Reasoning about Joint Action and Coalitional Ability in K_n with Intersection

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Abstract. In this paper we point out that standard PDL-like logics with intersection are useful for reasoning about game structures. In particular, they can express coalitional ability operators known from coalition logic and ATL. An advantage of standard, normal, modal logics is a well understood theoretical foundation and the availability of tools for automated verification and reasoning. We study a minimal variant, multi-modal K with intersection of modalities, interpreted over models corresponding to game structures. There is a restriction: we consider only game structures that are injective. We give a complete axiomatisation of the corresponding models, as well as a characterisation of key complexity problems. We also prove a representation theorem identifying the effectivity functions corresponding to injective games.

1 Introduction

Logics interpreted in *game structures*, enabling automated reasoning and verification of game theoretic properties of multi-agent systems, have received considerable interest in recent years. For computational reasons most such logics are modal logics [17]. One of the most popular approaches is reasoning about *coalitional ability*. Examples of logics in this category include *Coalition Logic* (CL) [14] and *Alternating-time Temporal Logic* (ATL) [2], and many extensions of these, which are interpreted in *game structures*. These logics have *coalition operators* of the form $[C]$ where C is a set of agents (a *coalition*), and $[C]\phi$ means that C can make ϕ true by choosing some joint action. On the other hand, van Benthem [15, 16] has pointed out that standard propositional dynamic logic (PDL) [9] is natural for reasoning about games. An advantage of using PDL-like languages is that they are theoretically well understood, with a range of mathematical and computational tools available.

In this paper we point out that standard PDL-like logics with *intersection* are useful for reasoning about games. In particular, they can express coalition operators. We study a minimal variant, multi-modal K with intersection of modalities ($K_n^{\cap}$), interpreted in game structures, and define an embedding of CL into $K_n^{\cap}$. $K_n^{\cap}$ is a fragment of Boolean Modal Logic [6] which has been extensively studied (and implemented) as a variant of propositional dynamic logic with intersection and also by researchers in description logic (see for example [12]).

Although other logics that are normal modal logics and/or have PDL-type operators and can express coalition operators have been studied recently [4, 11], these typically have non-standard syntactic operators and/or non-standard semantics (see Section 7). The focus in the current paper is on reasoning about joint action in game structures using a minimal language with the intersection operator.

The main contributions of the paper are, in our view, threefold. First, we give an interpretation of $K_n^{\cap}$ over models corresponding to game structures and give a sound and complete axiomatisation and characterisations of key complexity problems. There is a snag: this restricted model class does in fact not correspond to *all* game structures, only to *injective* game structures [7], i.e., game structures where two different strategy profiles never lead to the same outcome state. However, we argue that this is a minor limitation:

- Coalition Logic cannot discern between injective and non-injective game structures.
- Injectivity is a very common assumption in game theory. Indeed, the notion of outcome states is often (as in the standard textbook [13]) dispensed with altogether, and preferences defined directly over strategy profiles – implicitly defining an injective game.

Second, we show that the coalition logic operator can be expressed in $K_n^{\cap}$. Third, we prove a variant of Pauly's *representation theorem* [14, 8] for injective games: we characterise the class of effectivity functions that α-correspond to injective games.

We argue that the logic we study is interesting for several reasons. As a normal modal logic it has a well understood theoretical foundation, and it is well supported by tools for automated verification and reasoning as a fragment of standard computer science logics such as PDL. For example, model checking can be done using standard model checkers for PDL (with intersection). In contrast, CL is a non-normal modal logic, with only special purpose tool support (mainly tools developed for ATL) available. Our logic can also be seen as providing an additional set of model-checking and theorem-proving tools for CL. Finally, from a theoretical viewpoint, the logic establishes new connections between coalition logic and normal modal logics.

The paper is organised as follows. We start with introducing CL and $K_n^{\cap}$. We then, in Section 3, discuss the restriction of CL to injective game structures and prove the representation theorem. In Section 4 we define the interpretation of $K_n^{\cap}$ in game structures, together with translations from CL, before axiomatisation and complexity are discussed in Sections 5 and 6. We discuss related work and conclude in Section 7. Some of the proofs are found in a technical appendix.

2 Background

2.1 Coalition Logic

We give a brief introduction to Coalition Logic (CL) [14]. Let $N = \{1, \ldots, g\}$ be a finite set of agents, and Θ a set of propositional variables. The language of $\mathcal{L}_{CL}(N, \Theta)$

of CL [14] is defined as follows:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid [C]\phi$$

where $p \in \Theta$ and $C \subseteq N$. Derived propositional connectives are defined as usual. We write $\overline{C}$ for $N \setminus C$ and sometimes abuse notation and write a singleton $\{i\}$ as i. The language can be interpreted over *concurrent game structures* (CGSs) [2]. A CGS over N and Θ is a tuple $M = \langle S, V, Act, d, \delta \rangle$ where

- S is a set of *states*;
- V is a *valuation function*, assigning a set $V(q) \subseteq \Theta$ to each state $q \in S$;
- Act is a set of *actions*;
- For each $i \in N$ and $q \in S$, $d_i(q) \subseteq Act$ is a non-empty set of actions available to agent i in q. $D(q) = d_1(q) \times \cdots \times d_g(q)$ is the set of *full joint actions* in q. When $C \subseteq N$, $D_C(q) = \times_{i \in C} d_i(q)$ is the set of C-*actions* in q. If $\mathbf{a} \in D_C(q)$, a_i ($i \in C$) denotes the action taken from $d_i(q)$.
- δ is a *transition function*, mapping each state $q \in S$ and full joint action $\mathbf{a} \in D(q)$ to a state $\delta(q, \mathbf{a}) \in S$.

Let $\mathcal{M}_{cgs}(N, \Theta, Act)$ be the class of CGSs over N and Θ having Act as the set of actions.

A CGS can be seen as a state-transition system where the edges are labelled with joint actions, but also, alternatively, as an assignment of a *strategic game form* to each state. A *strategic game form* is a tuple $\langle N, \{\Sigma_i : i \in N\}, S, o\rangle$ where Σ_i is the *strategies* for $i \in N$, and $o : \times_{j \in N} \Sigma_j \to S$ is the *outcome function*. We write Σ_C for $\times_{i \in C} \Sigma_i$. When $\sigma_C \in \Sigma_C$, we use $(\sigma_C)_i$ to denote the strategy for i. A CGS can be seen as an assignment of a game $G(s) = \langle N, \{\Sigma_i^s : i \in N\}, S, o^s\rangle$ to each state $s \in S$, where $\Sigma_i^s = d_i(s)$ and $o^s(\mathbf{a}) = \delta(s, \mathbf{a})$.

Intuitively, the CL expression $[C]\phi$ means that a group of agents (*coalition*) C is *effective* for formula ϕ, i.e., that they can ensure that ϕ holds in the next state no matter what the other agents do. Formally, a formula ϕ is interpreted in a state s of CGS M as follows:

$M, s \models p \Leftrightarrow p \in V(s)$
$M, s \models \neg\phi \Leftrightarrow M, s \not\models \phi$
$M, s \models (\phi_1 \wedge \phi_2) \Leftrightarrow (M, s \models \phi_1$ and $M, s \models \phi_2)$
$M, s \models [C]\psi \Leftrightarrow$
$\exists \mathbf{a}_C \in D_C(s) \forall \mathbf{a}_{\overline{C}} \in D_{\overline{C}}(s),\ M, \delta(s, (a_1, \ldots, a_g)) \models \psi$

As effectivity is the only property of games relevant for the interpretation of coalition logic, we can in fact abstract away all other aspects of game structures. An *effectivity function* [14] over N and a set of states S is a function E that maps any coalition $C \subseteq N$ to a set of sets of states $E(C) \subseteq 2^S$. Given a strategic game form G, the *(α-)effectivity function* E_G of G is defined as follows:

$$X \in E_G(C) \text{ iff } \exists \sigma_C \in \Sigma_C \forall \sigma_{\overline{C}} \in \Sigma_{\overline{C}}, o(\sigma_C, \sigma_{\overline{C}}) \in X.$$

Which effectivity functions are the effectivity functions of strategic game forms? In [14] it is claimed that an effectivity function E is the α-effectivity function of a strategic game form iff E is *playable*:

1. $X \in E(C) \& X \subseteq Y \& Y \subseteq S \Rightarrow Y \in E(C)$ (*outcome monotonicity*);
2. $S \setminus X \notin E(\emptyset) \Rightarrow X \in E(N)$ (N*-maximality*);
3. $\emptyset \notin E(C)$ (*Liveness*);
4. $S \in E(C)$ (*Safety*);
5. $C \cap D = \emptyset \& X \in E(C) \& Y \in E(D) \Rightarrow X \cap Y \in E(C \cup D)$ (*superadditivity*).

