

↓Whitehead's (1929) 'connection' relation

$\mathcal{C}$-formulas: $\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad C(\tau_1, \tau_2) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$

a.k.a. $\mathcal{BRCC}$-8, still a fragment of $\mathcal{S}4_u$



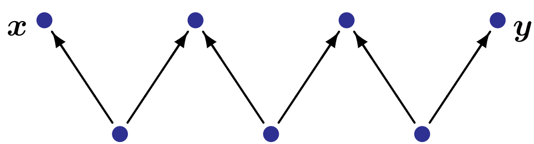
Sat$(\mathcal{C}, \text{Reg}) =$ **Sat**$(\mathcal{C}, \text{AlekReg})$ and this set is **NP**-complete; $\neq$ **Sat**$(\mathcal{C}, \mathbf{RC}(\mathbb{R}^n))$

Connectedness

A topological space is **connected** iff
it is not the union of two non-empty, disjoint, open sets

$X \subseteq T$ is **connected in T** if either it is empty,
or the topological space X (with the subspace topology) is connected

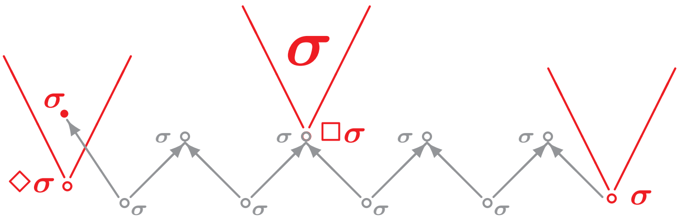
An **Aleksandrov space** induced by $\mathfrak{F} = (W, R)$ is **connected** iff
$\mathfrak{F}$ is connected as a **non-directed graph**
(any $x, y \in W$ are connected by a path along $R \cup R^{-1}$)



Logics with connectedness predicates

$\mathcal{S}4_u c$-, $\mathcal{C}c$-, $\mathcal{B}c$-formulas:	$\varphi \quad ::= \quad \dots \quad \mid \quad \underbrace{c(\tau)}_{\tau \text{ is connected}} \quad \mid \quad \dots$
$\mathcal{S}4_u cc$, $\mathcal{C}cc$-, $\mathcal{B}cc$-formulas:	$\varphi \quad ::= \quad \dots \quad \mid \quad \underbrace{c^{\leq k}(\tau)}_{\tau \text{ has } \leq k \text{ components}} \quad \mid \quad \dots$

How does the new 'modal operator' $c(\sigma)$ work?

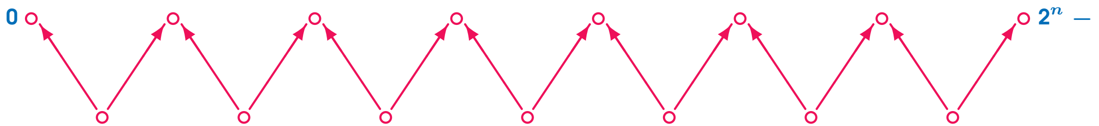


- How long can such σ-paths be?
- What can be expressed by means of c?
- What is the interaction between $c(\sigma_1), \dots, c(\sigma_n)$ and the Booleans?

$c(\sigma)$ can simulate PSpace Turing machines

Example: simulating a binary counter using predicates c and C

- Numbers: $\overbrace{(-r_n)\cdot\ldots\cdot(-r_1)}^{0},\ \overbrace{(-r_n)\cdot\ldots\cdot(-r_2)\cdot r_1}^{1},\ \ldots,\ \overbrace{r_n\cdot\ldots\cdot r_1}^{2^n-1}$
- $c(1)$: the whole space is connected
- $-r_n\cdot\ldots\cdot -r_1 \neq 0$, $r_n\cdot\ldots\cdot r_1 \neq 0$: there exist 0 and 2^n-1, and so a **path connecting them**



