Main Points

• A substructural epistemic logic of belief supported by evidence
• Evidence as a resource used in justifying beliefs and actions
• Support is not necessarily closed under classical consequence (resource-boundedness)

Overview

• Substructural epistemic logics – recent history and my contribution
• Motivating scenarios involving facts, evidence and beliefs
• Details of my approach
• Technical results – axiomatization and definability (briefly)
• Conclusion
Definition 1.1 (Language $\mathcal{L}_\square$)

- $p, \neg \varphi, \varphi \land \psi$;
- $\square \varphi$ as “the agent believes that $\varphi$”.

Definition 1.2 (Models for $\mathcal{L}_\square$)

$M = \langle P, E, V \rangle$

- $P$ is a non-empty set (“possible worlds”);
- $E \subseteq P \times P$ (“epistemic accessibility”);
- $V(p) \subseteq P$ for every variable $p$.

Truth and validity:

- $M, w \models \square \varphi$ iff $M, v \models \varphi$ for all $v$ such that $wE v$;
- $M \models \varphi$ iff $M, w \models \varphi$ for all $w \in P$. 
The Logical Omniscience Problem
(Hintikka, 1962, 1975)

Fact 1.3

\[ M \models \left( \bigwedge_{0 \leq i \leq n} \varphi_i \right) \rightarrow \psi \quad \Rightarrow \quad M \models \left( \bigwedge_{0 \leq i \leq n} \Box \varphi_i \right) \rightarrow \Box \psi \]
Definition 1.4 (Compatibility Models)

\[ M = \langle P, W, E, C, V \rangle \]

- \( C \subseteq P \times P \) ("compatibility") (Berto, 2015; Dunn, 1993);
- \( W \subseteq P \) such that \( u \in W \) only if \( uCx \leftrightarrow u = x \) ("worlds").

Truth and validity:

- \( M, x \models \neg \varphi \) iff \( M, y \not\models \varphi \) for all \( y \) such that \( xCy \);
- \( M \models \varphi \) iff \( M, w \models \varphi \) for all \( w \in W \).
Definition 1.4 (Compatibility Models)

\[ M = \langle P, W, E, C, V \rangle \]

- \( C \subseteq P \times P \) (“compatibility”) (Berto, 2015; Dunn, 1993);
- \( W \subseteq P \) such that \( u \in W \) only if \( uCx \leftrightarrow u = x \) (“worlds”).

Truth and validity:

- \( M, x \models \neg \varphi \) iff \( M, y \not\models \varphi \) for all \( y \) such that \( xCy \);
- \( M \models \varphi \) iff \( M, w \models \varphi \) for all \( w \in W \).

Example 1.5
Definition 1.6 (Epistemic Models)

\[ F = \langle P, \leq, L, R, S, C, V \rangle \]

- \( L \) is a \( \leq \)-closed subset of \( P \) (“logical states”)
- \( R \subseteq P^3 \) (“pooling of information”) (Beall et al., 2012)
- \( S \subseteq P^2 \) (“sources”)

Truth and validity

- \( M, x \models \varphi \rightarrow \psi \) iff for all \( y, z \), if \( M, y \models \varphi \) and \( Rxyz \), then \( M, z \models \psi \);
- \( M, x \models \Box \varphi \) iff there is \( ySx \) such that \( M, y \models \varphi \);
- \( M \models \varphi \) iff \( M, x \models \varphi \) for all \( x \in L \).
Some Problems of (Bílková et al., 2015)

- No explicit counterpart of the possible states of the environment;
- Models only “explicit” knowledge, construed as support by a source;
Some Problems of (Bílková et al., 2015)

• No explicit counterpart of the possible states of the environment;
• Models only “explicit” knowledge, construed as support by a source;

My Contribution

• Non-classical logics for evidence-based belief;
• A combination of modal substructural logics with normal modal logics based on a functional treatment of sources;
• General completeness theorem.
WHY, GOD, WHY?
Motivations

Example 2.1

Alice notices rain beating on her windowpane. She has her radio on, and news has just come on. Interestingly enough, the forecast for today calls for ‘sunny and pleasant’ weather.
Example 2.1

Alice notices rain beating on her windowpane. She has her radio on, and news has just come on. Interestingly enough, the forecast for today calls for ‘sunny and pleasant’ weather.
Motivations

Example 2.2

Alice is listening to the radio and she does not notice the rain outside. The weather forecast for today is ‘sunny and pleasant’. The forecast makes her believe that it is not raining. She also mishears a report about an accident that occurred on a canal. She thinks it took place on the canal surrounding Groningen’s city center. **Alice now believes that Groningen’s city center is surrounded by a canal.**
Example 2.2