However, it has recently been showed [8] that this claim is in fact not correct: there are playable effectivity functions over infinite sets, which are not α-effectivity functions of any strategic game forms. In [8] the result is also corrected: an effectivity function E is said to be *truly playable* iff it is playable and $E(\emptyset)$ has a *complete nonmonotonic core*. The *nonmonotonic core* $E^{nc}(C)$ of $E(C)$, for $C \subseteq N$, is defined as follows: $E^{nc}(C) = \{X \in E(C) : \neg\exists Y(Y \in E(C) \text{ and } Y \subset X)\}$. $E^{nc}(C)$ is *complete* if for every $X \in E(C)$ there exists $Y \in E^{nc}(C)$ such that $Y \subseteq X$. The corrected representation theorem [8] shows that E is the α-effectivity function of a strategic game form iff E is truly playable. A *coalition model* is a tuple $\mathcal{M} = \langle S, E, V\rangle$ where E gives a truly playable effectivity function $E(s)$ for each state $s \in S$, and V is a valuation function. The coalition logic language can alternatively be interpreted in a coalition model, as follows:

$$\mathcal{M}, s \models [C]\phi \text{ iff } \phi^{\mathcal{M}} \in E(s)(C)$$

where $\phi^{\mathcal{M}} = \{t \in S : \mathcal{M}, t \models \phi\}$. It is easy to see that the two semantics coincide: $M, s \models \phi$ iff $\mathcal{M}_\alpha, s \models \phi$ for all ϕ, where $M = (S, V, Act, d, \delta)$ and $\mathcal{M}_\alpha = (S, E_\alpha, V)$ and $E_\alpha(s) = E_{G(s)}$.

2.2 Multi-modal K with Intersection of Modalities

Given a finite set of atomic modalities Π^0 of cardinality n and a countably infinite set of propositional atoms Θ, the formulae $\phi \in \mathcal{L}_K^{\cap}(\Pi^0, \Theta)$ and modalities $\pi \in \Pi$ of multi-modal K with intersection of modalities ($K_n^{\cap}$) are defined as follows:

$$\phi ::= p \in \Theta \mid \neg\phi \mid \phi \wedge \phi \mid [\pi]\phi \qquad \pi ::= a \mid \pi \cap \pi$$

where $a \in \Pi^0$. As usual, $\langle\pi\rangle\phi$ is defined as $\neg[\pi]\neg\phi$, and derived propositional connectives are defined as usual.

A (Kripke) model for the language $\mathcal{L}_K^{\cap}(\Pi^0, \Theta)$ is a tuple $M = \langle S, V, \{R_\pi : \pi \in \Pi\}\rangle$ where

- S is a set of *states*;
- $V : \Theta \to 2^S$ is a *valuation function*;
- For each $\pi \in \Pi$, $R_\pi \subseteq S \times S$
- $R_{\pi_1 \cap \pi_2} = R_{\pi_1} \cap R_{\pi_2}$ (INT)

The interpretation of a formula in a state of a model is defined as follows (other clauses as usual):

$$M, s \models [\pi]\phi \text{ iff } \forall(s, s') \in R_\pi, M, s' \models \phi$$

3 Injective Games

The idea of interpreting $K_n^\cap$ in game structures is very simple: interpret a full joint action $\langle a_1, \ldots, a_g\rangle$ in a CGS as a set of g different ("atomic") edges, one for each agent-action combination. This gives us a $K_n^\cap$ model, where the atomic modalities are agent-action pairs. Full joint actions can be recovered by taking the intersection between the relations for two or more atomic modalities for different agents.

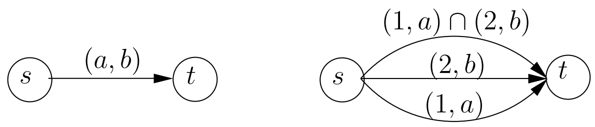

Fig. 1. CGS (left) and $K_n^\cap$ model (right).

For example, consider the CGS on the left in Figure 1. The corresponding $K_n^\cap$ model is shown to the right. Coalition operators can now be captured approximately as follows. If we for example want to say that there exists a joint action by agents 1 and 2, all executions of which result in the outcome p, in such a $K_n^\cap$ model, we can say something like $\bigvee_{a,b\in Act}[(1, a) \cap (2, b)]p$ (in addition we must check that the actions a and b are actually available in the current state, but that is straightforward).

However, there is a problem with this approach. Consider the CGS to the left in Figure 2. The approach above gives us the $K_n^\cap$ model to the right in the figure. This model has four atomic transitions from s to t: two labelled $(1, a_1)$ and $(2, b_1)$ which correspond to the full joint action (a_1, b_1), and two labelled $(1, a_2)$ and $(2, b_2)$ which correspond to (a_2, b_2). The full joint actions can be recovered by intersection of the atomic transitions, but the problem is that too much is "recovered" in this way: we get the spurious transitions (a_1, b_2) and (a_2, b_1) which are not present between these states in the original model.

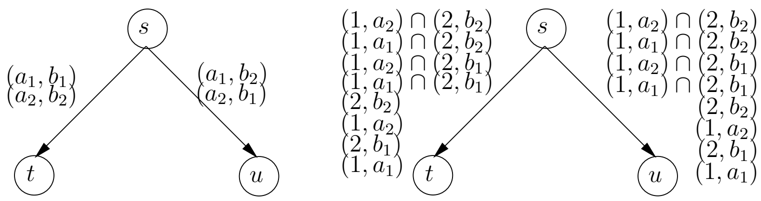

Fig. 2. CGS (left) and $K_n^\cap$ model (right). An arrow with more than one label represents a transition for each label.

The problem is that by decomposing a full joint action into individual actions, we lose information about which *combinations* of actions relate the two states. That information is crucial, e.g., for the interpretation of coalition operators. We call a CGS without two or more different full joint actions between the same two states, i.e., with an injective δ, *injective* (following [7]). Injective CGSs do not suffer this problem.

It is relatively straightforward to see that any CGS is equivalent, in the sense of satisfying the same coalition logic formulae, to an injective CGS: take the *tree-unfolding* of the model (see [1] for relevant definitions of tree-unfoldings, bisimulations, and invariance under bisimulation, for the ATL language which contains the CL language). The tree-unfolding, however, is a model with infinitely many states, which may be a problem, e.g., if we want to do model checking. Fortunately it turns out that every finite CGS (finite state space) is CL-equivalent to a finite, and even "small", injective CGS. The following theorem follows immediately from a result by Goranko [7, Proposition 12] (with some minor changes and amendments).

Theorem 1. *For every* CGS $M = \langle S, V, Act, d, \delta\rangle$ *there is an injective* CGS M' *with states* S' *such that* $S \subseteq S'$ *and for all* CL *formulae* ϕ *and states* $s \in S$*,* $M, s \models \phi$ *iff* $M', s \models \phi$*. Moreover, if* M *is finite, then* $|S'| \leq |S| + |\delta|$*.*

This makes it possible to translate a CGS into a $K_n^{\cap}$ model, such that we can recover a CGS that is CL-equivalent to the former from the latter. Before formally defining the translation in Section 4 we take a closer look at injective games; the reader mainly interested in the translation can skip directly to that section.