- that all numbers occur on this path (not necessarily in proper order) is ensured by polynomially-many constraints $\neg C(l,m)$ for $m \notin \{l-1, l, l+1\}$

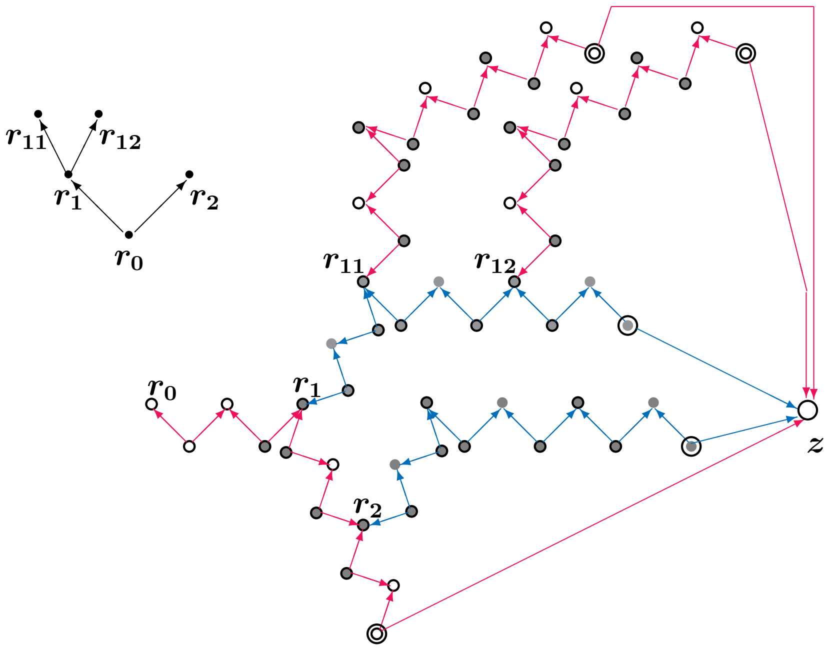
$\neg C\big((-r_i)\cdot r_j\cdot(-r_k), r_i\big)$, $i>j>k$; $\neg C\big(r_i\cdot r_{i-1}, (-r_i)\cdot r_{i-1}\big)$, $i>1$;

$\neg C\big(r_i\cdot(-r_j)\cdot r_k, (-r_i)\big)$, $i>j>k$; $\ldots$

$c(\sigma)$ can be exponentially long

$\mathcal{C}c$ is PSPACE-hard

$c(\sigma_1)$ and $c(\sigma_2)$ can simulate PSpace alternating TMs



- Two connectedness constraints $c(\sigma_0 + z)$, $c(\sigma_1 + z)$ and a number of $\neg C(\tau_1, \tau_2)$

$\mathcal{C}c$ is EXPTIME-hard

Contact $C(\tau_1, \tau_2)$ is expressible by means of $c(\sigma)$

$$\mathcal{C}c \models c(\tau_1) \land c(\tau_2) \longrightarrow \big[c(\tau_1 + \tau_2) \leftrightarrow C(\tau_1, \tau_2)\big]$$

Let φ be a $\mathcal{C}c$-formula

- $\varphi[C(\tau_1, \tau_2)]^+$ (positive occurrence of $C(\tau_1, \tau_2)$)

  eliminated using fresh variables t, t_1, t_2:

$$\varphi^* \quad = \quad \varphi[t = \mathbf{0}]^+ \quad \land \quad \big((t = \mathbf{0}) \quad \longrightarrow \quad c(t_1 + t_2) \ \land \bigwedge_{i=1,2} (t_i \subseteq \tau_i) \land c(t_i)\big)$$

- $\varphi[C(\tau_1, \tau_2)]^-$ (negative occurrence of $C(\tau_1, \tau_2)$)

  eliminated using fresh variables s, t, t_1, t_2:

$$\varphi^* \quad = \quad \big(\varphi[t \neq \mathbf{0}]^-\big)_{|s} \quad \land \quad \big(t \cdot s = \mathbf{0} \quad \longrightarrow \quad \neg c(t_1 + t_2) \ \land \bigwedge_{i=1,2} c(t_i) \land (\tau_i \cdot s \subseteq t_i)\big)$$