Alice is listening to the radio and she does not notice the rain outside. The weather forecast for today is ‘sunny and pleasant’. The forecast makes her believe that it is not raining. She also mishears a report about an accident that occurred on a canal. She thinks it took place on the canal surrounding Groningen’s city center. Alice now believes that Groningen’s city center is surrounded by a canal.
Example 2.3

Beth is taking a course in first-order logic. Let $p$ represent a sound and complete axiomatization and $q$ a rather complicated first-order theorem. $q$ is true in every possible world and it follows from Beth’s beliefs that $p$. But assume that Beth has never actually proved $q$. Her evidential situation does not support $q$, nor the fact that $q$ follows from Beth’s beliefs.
Example 2.3

Beth is taking a course in first-order logic. Let $p$ represent a sound and complete axiomatization and $q$ a rather complicated first-order theorem. $q$ is true in every possible world and it follows from Beth’s beliefs that $p$. But assume that Beth has never actually proved $q$. Her evidential situation does not support $q$, nor the fact that $q$ follows from Beth’s beliefs.
Example 2.4

Assume that Carol is conducting an experiment. Carol’s evidential situation may be seen as comprising of her background knowledge, the lab, the experiment and its results, together with Carol’s interpretation of the results. Assume that, in fact the experiment does not support a conclusion \( p \), but Carol assumes that it does. As a result, Carol believes that \( p \).
Example 2.4

Assume that Carol is conducting an experiment. Carol’s evidential situation may be seen as comprising of her background knowledge, the lab, the experiment and its results, together with Carol’s interpretation of the results. Assume that, in fact the experiment does not support a conclusion $p$, but Carol assumes that it does. As a result, Carol believes that $p$. 
The Language $\mathcal{L}_B$

Definition 3.1

\[ \varphi ::= p \mid \top \mid \bot \mid t \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \otimes \varphi \mid \varphi \rightarrow \varphi \mid \Box \varphi \mid A\varphi \]

- $\Box \varphi$ as “The agent implicitly believes that $\varphi$”;
- $A\varphi$ as “The body of evidence available to the agent supports $\varphi$”;
and
- $B\varphi \overset{\text{def}}{=} \Box \varphi \land A\varphi$ (Fagin and Halpern, 1988).
**Substructural Frames**

**Definition 3.2 (Weakly Commutative Simple Frames)**

\[ F = \langle P, \leq, L, R, C \rangle \]

- \( \langle P, \leq \rangle \) is a poset with a non-empty domain \( P \);
- \( L \subseteq P \) is \( \leq \)-closed (\( x \in L \) and \( x \leq y \) only if \( y \in L \));
- \( x \leq y \iff (\exists z \in L). R z x y \)
- \( R x y z \) and \( x' \leq x \) and \( y' \leq y \) and \( z \leq z' \implies R x' y' z' \)
- \( R x y z \implies R y x z \)
- \( C x y \) and \( x' \leq x \) and \( y' \leq y \implies C x' y' \)
- \( C x y \implies C y x \)
Substructural Models

Definition 3.3

\( M = \langle F, V \rangle \), \( V(p) \) is \( \leq \)-closed

- \( x \models p \) iff \( x \in V(p) \)
- \( x \models t \) iff \( x \in L \)
- \( x \models \neg \varphi \) iff for all \( y \), \( Cxy \) implies \( y \not\models \varphi \)
- \( x \models \varphi \otimes \psi \) iff there are \( y, z \) such that \( Ryzx \) and \( y \models \varphi \) and \( z \models \psi \)
- \( x \models \varphi \rightarrow \psi \) iff for all \( y, z \), if \( Rxyz \) and \( y \models \varphi \), then \( z \models \psi \)
- \( \varphi \) is \( L \)-valid in \( M \) \( (M \models^L \varphi) \) iff \( x \models \varphi \) for all \( x \in L \)

Commutative distributive non-associative full Lambek calculus with a simple negation \textbf{DFNLe}. 
Definition 3.4

$w \in P$ is a world in $F$ iff (for all $x, y$)

1. $C_{ww}$
2. $C_{wx}$ implies $x \leq w$
3. $R_{www}$
4. $R_{wxy}$ implies $x \leq w \leq y$
5. $R_{xyw}$ implies $x \leq w$ and $y \leq w$
Definition 3.4

\( w \in P \) is a **world in** \( F \) iff (for all \( x, y \))