3.1 Effectivity Functions and Representation

Although coalition logic cannot discern between injective games and non-injective games, there is still another pertinent question if we want to restrict our attention to injective games: the question of representation using effectivity functions. Which truly playable effectivity functions correspond to injective games? The answer is not necessarily "all": this is similar to the relationship between playable and truly playable effectivity functions [8]; the latter is a proper subset of the former while coalition logic cannot discern between the two. Indeed, not all truly playable effectivity functions are the α-effectivity functions of injective games:

Example 1. Let $N = \{1, 2\}$ and E be defined as follows:

$$E(\emptyset) = E(1) = E(2) = \{\{s, t\}\} \quad E(\{1, 2\}) = \{\{s\}, \{t\}, \{s, t\}\}$$

(where $s \neq t$). The reader can check that E is truly playable. However, it is not the α-effectivity function of an injective game. For assume it is, that $E = E_G$ for some G.. Because of Safety, the game has exactly two states s and t. Together with the fact that $\{s\}, \{t\} \in E(\{1, 2\})$, that means that one of the agents must have exactly one strategy, and the other exactly two: all other combinations violates injectivity of a two-state game. Wlog. assume that $\Sigma_1 = \{\sigma_1\}$ and $\Sigma_2 = \{\sigma_1', \sigma_2'\}$. $\{s\} \in E_G(\{1, 2\})$ implies that $o(\sigma_1, \sigma_1') = s$ or $o(\sigma_1, \sigma_2') = s$; wlog. assume the former. Then $\{t\} \in E_G(\{1, 2\})$ implies that $o(\sigma_1, \sigma_2') = t$. But that means that $\{s\}, \{t\} \in E_G(2)$, which is not the case.

We now state and prove a representation theorem (Theorem 2) for injective games. An effectivity function is *injectively playable* iff it is playable and (for all C, i, j, X, Y):

$$E(C) \text{ has a complete nonmonotonic core} \tag{1}$$

$$E^{nc}(C) = \{\bigcap_{i \in C} X_i : X_i \in E^{nc}(i)\} \quad C \neq \emptyset \tag{2}$$

$$X, Y \in E^{nc}(i) \text{ and } X \neq Y \Rightarrow X \cap Y = \emptyset \tag{3}$$

$$X \in E^{nc}(j) \text{ and } x \in X \Rightarrow \exists Y \in E^{nc}(i), x \in Y \tag{4}$$

Injective playability extends the true playability requirement of a complete nonmonotonic core [8] from the empty coalition to *all* coalitions. As a result, $E(C)$ is completely determined by its nonmonotonic core (stated formally in the following Lemma). In addition, there are some restrictions on the structure of the core. None of the additional properties of injective playability, (1)–(4), hold in general for truly playable effectivity functions (in particular, true playability does not imply complete nonmonotonic core for non-empty coalitions).

Lemma 1. *Let E be an outcome monotonic effectivity function. $E(C)$ has a complete nonmonotonic core iff $E(C) = \{X : Y \subseteq X, Y \in E^{nc}(C)\}$.*

Lemma 2. *If E is injectively playable, then:*

$$(\forall_{i \in N} X_i \in E^{nc}(i)) \Rightarrow |\bigcap_{i \in N} X_i| = 1 \tag{5}$$

$$E^{nc}(\emptyset) = \{Z\} \text{ where } Z = \bigcup E^{nc}(N) \tag{6}$$

Before proving the main result (Th. 2) we need the following lemma.

Lemma 3. *If E_G is the α-effectivity function of an injective game $G = (N, \{\Sigma_i : i \in N\}, o, S)$, then for all $C \subseteq N$:*

$$E_G^{nc}(C) = \{ \{o(\sigma_C, \sigma_{\overline{C}}) : \sigma_{\overline{C}} \in \Sigma_{\overline{C}}\} : \sigma_C \in \Sigma_C\}$$

Theorem 2. *An effectivity function E is injectively playable iff it is the α-effectivity function of some injective game G.*

Proofs are found in the appendix.

4 Multi-modal K with Intersection for Games

We now show how $K_n^\cap$ formulae can be interpreted in game structures, by identifying a class of $K_n^\cap$ models corresponding to (injective) game structures.

4.1 Joint Action Models

Let Act be a finite set of actions and N a set of g agents. Define a set of atomic modalities as follows:

$$\Pi^0_{ActN} = N \times Act$$

Intuitively, an atomic modality is a pair (i, a) intended to represent an action and the agent that executes that action. We will call an atomic modality in Π^0_{ActN} an *individual action*, and a composite modality $\pi = \pi_1 \cap \pi_2$ a *joint action*. Since the intersection operation is associative, we can write any joint action π as an intersection of a set of individual actions: $\pi = (i_1, a_1) \cap \cdots \cap (i_k, a_k)$. Joint actions of the form $(1, a_1) \cap \ldots \cap (g, a_g)$ with one individual action for *every* agent in N will be called *complete (joint) actions*.

The following are some properties of $K^{\cap}_n$ models over Π^0_{ActN} that will be of particular interest. We say that an action $a \in Act$ is *enabled* for agent i in a state s iff there is a state s' such that $(s, s') \in R_{(i,a)}$.

Seriality (SER) For any state s and agent i, at least one action is enabled in s for i.
Independent Choice (IC) For any state s, agents $C = \{i_1, \ldots, i_k\}$ and actions $a_1, \ldots, a_k \in Act$, if for every j a_j is enabled for i_j in s, then there is a state s' such that $(s, s') \in R_{(i_1,a_1)\cap\cdots\cap(i_k,a_k)}$.
Deterministic Joint Actions (DJA) For any complete joint action α and states s, s_1, s_2, $(s, s_1), (s, s_2) \in R_\alpha$ implies that $s_1 = s_2$.
Unique Joint Actions (UJA) For any complete joint actions α and β and states s, t, if $(s, t) \in R_\alpha \cap R_\beta$ then $\alpha = \beta$.

A $K^{\cap}_n$ model over Π^0_{ActN} (where Act is finite) is a *joint action model* if it satisfies the properties SER, IC, DJA, and UJA.

Given (finite) Act and N, we now translate any CGS over Act and N into a joint action model.

Definition 1. *Given an injective* CGS $M = \langle S, V, Act, d, \delta \rangle \in \mathcal{M}_{cgs}(N, \Theta, Act)$ *where Act is finite, the corresponding joint action model* $\hat{M} = \langle S, V, \{R_\pi : \pi \in \Pi\} \rangle$ *over* Θ *and* Π^0_{ActN} *is defined as follows:*

- $R_{(i,a)} = \{(s, s') : \exists \mathbf{a} \in D(s)$ *s.t.* $a_i = a$ *and* $s' = \delta(s, \mathbf{a})\}$, *when* $(i, a) \in \Pi^0_{ActN}$
- $R_{\pi_1 \cap \pi_2} = R_{\pi_1} \cap R_{\pi_2}$

We use the following property to show that M^N is indeed a joint action model.

Lemma 4. *Let* $M = \langle S, V, Act, d, \delta \rangle$ *be injective and* $\hat{M} = \langle S, V, \{R_\pi : \pi \in \Pi\} \rangle$ *be the corresponding joint action model, and let* $\pi = (i_1, a_1) \cap \ldots \cap (i_k, a_k)$. *Then* $(s, t) \in R_\pi$ *iff there is an* $\mathbf{a}' \in D(s)$ *such that* $a'_{i_j} = a_j$ *for all* $1 \le j \le k$ *and* $\delta(s, \mathbf{a}') = t$.

Proof. $(s, t) \in R_\pi$ iff there are $\mathbf{a}^1, \ldots, \mathbf{a}^k \in D(s)$ such that for all $1 \le j \le k$: $a^j_{i_j} = a_j$ and $t = \delta(s, \mathbf{a}^j)$. Since M is injective, it must be the case that $\mathbf{a}^1 = \cdots = \mathbf{a}^k$.

Lemma 5. $\hat{M}$ *is a joint action model.*

Proof. $\hat{M}$ is a proper $\mathcal{L}_K^{\cap}(\Pi^0_{ActN}, \Theta)$ model by definition. SER holds because $d_i(s)$ is always non-empty. IC holds because $D(s)$ is defined as the cross product of $d_i(s)$ for all i. For DJA, $(s, s_1), (s, s_2) \in R_\alpha$ implies that $\delta(s, \mathbf{a}) = s_1 = s_2$ by Lemma 4. For UJA, $(s, t) \in R_\alpha \cap R_\beta$ implies that $\delta(s, \mathbf{a}) = t$ and $\delta(s, \mathbf{b}) = t$ by Lemma 4, which by injectivity implies that $\mathbf{a} = \mathbf{b}$ which again implies that $\alpha = \beta$.