φ is satisfiable iff φ^* is satisfiable

$\mathcal{B}c$ is EXPTIME-hard

Complexity results

- $\mathcal{S}4_u c$ can be embedded into $\mathcal{CPDL}$ **with nominals**, which is ExpTime-complete [De Giacomo, 1995]

$\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{S}4_u c$ are all ExpTime-complete

- The component counting predicates $c^{\leq k}(\tau)$ increase complexity:

$\mathcal{B}cc$, $\mathcal{C}cc$, $\mathcal{S}4_u cc$ are all NExpTime-complete

Open problems

- Axiomatise the connectedness predicate.
- Interpretations over $\mathbb{R}^2$ with 'tame' regions.
- Query languages.
- ...

Étude II: Description Logic

Description Logic: a family of formal languages tailored for representing knowledge about concepts and concept hierarchies

- Invented in 1980s as logic-based formalisations of semantic networks & frames

  Identified in 1990s as a **'notational variant'** of modal and hybrid logics

- Has become very popular as **ontology languages** (Snomed, GO, NCI, ...)

  SNOMED will be a core component of the NHS Connecting for Health IT project; already worth £6.5 billion

- Was recognised as the **'cornerstone of the Semantic Web'** for providing a formal basis for the Web Ontology Language (OWL)

- Applications in database integration, querying via ontologies, etc.

Description logic $\mathcal{ALCQI}$ (simplified OWL)

Vocabulary: $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an **interpretation**

- individuals $a_0, a_1, \dots$ (e.g., john, mary) $\quad a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
- concept names $A_0, A_1, \dots$ (e.g., Person, Female) $\quad A_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
- role names $R_0, R_1, \dots$ (e.g., hasChild, loves) $\quad R_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
- roles

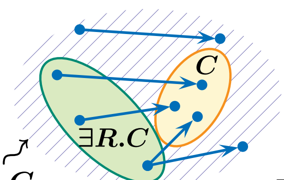
$$R \quad ::= \quad R_i \quad | \quad R_i^-$$

$$(R_i^-)^{\mathcal{I}} = \{(y, x) \mid (x, y) \in R_i^{\mathcal{I}}\}$$

- concepts

$$C \quad ::= \quad A_i \quad | \quad \neg C \quad | \quad C_1 \sqcap C_2 \quad | \quad \exists R.C \quad | \quad \forall R.C \quad | \quad \geq q R.C$$

$\Diamond_R C$



$\Box_R C$

$\forall R.C \equiv \neg(\exists R.\neg C)$

'there are at least q distinct R-successors that are in C'

Description logic $\mathcal{ALCQI}$ (cont.)

knowledge base $\mathcal{K}$ = TBox $\mathcal{T}$ + ABox $\mathcal{A}$

- $\mathcal{T}$ is a set of **terminological axioms** of the form $C \sqsubseteq D$
- $\mathcal{A}$ is a set of **assertions** of the form $C(a)$ and $R(a, b)$