1. \( Cww \)
2. \( Cwx \) implies \( x \leq w \)
3. \( Rwww \)
4. \( Rwxy \) implies \( x \leq w \leq y \)
5. \( Rxyw \) implies \( x \leq w \) and \( y \leq w \)

Lemma 3.5 (Extensionality and Logicality of Worlds)

1. **worlds** \( \subseteq \mathcal{L} \)
2. \( w \models \neg \varphi \) iff \( w \not\models \varphi \)
3. \( w \models \varphi \rightarrow \psi \) iff \( w \not\models \varphi \) or \( w \models \psi \)
4. \( w \models \varphi \otimes \psi \) iff \( w \models \varphi \) and \( w \models \psi \)
Evidence Frames

Definition 3.6

\[ \mathcal{G} = \langle F, W, E, | \cdot | \rangle \]

- \( W \subseteq P \) is a set of worlds in \( F \)
- \( Exy \) and \( x' \leq x \) and \( y \leq y' \) \( \implies Ex'y' \)
- \( Exy \) and \( Wx \) \( \implies Wy \)
- \( x \leq y \) \( \implies |x| \leq |y| \)
Definition 3.7

\( \mathcal{M} = \langle \mathcal{F}, V \rangle \), \( V(p) \) is \( \leq \)-closed

- \( x \models \Box \varphi \) iff for all \( y \), \( E_{xy} \) implies \( y \models \varphi \)
- \( x \models A \varphi \) iff \( |x| \models \varphi \)
- \( \varphi \) is valid in \( \mathcal{M} \) (\( \mathcal{M} \models \varphi \)) iff \( x \models \varphi \) for all \( x \in W \)
\[ \wedge \varphi(n) \rightarrow \psi \text{ and } \wedge B\varphi(n) \rightarrow B\psi \]

\[ w_i \text{ are “local”. } x_i \leq y \text{ iff } x_i = y, Rx_1x_1x_1, Rx_1x_2x_2 \text{ and } Rx_2x_1x_2, \text{ while } Cx_ix_j \text{ for all } i,j \in \{1,2\}. L = \{w_1,w_2,x_1\}. \]
Some Valid Schemas

1. Propositional tautologies (in $\mathcal{L}_B$) and Modus Ponens
2. $(\varphi \otimes \psi) \leftrightarrow (\varphi \land \psi)$
3. $t \leftrightarrow T$
4. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$
5. $\varphi / \Box \varphi$
6. $T \rightarrow A\top$ and $A\bot \rightarrow \bot$
7. $(\land A\varphi_{(n)}) \leftrightarrow A(\land \varphi_{(n)})$
8. $(\lor A\varphi_{(n)}) \leftrightarrow A(\lor \varphi_{(n)})$
9. If $\mathcal{M} \models^L \land \varphi_{(n)} \rightarrow \lor \psi_{(m)}$, then $\mathcal{M} \models \land A\varphi_{(n)} \rightarrow \lor A\psi_{(m)}$ and $\mathcal{M} \models \land B\varphi_{(n)} \rightarrow B \lor \psi_{(m)}$
Axiomatization of $K + DFNLe$

$l$-axioms

- $\phi \rightarrow \phi$
- $\phi \land \psi \rightarrow \phi$ and $\phi \land \psi \rightarrow \psi$
- $\phi \rightarrow \phi \lor \psi$ and $\psi \rightarrow \phi \lor \psi$
- $\phi \rightarrow T$ and $\bot \rightarrow \phi$
- $\phi \land (\psi \lor \chi) \rightarrow (\phi \land \psi) \lor (\phi \land \chi)$
- $\top \rightarrow AT$ and $A\bot \rightarrow \bot$

$r$-axioms

- propositional tautologies in $\mathcal{L}_B$
- $\Box (\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$
- $\phi \land \psi \leftrightarrow \phi \otimes \psi$
- $t \leftrightarrow \top$

$l$-rules

- $\phi, \phi \rightarrow \psi / \psi$
- $\phi \rightarrow \psi, \psi \rightarrow \chi / \phi \rightarrow \chi$
- $\chi \rightarrow \phi, \chi \rightarrow \psi / \chi \rightarrow (\phi \land \psi)$
- $\phi \rightarrow \chi, \psi \rightarrow \chi / (\phi \lor \psi) \rightarrow \chi$
- $\phi \rightarrow (\psi \rightarrow \chi) // (\psi \otimes \phi) \rightarrow \chi$
- $\phi \rightarrow (\psi \rightarrow \chi) // \psi \rightarrow (\phi \rightarrow \chi)$
- $t \rightarrow \phi // \phi$
- $\phi \rightarrow \neg \psi // \psi \rightarrow \neg \phi$
- $\land \phi(n) \rightarrow \lor \phi(m) /$
- $\land A\phi(n) \rightarrow \lor A\phi(m)$, for $n, m \geq 1$
- $\land \phi(n) \rightarrow \psi / \land \Box \phi(n) \rightarrow \Box \psi$, for $n \geq 1$