Thus, we get a direct interpretation of formulae $\phi \in \mathcal{L}_K^{\cap}(\Pi^0_{ActN}, \Theta)$ in injective CGSs $M \in \mathcal{M}_{cgs}(N, \Theta, Act)$: $M, s \models \phi$ iff $\hat{M}, s \models \phi$.

4.2 Embedding of CL

Given a coalition logic formula $\phi \in \mathcal{L}_{CL}(N, \Theta)$ and a finite set of actions Act, we define the translation $\phi' \in \mathcal{L}_K^{\cap}(\Pi^0_{ActN}, \Theta)$ as follows:

$$\begin{array}{ll} p' & \equiv p \\ (\neg\phi)' & \equiv \neg\phi' \\ (\phi_1 \wedge \phi_2)' & \equiv \phi_1' \wedge \phi_2' \\ ([\{i_1, \ldots, i_k\}]\phi)' & \equiv \bigvee_{a_1, \ldots, a_k \in Act} \bigwedge_{1 \leq j \leq k} \langle (i_j, a_j) \rangle \top \\ & \wedge [(i_1, a_1) \cap \ldots \cap (i_k, a_k)]\phi' \end{array}$$

The translation of the CL formula $[C]\phi$ says that there is an action for each agent in C, such that (i) the actions are enabled and (ii) for all possible states resulting from executing the actions at the same time (the translation of) ϕ holds. The translation assumes that the set of possible actions Act is given and that it is finite. In model checking this can be obtained directly from the model.

Theorem 3. *Let Θ, N and Act (finite) be fixed. For any $\phi \in \mathcal{L}_{CL}(N, \Theta)$ and any injective* CGS *$M \in \mathcal{M}_{cgs}(N, \Theta, Act)$ and any state s in M:*

$$M, s \models \phi \text{ iff } \hat{M}, s \models \phi'$$

Proof. Let Θ, N and Act (finite) be given, $\phi \in \mathcal{L}_{CL}(N, \Theta)$ and $M \in \mathcal{M}_{cgs}(N, \Theta, Act)$ be injective and s in M. The proof is by induction on the structure of ϕ. The only interesting case is when $\phi = [C]\psi$ where $C = \{i_1, \ldots, i_k\}$. By Lemma 4 $\hat{M}, s \models ([C]\psi)'$ iff there is a $\langle a_1, \ldots, a_k \rangle \in D_C(s)$ such that for all $(s, t) \in R_{(i_1, a_1) \cap \ldots (i_k, a_k)}$ $\hat{M}, t \models \psi'$ iff there is a $\langle a_1, \ldots, a_k \rangle \in D_C(s)$ such that for all $\mathbf{a}' \in D(s)$ with $\mathbf{a}'_{i_j} = a_j$ $(1 \leq j \leq k)$ $\hat{M}, \delta(s, \mathbf{a}') \models \psi'$ iff by the induction hypothesis there is a $\langle a_1, \ldots, a_k \rangle \in D_C(s)$ such that for all $\mathbf{a}' \in D(s)$ with $\mathbf{a}'_{i_j} = a_j$ $(1 \leq j \leq k)$ $M, \delta(s, \mathbf{a}') \models \psi$ iff $M, s \models [\{i_1, \ldots, i_k\}]\psi$.

The following follows immediately from Theorems 3 and 1.

Corollary 1. *Let Θ, N and Act (finite) be fixed. For any $\phi \in \mathcal{L}_{CL}(N, \Theta)$ and any* CGS *$M \in \mathcal{M}_{cgs}(N, \Theta, Act)$ there is a joint action model $\overline{M}$ over Π^0_{ActN} such that for any state s in M:*

$$M, s \models \phi \text{ iff } \overline{M}, s \models \phi'$$

5 Axiomatisation of joint action models

We now give an axiomatisation for the language $\mathcal{L}_K^{\cap}(\Pi^0_{ActN}, \Theta)$ and prove that it is sound and complete with respect to the class of all joint action models over Π^0_{ActN} and Θ. The axiom system $\mathcal{S}$ is defined as follows:

K $[\pi](\phi \rightarrow \psi) \rightarrow ([\pi]\phi \rightarrow [\pi]\psi)$
A1 $\bigvee_{a \in Act} \langle (i,a) \rangle \top$
A2 $\langle \pi \rangle \phi \rightarrow \bigvee_{a \in Act} \langle \pi \cap (i,a) \rangle \phi$
A3 $\bigwedge_{i \in N} \langle (i,a_i) \rangle \top \rightarrow \langle (1,a_1) \cap \ldots \cap (g,a_g) \rangle \top$
A4 $\langle (1,a_1) \cap \cdots \cap (g,a_g) \rangle \phi \rightarrow [(1,a_1) \cap \ldots \cap (g,a_g)]\phi$
A5 $[\pi]\phi \rightarrow [\pi \cap \pi']\phi$
A6 $[(i,a) \cap (i,b)]\bot$ when $a \neq b$
MP From $\phi \rightarrow \psi$ and ϕ infer ψ
G From ϕ infer $[\pi]\phi$

K, **MP** and **G** says that the $[\pi]$ modalities are normal. **A1** says that at least one action is enabled for each agent in every state, **A2** says that if there is a joint action for some agents that can ensure ϕ, then any agent can do some action at the same time such that ϕ is still ensured, **A3** says that all joint actions composed of enabled individual actions are enabled, **A4** says that complete joint actions are deterministic, **A5** is the standard axiom for intersection, and **A6** says that an agent cannot do two actions simultaneously.

Theorem 4. *The axiom system $\mathcal{S}$ is sound and complete wrt. all joint action models.*

Proof. Soundness is straightforward. To prove completeness, we introduce some conventions and auxiliary concepts and show some intermediate properties. When $\pi = (i_1,a_1) \cap \cdots \cap (i_k,a_k)$ and $\pi' = (j_1,b_1) \cap \cdots \cap (j_l,b_l)$ are joint actions, we write $\pi \leq \pi'$ to denote that for every $1 \leq u \leq k$ there is a $1 \leq v \leq l$ such that $i_u = j_v$ and $a_u = b_v$. Recall that a complete joint action is an expression of the form $(1,a_1) \cap \cdots \cap (g,a_g)$ where $a_i \in Act$, giving one action for each agent in the system. Let JA denote the (finite) set of complete joint actions. We use $\alpha, \beta, \ldots$ to denote complete joint actions.

We will make use of a notion of *pseudomodels*, which only have transition relations for complete joint actions. Formally, a pseudomodel is a tuple $(S, \{R_\alpha : \alpha \in JA\}, V)$ where: S is a set of states, $R_\alpha \subseteq S \times S$ for each $\alpha \in JA$ and $V : \Theta \rightarrow 2^S$.

First, we construct the *canonical* pseudomodel $M^c = (S^c, \{R^c_\alpha : \alpha \in JA\}, V^c)$ as follows:

- S^c is the set of $\mathcal{L}_K^{\cap}(\Pi^0_{ActN}, \Theta)$-maximal $\mathcal{S}$-consistent sets Γ
- $V^c(p) = \{\Gamma : p \in \Gamma\}$
- $R^c_\alpha \Gamma \Gamma'$ iff for any ψ, if $\psi \in \Gamma'$ then $\langle \alpha \rangle \psi \in \Gamma$

Lemma 6 (Existence Lemma). *For any $s \in S^c$, if $\langle \alpha \rangle \gamma \in s$ for some $\alpha \in JA$, then there is an s' such that $(s,s') \in R^c_\alpha$ and $\gamma \in s'$.*

The proof of the existence lemma is as in standard normal modal logic.