Reasoning: combined complexity

– satisfiability, subsumption $\mathcal{K} \models C \sqsubseteq D$ ExpTime

– instance checking $\mathcal{K} \models C(a)$ data (ABox) complexity

– conjunctive query answering $\mathcal{K} \models q(\vec{a})$ coNP

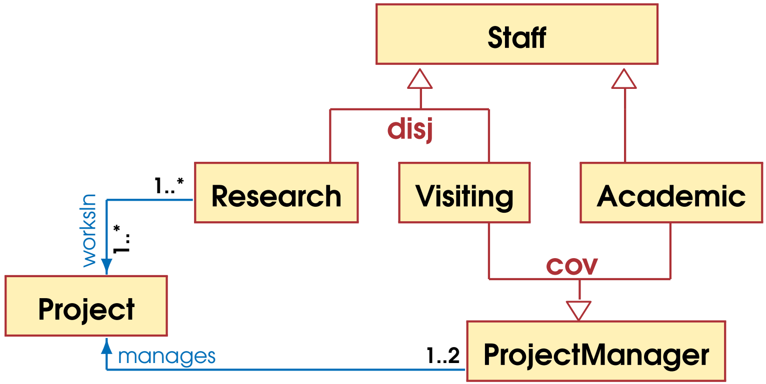
$q(\vec{a}) = \exists \vec{y} \varphi(\vec{a}, \vec{y})$, φ a conjunction of atoms

Reasoners: **RACER**, **FaCT++**, **Pellet** cope well with subsumption & instance checking

What about **'islands of tractability'** in DL?

DL 'islands of tractability': *DL-Lite* family

- Typical conceptual database schema (a fragment):



Translating into DL:

$$\exists \text{manages}.\top \sqsubseteq \text{ProjectManager}$$
$$\exists \text{manages}^{-}.\top \sqsubseteq \text{Project}$$
$$\text{Project} \sqsubseteq \exists \text{manages}^{-}.\top$$
$$\geq 3 \ \text{manages}^{-}.\top \sqsubseteq \bot$$

$$\text{Research} \sqcap \text{Visiting} \sqsubseteq \bot$$
$$\text{Academic} \sqsubseteq \text{ProjectManager}$$
$$\text{ProjectManager} \sqsubseteq \text{Academic} \sqcup \text{Visiting}$$
$$\ldots$$

DL 'islands of tractability': *DL-Lite* family (cont.)

1. *DL-Lite*$_{bool}$

$$R ::= P \mid P^-$$
$$B ::= \bot \mid A \mid \geq qR$$
$$C ::= B \mid \neg C \mid C_1 \sqcap C_2$$

combined complexity: **NP**
data comp. instance: **LogSpace**
data comp. query: **coNP**

TBox axioms $C_1 \sqsubseteq C_2$ ABox assertions: $C(a)$, $R(a,b)$

2. *DL-Lite*$_{horn}$

combined complexity: **P**
data comp. instance: **LogSpace**
data comp. query: **LogSpace**

TBox axioms $B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B$

3. *DL-Lite*$_{krom}$

comb. comp.: **NLogSpace**
d.c. instance: **LogSpace**
d.c. query: **coNP**

TBox axioms $B_1 \sqsubseteq B_2$ (subclass) $B_1 \sqsubseteq \neg B_2$ (disjointness) $\neg B_1 \sqsubseteq B_2$

NB: these complexity results are closely related to complexity of reasoning in fragments of propositional logic

Σ-inseparability

$\mathcal{T}_1$:

Research $\sqcap$ Visiting $\sqsubseteq \bot$	Academic $\sqsubseteq$ $=1$ teaches
$\exists$teaches $\sqsubseteq$ Academic $\sqcup$ Research	$\exists$writes $\sqsubseteq$ Academic $\sqcup$ Research
Research $\sqsubseteq$ $\exists$worksIn	$\exists$worksIn$^-$ $\sqsubseteq$ Project
Project $\sqsubseteq$ $\exists$manages$^-$	$\exists$manages $\sqsubseteq$ Academic $\sqcup$ Visiting

$\mathcal{T}_2 = \mathcal{T}_1 \cup \{\text{Visiting} \sqsubseteq\ \geq 2\,\text{writes}\}$, $\quad \Sigma = \{\text{teaches}\}$

- Are $\mathcal{T}_1$ and $\mathcal{T}_2$ Σ-concept inseparable?
  YES

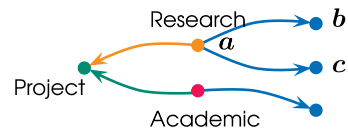
  $\mathcal{T}_1 \not\models \text{Visiting} \sqsubseteq\ = 1\,\text{teaches}$
  $\mathcal{T}_2 \models \text{Visiting} \sqsubseteq\ = 1\,\text{teaches}$