$r$-rules

- Modus Ponens
- $\phi / \Box \phi$
Proofs

are ordered couples of sequences of $\mathcal{L}_B$-formulas:

1. If $\overrightarrow{X_n} \vdash \overrightarrow{X_m}$ is a proof and $\phi$ is a $l$-axiom, then $\overrightarrow{X_n}\phi \vdash \overrightarrow{X_m}$ is a proof $(n,m \geq 0)$

2. If $\overrightarrow{X_n} \vdash \overrightarrow{X_m}$ is a proof and $\phi$ is a $r$-axiom, then $\overrightarrow{X_n} \vdash \overrightarrow{X_m}\phi$ is a proof $(n,m \geq 0)$

3. If $\overrightarrow{X_n} \vdash \overrightarrow{X_m}$ is a proof such that $\overrightarrow{X_n}$ contains $\phi_1, \ldots, \phi_n$ and $\phi_1, \ldots, \phi_n / \psi$ is a $l$-rule, then $\overrightarrow{X_n}\psi \vdash \overrightarrow{X_m}$ is a proof

4. If $\overrightarrow{X_n} \vdash \overrightarrow{X_m}$ is a proof such that $\overrightarrow{X_m}$ contains $\phi_1, \ldots, \phi_n$ and $\phi_1, \ldots, \phi_n / \psi$ is a $r$-rule, then $\overrightarrow{X_n} \vdash \overrightarrow{X_m}\psi$ is a proof

5. If $\overrightarrow{X_n}\psi \vdash \overrightarrow{X_m}$ is a proof, then $\overrightarrow{X_n}\psi \vdash \overrightarrow{X_m}\psi$ is a proof (“the jump rule”)

$\phi$ is provable ($\vdash \phi$) iff there is a proof $\overrightarrow{X_n} \vdash \overrightarrow{X_m}\phi$.

**Theorem 4.1**

$\vdash \phi$ iff $M \models \phi$ for all $M$. 
### Definability

<table>
<thead>
<tr>
<th>Schema</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>□φ → Aφ</td>
<td>Wx ∧</td>
</tr>
<tr>
<td>Aφ → □φ</td>
<td>Wx ∧ Sxy →</td>
</tr>
<tr>
<td>Aφ → φ</td>
<td>Wx →</td>
</tr>
<tr>
<td>Aφ → AAφ</td>
<td>Wx →</td>
</tr>
<tr>
<td>Aφ → ¬A¬φ</td>
<td>Wx → C</td>
</tr>
<tr>
<td>¬Aφ → A¬Aφ</td>
<td>Wx ∧ C</td>
</tr>
<tr>
<td>Aφ → □Aφ</td>
<td>Wx ∧ Sxy →</td>
</tr>
<tr>
<td>¬Aφ → □¬Aφ</td>
<td>Wx ∧ Sxy →</td>
</tr>
<tr>
<td>□φ → A□φ</td>
<td>Wx ∧ S</td>
</tr>
<tr>
<td>¬□φ → A¬□φ</td>
<td>Wx ∧ Sxy ∧ C</td>
</tr>
<tr>
<td>□φ → φ</td>
<td>Wx → Sxx</td>
</tr>
<tr>
<td>□φ → □□φ</td>
<td>Wx ∧ Sxy ∧ Syz → Sxz</td>
</tr>
<tr>
<td>¬□φ → □¬□φ</td>
<td>Wx ∧ Sxy ∧ Sxz → Syz</td>
</tr>
</tbody>
</table>
More Details On

• definability
• strong completeness of extensions
• non-classical relational belief revision
• outline of informational dynamics

Can Be Found In

• “Substructural Epistemic Logics”, to appear in the *Journal of Applied Non-Classical Logics*,
• “Epistemic Extensions of Modal Distributive Substructural Logics”, to appear in the *Journal of Logic and Computation*,
• “Information, Awareness and Substructural Logics”, in *Proc. of WoLLIC 2013*.  

Future Work

- A fuller development of substructural models of information dynamics and action;
- Group-epistemic modalities in the substructural setting;
- Combinations with related approaches;
- Applications in Phil, CS etc.
THANK YOU!
References I