Now let ϕ be a consistent formula; we show that it is satisfied in some joint action model. Let $x \in S^c$ be such that $\phi \in x$. We now take the *unravelling* of the canonical pseudomodel around x. The pseudomodel $M^x = (S^x, \{R^x_\alpha : \alpha \in JA\}, V^x)$ is defined as follows:

– S^x is the set of all finite sequences $(s_0, R^c_{\alpha_0}, s_1 \ldots, R^c_{\alpha_{k-1}}, s_k)$ such that $s_0, \ldots, s_k \in S^c$, $s_0 = x$, and $(s_i, s_{i+1}) \in R^c_{\alpha_i}$ for all $0 \leq i \leq k-1$.
– $(s, u) \in R^x_\alpha$ iff $s = (x, R^c_{\alpha_0}, s_1 \ldots, R^c_{\alpha_{k-1}}, s_k)$ and $u = (x, R^c_{\alpha_0}, s_1 \ldots, R^c_{\alpha_{k-1}}, s_k, R^c_\alpha, s_{k+1})$ for some $s_{k+1} \in S^c$.
– $V^x(p) = \{(x, R^c_{\alpha_0}, s_1 \ldots, R^c_{\alpha_{k-1}}, s_k) : s_k \in V^c(p)\}$

We now transform the pseudomodel M^x into a (proper) model $M = (S^x, \{R_\pi : \pi \in \Pi\}, V^x)$ as follows:

– $R_{(i,a)} = \bigcup_{\alpha \in JA, (i,a) \leq \alpha} R^x_\alpha$
– $R_{\pi_1 \cap \pi_2} = R_{\pi_1} \cap R_{\pi_2}$

Lemma 7. *$R_\alpha = R^x_\alpha$, for any complete joint action $\alpha \in JA$.*

Proof. Let $\alpha = (1, a_1) \cap \cdots \cap (g, a_g)$, and observe that

$$R_\alpha = (\bigcup_{(1,a_1) \leq \alpha'} R^x_{\alpha'}) \cap \cdots \cap (\bigcup_{(g,a_g) \leq \alpha'} R^x_{\alpha'}).$$

First, assume that $(s, t) \in R^x_\alpha$. It follows immediately that $(s, t) \in \bigcup_{(i,a_i) \subseteq \alpha'} R^x_{\alpha'}$ for all i by taking $\alpha' = \alpha$. Second, assume that $(s, t) \in R_\alpha$, i.e., that there are $\alpha_1, \ldots, \alpha_g$ such that $(i, a_i) \leq \alpha_i$ for all i and $(s, t) \in R^x_{\alpha_1} \cap \cdots \cap R^x_{\alpha_g}$. By construction of M^x, that implies that $\alpha_1 = \cdots = \alpha_g$. So $\alpha_1 = \cdots = \alpha_g = \alpha$.

Let $last(s)$ denote the last element $s_k \in S^c$ in a sequence $s \in S^x$.

Lemma 8 (Truth Lemma). *For any s and ψ, $M, s \models \psi$ iff $\psi \in last(s)$.*

Proof. The proof is by induction on the structure of ψ. The cases for propositional atoms and Boolean connectives are straightforward, so let $\psi = \langle\pi\rangle\gamma$. Let $\pi = (i_1, a_1) \cap \cdots \cap (i_k, a_k)$ for some $1 \leq k \leq n$.

First assume that $i_l = i_m = i$ and $a_l \neq a_m$ for some $l \neq m$. Observe that $R_\pi = (\bigcup_{(i_1,a_1) \leq \alpha'} R^x_{\alpha'}) \cap \cdots \cap (\bigcup_{(i_k,a_k) \leq \alpha'} R^x_{\alpha'})$. Let α_1 and α_2 be arbitrary complete joint actions such that $(i_l, a_l) \leq \alpha_1$ and $(i_m, a_m) \leq \alpha_2$. $\alpha_1 \neq \alpha_2$ since $i_l = i_m$ and $a_l \neq a_m$. By construction of M^x, $R^x_{\alpha_1} \cap R^x_{\alpha_2} = \emptyset$. Thus, $R_\pi = \emptyset$ (since α_1 and α_2 were arbitrary), and $M, s \not\models \langle\pi\rangle\gamma$ for any γ. On the other hand, by **A6** $\neg\langle(i, a_l) \cap (i, a_m)\rangle\gamma \in last(s)$ for any γ, and it follows by **A5** that $\neg\langle\pi\rangle\gamma \in last(s)$. Thus, the Lemma holds in this case and we henceforth assume that $a_l = a_m$ whenever $i_l = i_m$.

For the implication to the right, assume that $M, s \models \langle\pi\rangle\gamma$, i.e., that $M, t \models \gamma$ for some $(s, t) \in R_\pi = \bigcup_{(i_1,a_1) \leq \alpha'} R^x_{\alpha'} \cap \cdots \cap \bigcup_{(i_k,a_k) \leq \alpha'} R^x_{\alpha'}$. Thus, for each $1 \leq j \leq k$, there is an α_j such that $(s, t) \in R^x_{\alpha_j}$ and $(i_j, a_j) \in \alpha_j$. By construction of M^x, $\alpha_1 = \cdots = \alpha_k = \alpha$ (the state t has only one “incoming” transition). Thus, $(s, t) \in R^x_\alpha$ with $\pi \leq \alpha$. Assume that $s = (x, R^c_{\alpha_0}, s_1, \ldots, R^c_{\alpha_{k-1}}, s_k)$. Then $t = (x, R^c_{\alpha_0}, s_1, \ldots, R^c_{\alpha_{k-1}}, s_k, R^c_\alpha, s_{k+1})$ for some s_{k+1} such that $(s_k, s_{k+1}) \in R^c_\alpha$. By the induction hypothesis $\gamma \in last(t) = s_{k+1}$. By construction of R^c_α, $\langle\alpha\rangle\gamma \in s_k = last(s)$. By (repeated applications of) **A5**, $\langle\pi\rangle\gamma \in s_k = last(s)$.

For the implication to the left, let $s = (x, R^c_{\alpha_0}, s_1, \ldots, R^c_{\alpha_{k-1}}, s_k)$ and let $\langle\pi\rangle\gamma \in last(s) = s_k$. Let $X = \{l_0, \ldots, l_m\}$ be the agents not mentioned in π, i.e., $X = \{l \in$

$N : l \neq i_j, 1 \leq j \leq n\}$. Let $\pi_0 = \pi$, and $\pi_{j+1} = \pi_j \cap (l_j, a_j)$ for each $0 \leq j \leq m$ where $a_j \in Act$ is such that $\langle \pi_j \cap (l_j, a_j) \rangle \gamma \in s_k$. The existence of such a_js are ensured by axiom **A2**. Finally, let $\alpha = \pi_{m+1}$. This construction together with the assumption that $a_l = a_m$ whenever $i_k = i_m$ ensures that α is a complete joint action. By the fact that $\langle \alpha \rangle \gamma \in s_k$ and the existence lemma, there is a state $s_{k+1} \in S^c$ such that $(s_k, s_{k+1}) \in R^c_\alpha$ and $\gamma \in s_{k+1}$. Let $t = (x, R^c_{\alpha_0}, s_1, \ldots, R^c_{\alpha_{k-1}}, s_k, R^c_\alpha, s_{k+1})$; $t \in S^x$ and $(s, t) \in R^x_\alpha$ by definition of M^x. By Lemma 7 $(s, t) \in R_\alpha$, and from the fact that $\pi \leq \alpha$ it is easy to see that $R_\alpha \subseteq R_\pi$ by definition of R_π, so $(s, t) \in R_\pi$. Since $\gamma \in last(t)$, by the induction hypothesis $M, t \models \gamma$, and thus $M, s \models \langle \pi \rangle \gamma$.

Lemma 9. *M is a joint action model.*

Proof. INT: Immediate from the definition.

SER: Let s be a state and i an agent. From **A1** and **A3** there is some $\alpha = (1, a_1) \cap \cdots \cap (g, a_g)$ such that $\langle \alpha \rangle \top \in s$. From the truth lemma and Lemma 7 there is a s' such that $(s, s') \in R_\alpha = R^x_\alpha$. From the definition of $R_{(i,a_i)}$, $R^x_\alpha \subseteq R_{(i,a_i)}$, and thus $(s, s') \in R_{(i,a_i)}$.

IC: Let s be a state, $C = \{i_1, \ldots, i_k\}$ a coalition, and assume that for each j, (i_j, a_{i_j}) is enabled for i_j in s. Let $\pi = (i_1, a_{i_1}) \cap \cdots \cap (i_k, a_{i_k})$. By the truth lemma, $\bigwedge_{i_j \in C} \langle (i_j, a_{i_j}) \rangle \top \in last(s)$. For each $i_j \in N \setminus C$, let a_{i_j} be such that $\langle (i_j, a_{i_j}) \rangle \top \in last(s)$ – existing by **A1**. Let $\alpha = (1, a_1) \cap \cdots \cap (g, a_g)$. By **A3** $\langle \alpha \rangle \top \in last(s)$, and by **A5** $\langle \pi \rangle \top \in last(s)$. By the truth lemma, there is an s' such that $(s, s') \in R_\pi$.