- Are $\mathcal{T}_1$ and $\mathcal{T}_2$ Σ-query inseparable? $\quad \mathcal{A} := \{\text{teaches}(a,b), \text{teaches}(a,c)\}$

NO: $\quad q := \exists x\, (=1\,\text{teaches})(x)$

$(\mathcal{T}_1, \mathcal{A}) \not\models q \quad$ but $\quad (\mathcal{T}_2, \mathcal{A}) \models q$

$(\mathcal{T}_2, \mathcal{A})$:



- Σ-inseparability (both concept and query) for *DL-Lite$_{bool}$* is **Π_2^p-complete**
- Σ-inseparability (both concept and query) for *DL-Lite$_{horn}$* is **coNP-complete**

Off-the-shelf QBF solvers (sKizzo, Quaffle, QuBE) show reasonable performance

Σ-inseparability and uniform interpolation

$\mathcal{L}$ **has uniform interpolation** (logic) or **admits forgetting** (CS) if

$$\forall \mathcal{T}\ \forall \Sigma\ \ \exists \mathcal{T}_\Sigma\ \ \big(\mathbf{sig}(\mathcal{T}_\Sigma) \subseteq \Sigma\ \wedge\ \mathcal{T} \text{ and } \mathcal{T}_\Sigma \text{ are } \Sigma\text{-concept inseparable}\big)$$

- *DL-Lite$_{bool}$* and *DL-Lite$_{horn}$* have uniform interpolation

> $\mathcal{T}$ Σ-concept entails $\mathcal{T}'$ iff $\mathcal{T} \models C_1 \sqsubseteq C_2$, for all $(C_1 \sqsubseteq C_2) \in \mathcal{T}'_\Sigma$
> where $\mathcal{T}'_\Sigma$ is a uniform interpolant of $\mathcal{T}'$ w.r.t. Σ

DL-Lite$^u_{bool}$ extends *DL-Lite$_{bool}$* with the **existential modality** $\exists u.C$ (or $C \neq \bot$)

- *DL-Lite$^u_{bool}$* has uniform interpolation

> $\mathcal{T}$ Σ-query entails $\mathcal{T}'$ in *DL-Lite$_{bool}$* iff $\mathcal{T} \models C_1 \sqsubseteq C_2$, for all $(C_1 \sqsubseteq C_2) \in \mathcal{T}'_\Sigma$
> where $\mathcal{T}'_\Sigma$ is a uniform interpolant of $\mathcal{T}'$ w.r.t. Σ in *DL-Lite$^u_{bool}$*

In the previous example: $= 1$ teaches $\neq \bot$

What is the size of uniform interpolants?

Delicate balance: either numbers restrictions or role inclusions

DL-Lite$_{core}$ with axioms of the form $B_1 \sqsubseteq B_2$ is **NLogSpace**-complete

DL-Lite$_{core}$ **+** role inclusions $R_1 \sqsubseteq R_2$ is **ExpTime**-complete

Example: $A_1 \sqcap A_2 \sqsubseteq C$ can be simulated by the axioms:

- $A_1 \sqsubseteq \exists R_1$ $\quad A_2 \sqsubseteq \exists R_2$
- $R_1 \sqsubseteq R_{12}$ $\quad R_2 \sqsubseteq R_{12}$ $\quad \geq 2\, R_{12} \sqsubseteq \bot$
- $\exists R_1^- \sqsubseteq \exists R_3^-$ $\quad \exists R_3 \sqsubseteq C$
- $R_3 \sqsubseteq R_{23}$ $\quad R_2 \sqsubseteq R_{23}$ $\quad \geq 2\, R_{23}^- \sqsubseteq \bot$

DL-Lite **+** role inclusions **–** number restrictions is fine again

Proofs and other results are available at

http://www.dcs.bbk.ac.uk/~michael