DJA: Assume that $s = (x, R^c_{\alpha_0}, s_1, \ldots, R^c_{\alpha_{k-1}}, s_k)$. Let $(s, s_1), (s, s_2) \in R_\alpha$, where α is complete. By Lemma 7, there are $s^1_{k+1}, s^2_{k+1} \in S^c$ such that $s_1 = (x, R^c_{\alpha_0}, s_1, \ldots, R^c_{\alpha_{k-1}}, s_k, R^c_\alpha, s^1_{k+1})$, $s_2 = (x, R^c_{\alpha_0}, s_1, \ldots, R^c_{\alpha_{k-1}}, s_k, R^c_\alpha, s^2_{k+1})$, and $(s_k, s^1_{k+1}), (s_k, s^2_{k+1}) \in R^c_\alpha$. Assume that $s_1 \neq s_2$, i.e., since s_1 and s_2 are identical up to the last state, that $s^1_{k+1} \neq s^2_{k+1}$. By the definition of S^c, there must be a formula $\psi \in s^1_{k+1}$ such that $\neg\psi \in s^2_{k+1}$. By the truth lemma, $M, s_1 \models \psi$ and $M, s_2 \models \neg\psi$ and thus $M, s \models \langle \alpha \rangle \psi \wedge \langle \alpha \rangle \neg\psi$. By the truth lemma again, $\langle \alpha \rangle \psi, \langle \alpha \rangle \neg\psi \in last(s)$. By **A4**, $[\alpha]\psi, [\alpha]\neg\psi \in last(s)$. But $\langle \alpha \rangle \psi, [\alpha]\neg\psi \in last(s)$ contradicts, via standard modal reasoning, the fact that $last(s)$ is consistent. Thus, $s_1 = s_2$.

UJA: Immediate by Lemma 7 and construction of M^x.

Since $\phi \in x = last(x)$, ϕ is satisfied in a joint action model by Lemma 8 and 9. This concludes the completeness proof.

6 Complexity

We show that the complexity of deciding satisfiability in joint action models of $\mathcal{L}^{\cap}_K(\Pi^0_{ActN}, \Theta)$ formulae is in PSPACE. The proof uses ideas from [12].

In what follows, we assume wlog. that formulas do not contain diamond modalities and disjunctions. Given a set of formulas X, we use $Cl(X)$ to denote the smallest set containing all subformulas of formulas in X such that:

(a) for each agent i and action a, $[(i, a)]\bot \in Cl(X)$
(b) for every complete joint action $\alpha \in JA$, $[\alpha]\bot \in Cl(X)$

(c) if $\neg[(1,a_1),\ldots,(g,a_g)]\psi \in Cl(X)$, then $[(1,a_1),\ldots,(g,a_g)] \sim \psi \in Cl(X)$, where $\sim \psi = \neg\psi$ if ψ is not of the form $\neg\chi$, and $\sim \psi = \chi$ otherwise
(d) for each i and $a \neq b$, $[(i,a) \cap (i,b)]\bot \in Cl(X)$
(e) if $\psi \in Cl(X)$, then $\sim \psi \in Cl(X)$

The following procedure Tab is based on the $K_\omega^{\cap\cup}$-World proc. of [12]. For sets of formulas Δ and S where S is closed as above, $Tab(\Delta, S)$ returns true iff

(A) Δ is a maximally propositionally consistent subset of S, that is, for each $\neg\psi \in S$, $\psi \in \Delta$ iff $\neg\psi \notin \Delta$ and for each $\psi_1 \wedge \psi_2 \in S$, $\psi_1 \wedge \psi_2 \in \Delta$ iff $\psi_1 \in \Delta$ and $\psi_2 \in \Delta$.
(B) There is a partition of the set $\{\neg[\pi]\psi : \neg[\pi]\psi \in \Delta\}$ into sets W_α (at most one for each $\alpha \in JA$) such that if $\neg[\pi]\psi \in W_\alpha$ then $\pi \leq \alpha$ and
  (i) $\neg\psi \in \Delta_\alpha$
  (ii) for each π' and ψ', if $[\pi']\psi' \in \Delta$ and $\pi' \leq \alpha$, then $\psi' \in \Delta_\alpha$
  (iii) $Tab(\Delta_\alpha, S')$ returns true, where $S' = Cl(\{\psi' : [\pi']\psi' \in \Delta \text{ and } \pi' \leq \alpha\} \cup \{\neg\psi : \neg[\pi]\psi \in W_\alpha\})$
(C) for each $i \in N$, $\neg[(i,a)]\bot \in \Delta$ for some $a \in Act$
(D) if $\neg[(1,a_1)]\bot,\ldots,\neg[(g,a_g)]\bot \in \Delta$, then $\neg[(1,a_1)\cap\ldots\cap(g,a_g)]\bot \in \Delta$
(E) if $\neg[\alpha]\psi \in \Delta$, then $[\alpha] \sim \psi \in \Delta$
(F) for every $i \in N$ and $a, b \in Act$ such that $a \neq b$, $[(i,a) \cap (i,b)]\bot \in \Delta$

We require $Tab(\Delta, S)$ to terminate when the only modal formulas in S are those introduced by the clauses (a), (b) and (d) of the definition of $Cl(X)$. Note that otherwise formulas of the form $\neg[\alpha]\bot$ will continue triggering new calls to $Tab(\Delta, S)$.

Lemma 10. *A formula ϕ is satisfiable in a joint action model iff there exists $\Delta \subseteq Cl(\phi)$ with $\phi \in \Delta$ such that $Tab(\Delta, Cl(\phi))$ returns true.*

Proof. One direction is easy. For the other direction, we will show how to construct a model for ϕ if there exists $\Delta \subseteq Cl(\phi)$ with $\phi \in \Delta$ such that $Tab(\Delta, Cl(\phi))$ returns true. Suppose for ϕ such Δ exists, and let us call it Δ_0. The model M and state s_0 satisfying ϕ are constructed as follows. Each Δ in successive recursive calls of $Tab(\Delta, S)$ corresponds to a (partial specification of a) state. The existence of propositional assignment satisfying formulas in Δ is ensured by clause (A). The initial state s_0 corresponds to Δ_0. In each Δ, each formula of the form $\neg[\pi]\psi$ by the clause (B) belongs to a set W_α and has a 'witness' Δ_α for $\neg\psi$ accessible by a complete joint action α such that $\pi \leq \alpha$ and $\neg\psi \in \Delta_\alpha$. In the model we stipulate $R_\alpha(s_\Delta, s_{\Delta_\alpha})$ holds together with $R_{\pi'}(s_\Delta, s_{\Delta_\alpha})$ for every $\pi' \leq \alpha$. The rest of clause (B) makes sure that Δ_α contains all formulas ψ' such that $[\pi']\psi' \in \Delta$, which makes sure both that the truth definition for $[\pi]$ and the semantics of intersection work as expected. This part is almost identical to the proof for $K_\omega^{\cap}$ (apart from requiring a unique α-successor). All we need to prove is that in addition, the resulting model satisfies the properties of seriality, independent choice, determinism for complete joint actions, and uniqueness of joint actions.

SER is trivial by clause (C). When we terminate the procedure, to ensure seriality we add one more successor state for each α with an α loop to itself. This modification will not affect the truth of ϕ in s_0 because it is at a modal distance from s_0 which is greater than the modal depth of ϕ. IC is ensured by clause (D). DJA is ensured by (B); the existence of partition is enabled by (E) which makes the set of formulas $\sim \psi$ for $\neg[\alpha]\psi \in \Delta$ consistent. UJA is ensured by (F); namely there is no Δ' accessible by $\alpha \cap \alpha'$ from Δ, where $\alpha, \alpha' \in JA$ and $\alpha \neq \alpha'$; otherwise by clause (F) for some agent i which performs a different action in α and α', $[(i, a) \cap (i, b)]\bot \in \Delta$ and hence $\bot \in \Delta'$, but by the definition of the procedure then it cannot return true for Δ'.

Theorem 5. *The complexity of satisfiability problem of formulas in joint action models is PSPACE-complete.*

Proof. Satisfiability is decided by $Tab(\Delta, Cl(\phi))$ by the previous lemma. To see that $Tab(\Delta, Cl(\phi))$ requires polynomial space, consider the size of $Cl(\phi)$. The set of subformulas of ϕ is clearly polynomial in $|\phi|$. The number of formulas added to $Cl(\phi)$ by clause (a) is gm, the number of formulas added by clause (b) is m^g, the number of formulas added by clause (d) is gm^2, and (c) and (e) at most double the number of formulas in $Cl(\phi)$. Note that g and m are constant factors, hence the size of $Cl(\phi)$ and Δ is polynomial in $|\phi|$. PSPACE-hardness follows from K being PSPACE-complete.

The following is an immediate consequence of the result for model checking complexity of PDL with intersection [10]:

Theorem 6. *Model checking the $\mathcal{L}_K^{\cap}(\Pi^0_{ActN}, \Theta)$ language in joint action models is in PTIME.*

The complexity results above are encouraging from the point of view of using the logic of joint actions for verifying properties of game structures using standard theorem-proving and model-checking tools for normal modal logic. However, verification of properties involving coalitional ability comes at the price of performing a translation from CL to the language of $K_n^{\cap}$. The size of the translation may grow exponentially in the size of the input formula (nested coalition modalities give rise to nested disjunctions over all possible actions).

7 Discussion

In this paper we defined and studied a class of $K_n^{\cap}$ models corresponding to the class of concurrent game structures that are (1) injective and (2) parameterised by a fixed and finite set of actions, and showed that on this model class coalition modalities can be expressed in the $K_n^{\cap}$ language. Along the way we proved a representation theorem for injective games (this result holds also games with infinite sets of actions).

As mentioned in the introduction, the idea of interpreting PDL-like languages in games is not new. However, we are not aware of existing completeness or complexity results for K_n with intersection interpreted in game structures, nor on using intersection to capture the coalition operator.

[15] uses propositional dynamic logic (PDL) interpreted directly in extensive-form games, and also suggests extending the language with a "forcing" operator $\{G, i\}\phi$, with the meaning that agent i has a strategy in game G which forces a set of outcomes

that all will satisfy ϕ. However, the forcing operator is not defined in terms of intersection, and the operator is only defined for singleton coalitions. [4] have already shown that coalition logic can be embedded in a normal modal logic, namely in a variant of STIT (*seeing-to-it-that*) logic [3]. While this is a valuable result for several reasons, we argue that embedding in $K_n^\cap$ is of additional interest because the latter is a more standard logic (see the introduction). A closely related work is [11], which sets out from a similar starting point as the current paper: defining a "minimalistic" logical framework based on PDL that is interpreted in models where agents perform joint actions. *Deterministic Dynamic Logic of Agency* ($\mathcal{DDLA}$) [11] has modalities of the form $\langle i : a \rangle$ where i is an agent and a is an action, very similar to the modalities in the current paper in other words, and is shown to embed coalition logic. The interpretation of the modalities is slightly different: $\langle i : a \rangle\phi$ informally means that "i performs action a and ϕ holds afterwards". The formal interpretation is not a standard PDL interpretation. Also the language is not standard PDL; it includes a modality $\diamond$ that quantifies over actions. The language does not use intersection. In contrast, the current paper has focused on reasoning about joint action using only standard PDL modalities and operators, in particular intersection. We leave the precise relationship between the two logics to future work.

In this paper we studied a "minimal" language with intersection, sufficient to capture the coalition operators. For future work, extensions of the language with other PDL operators would be of interest, building on existing results on PDL with intersection such as [5].

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Appendix: some proofs

Proof (Lemma 1). For the implication to the right, assume that $E(C)$ has a complete nonmonotonic core. $E(C) \subseteq \{X : Y \subseteq X, Y \in E^{nc}(C)\}$ is immediate. If $Y \subseteq X$ and $Y \in E^{nc}(C)$, then $X \in E(C)$ by outcome monotonicity. The implication to the left is immediate.

Proof (Lemma 2). (5) Let $X_i \in E^{nc}(i)$ for each $i \in N$. By (2), $\bigcap_{i\in N} X_i \in E^{nc}(N)$, and by true playability and [8, Proposition 5] there is an $x \in \bigcap_{i\in N} X_i$ such that $\{x\} \in E(N)$. That means that $\{x\} = \bigcap_{i\in N} X_i$; because $\bigcap_{i\in N} X_i \neq \{x\}$ contradicts the facts that $\bigcap_{i\in N} X_i \in E^{nc}(N)$, $\{x\} \in E(N)$ and $x \in \bigcap_{i\in N} X_i$.

(6) Let $Z = \bigcup E^{nc}(N)$. Since injective playability implies true playability, we know that $E^{nc}(\emptyset) = \{Z'\}$ for some Z' [8, Proposition 5]. We show that $Z' = Z$. We have that $S \setminus Z \notin E(N)$; otherwise there would be a $X \subseteq S \setminus Z$ such that $X \in E^{nc}(N)$ (by (1)) and thus $X \subseteq Z$ by definition of Z, which together with the fact that $X \neq \emptyset$ (Liveness) is a contradiction. By N-maximality, $S \setminus (S \setminus Z) = Z \in E(\emptyset)$. Thus,

$Z' \subseteq Z$ by (1). Assume, towards a contradiction, that $Z \not\subseteq Z'$, i.e., that there is an $x \in Z$ such that $x \notin Z'$. That $x \in Z$ means that there is an $X \in E^{nc}(N)$ with $x \in X$. Let $X' = X \setminus \{x\}$. That $Z' \in E(\emptyset)$ and $X \in E(N)$ implies by superadditivity that $Z' \cap X \in E(N)$, and by the fact that $x \notin Z'$ we have that $Z' \cap X \subseteq X'$. By outcome monotonicity, $X' \in E(N)$. But that contradicts the fact that $X \in E^{nc}(N)$. Thus, $Z = Z'$.

Proof (Lemma 3). $X \in E^{nc}_G(C)$ iff $\exists \sigma_C \forall \sigma_{\overline{C}} o(\sigma_C, \sigma_{\overline{C}}) \in X$ and there is no $Y \in E_G(C)$ such that $Y \subset X$. Let $P = \{\ \{o(\sigma_C, \sigma_{\overline{C}}) : \sigma_{\overline{C}} \in \Sigma_{\overline{C}}\} : \sigma_C \in \Sigma_C\}$.

First, let $X \in E^{nc}_G(C)$ and let σ_C be as above (a witness for X). Let $Y = \{o(\sigma_C, \sigma_{\overline{C}}) : \sigma_{\overline{C}} \in \Sigma_{\overline{C}}\}$. $Y \in P$. $Y \subseteq X$. We have that $Y \in E_G(C)$ (by definition of α-effectivity), so by the fact that $X \in E^{nc}_G(C)$ it follows that $Y \not\subset X$ and thus that $Y = X$.

Second, let $X = \{o(\sigma_C, \sigma_{\overline{C}}) : \sigma_{\overline{C}} \in \Sigma_{\overline{C}}\} \in P$ for some σ_C. $X \in E_G(C)$. Assume towards a contradiction that there is a $Y \in E_G(C)$ such that $Y \subset X$. Thus there is a $\sigma'_C \in \Sigma_C$ such that for all $\sigma'_{\overline{C}} \in \Sigma_{\overline{C}}$, $o(\sigma'_C, \sigma'_{\overline{C}}) \in Y$. $\sigma_C \neq \sigma'_C$; otherwise $X \subseteq Y$, a contradiction. $Y \neq \emptyset$, so there is a $y \in Y \cap X$. In other words, there are $\sigma_{\overline{C}}$ and $\sigma'_{\overline{C}}$ such that $o(\sigma_C, \sigma_{\overline{C}}) = o(\sigma'_C, \sigma'_{\overline{C}}) = y$. But this contradicts the fact that G is injective, since $\sigma_C \neq \sigma'_C$. Thus, there is no such Y, and $X \in E^{nc}_G(C)$.

Proof (Theorem 2). First, let E_G be the α-effectivity function of some injective game G. We show that E_G is injectively playable. It is immediate from [14] that E_G is playable.

In order to show (1), let $X \in E_G(C)$, i.e., there is a σ_C such that for all $\sigma_{\overline{C}}$ $o(\sigma_C, \sigma_{\overline{C}}) \in X$. Let $Y = \{o(\sigma_C, \sigma_{\overline{C}}) : \sigma_{\overline{C}} \in \Sigma_{\overline{C}}\}$. $Y \subseteq X$, and $Y \in E^{nc}_G(C)$ by Lemma 3.

In order to show (2), assume that $|C| \geq 2$, since (2) holds trivially for $|C| = 1$. For one direction, let $X \in E^{nc}_G(C)$. By Lemma 3, $X = \{o(\sigma_C, \sigma_{\overline{C}}) : \sigma_{\overline{C}} \in \Sigma_{\overline{C}}\}$ for some σ_C. Let, for each $i \in C$, $\sigma_i = (\sigma_C)_i$ and $X_i = \{o(\sigma_i, \sigma_{\overline{i}}) : \sigma_{\overline{i}} \in \Sigma_{\overline{i}}\}$. $X_i \in E^{nc}_G(i)$ by Lemma 3. $X \subseteq \bigcap_{i \in C} X_i$. We must show that $\bigcap_{i \in C} X_i \subseteq X$. Let $x \in \bigcap_{i \in C} X_i$. For each $i \in C$, there exists some $\sigma_{\overline{i}}$ such that $x = o(\sigma_i, \sigma_{\overline{i}})$. For any arbitrary $i, j \in C$, $i \neq j$, from $o(\sigma_j, \sigma_{\overline{j}}) = x = o(\sigma_i, \sigma_{\overline{i}})$ we get that $(\sigma_{\overline{j}})_i = \sigma_i$ by injectivity. Thus, $o(\sigma_C, \sigma_{\overline{C}}) = o(\sigma_j, \sigma_{\overline{j}})$, for all $j \in C$ and some $\sigma_{\overline{C}}$, and thus $x \in X$.

For the other direction of (2), let $X = \bigcap_{i \in C} X_i$ with $X_i \in E^{nc}_G(i)$. Again, for each $i \in C$, $X_i = \{o(\sigma_i, \sigma_{\overline{i}}) : \sigma_{\overline{i}} \in \Sigma_{\overline{i}}\}$ for some σ_i. Let σ_C be defined by $(\sigma_C)_i = \sigma_i$. Let $Y = \{o(\sigma_C, \sigma_{\overline{C}}) : \sigma_{\overline{C}} \in \Sigma_{\overline{C}}\}$. $Y \in E^{nc}_G(C)$ by Lemma 3. We show that $Y = X$. First, let $\sigma_{\overline{C}} \in \Sigma_{\overline{C}}$ be arbitrary. Since $(\sigma_C)_i = \sigma_i$, $o(\sigma_C, \sigma_{\overline{C}}) \in X_i$ for all $i \in C$, and $o(\sigma_C, \sigma_{\overline{C}}) \in X$. Thus, $Y \subseteq X$. Let $x \in X$. For each $i \in C$, there is some $\sigma_{\overline{i}}$ such that $o(\sigma_i, \sigma_{\overline{i}}) = x$. We can now reason as above. Let $i, j \in C$, $i \neq j$. From $o(\sigma_j, \sigma_{\overline{j}}) = x = o(\sigma_i, \sigma_{\overline{i}})$, we get that $(\sigma_{\overline{j}})_i = \sigma_i$ by injectivity. Thus, $o(\sigma_j, \sigma_{\overline{j}}) = o(\sigma_C, \sigma_{\overline{C}})$, for some arbitrary $j \in C$ and some $\sigma_{\overline{C}}$, and thus $x \in Y$. Thus, $X \subseteq Y$.

In order to show that (3) holds, let $X \neq Y \in E^{nc}_G(i)$. By Lemma 3, there are σ_i, σ'_i such that $X = \{o(\sigma_i, \sigma_{\overline{i}}) : \sigma_{\overline{i}} \in \Sigma_{\overline{i}}\}$ and $Y = \{o(\sigma'_i, \sigma_{\overline{i}}) : \sigma_{\overline{i}} \in \Sigma_{\overline{i}}\}$. Assume that $x \in X \cap Y$, i.e., that $o(\sigma_i, \sigma_{\overline{i}}) = o(\sigma'_i, \sigma'_{\overline{i}})$ for some $\sigma_{\overline{i}}$ and $\sigma'_{\overline{i}}$. Since the game is injective that means that $\sigma_i = \sigma'_i$, but that contradicts the fact that $X \neq Y$. Thus, $X \cap Y = \emptyset$.

In order to show that (4) holds, let $X \in E^{nc}_G(j)$ and $x \in X$. By Lemma 3, there is a σ_j such that $X = \{o(\sigma_j, \sigma_{\overline{j}}) : \sigma_{\overline{j}} \in \Sigma_{\overline{j}}\}$. In particular, $x = o(\sigma_j, \sigma_{\overline{j}})$ for some $\sigma_{\overline{j}}$. Let $\sigma_i = (\sigma_{\overline{j}})_i$, and let $Y = \{o(\sigma_i, \sigma_{\overline{i}}) : \sigma_{\overline{i}} \in \Sigma_{\overline{i}}\}$. $x \in Y$, and $Y \in E^{nc}_G(i)$ by Lemma 3.

Second, let E be an injectively playable effectivity function over N and S. We construct a game $G = (N, \{\Sigma_i : i \in N\}, o, S)$ as follows:

$$\Sigma_i = E^{nc}(i) \qquad o(X_1, \ldots, X_g) = x \quad \text{where} \quad \{x\} = \bigcap_{i \in N} X_i$$

The property (5) (Lemma 2) ensures that the game is well defined. To see that G is injective, assume that $o(X_1, \ldots, X_g) = o(X'_1, \ldots, X'_g) = x$. That means that, for each i, $x \in X_i \cap X'_i$, and by (3) it follows that $X_i = X'_i$. Thus, G is injective.

We must show that $E_G = E$. By (1), outcome monotonicity and Lemma 1 it suffices to show that $E^{nc}_G(C) = E^{nc}(C)$ for all $C \subseteq N$.

First assume that $C \neq \emptyset$. For any X, $X \in E^{nc}_G(C)$ iff (by Lemma 3) $\exists \sigma_C$ such that $X = \{o(\sigma_C, \sigma_{\overline{C}}) : \sigma_{\overline{C}} \in \Sigma_{\overline{C}}\}$ iff $\exists \{X_i \in E^{nc}(i) : i \in C\}$ such that $X = \{x : \{x\} = \bigcap_{i \in N} X_i, X_j \in E^{nc}(j), j \in N \setminus C\}$. On the other hand, $X \in E^{nc}(C)$ iff $\exists \{X_i \in E^{nc}(i) : i \in C\}$ such that $X = \bigcap_{i \in C} X_i$, by (2). Thus, let, for each $i \in C$, $X_i \in E^{nc}(i)$. It suffices to show that $\{x : \{x\} = \bigcap_{i \in N} X_i, X_j \in E^{nc}(j), j \in N \setminus C\} = \bigcap_{i \in C} X_i$. For inclusion towards the left, assume that $x \in \bigcap_{i \in C} X_i$. If $C = N$ we are done. Otherwise, from (4) it follows that there is a $X_j \in E^{nc}(j)$ such that $x \in X_j$, for every $j \in N \setminus C$. Thus, $x \in \bigcap_{i \in N} X_i$. For inclusion towards the right, let $\{x\} = \bigcap_{i \in N} X_i$ for some $\{X_i \in E^{nc}(i) : i \in N \setminus C\}$. It immediately follows that $x \in \bigcap_{i \in C} X_i$.

Second, consider the case that $C = \emptyset$. $X \in E^{nc}_G(\emptyset)$ iff (by Lemma 3) $X = \{o(\sigma_N) : \sigma_N \in \Sigma_N\}$ iff $X = \{x : \{x\} = \bigcap_{i \in N} X_i, X_i \in E^{nc}(i)\}$ iff (by (2)) $X = \bigcup E^{nc}(N)$ iff (by Lemma 2) $X \in E^{nc}(\emptyset)$.