Mathematics for Computer Scientists

Lecture notes for the module G51MCS

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Chapter 1

Logic

Human language is the most sophisticated system of communication in the known universe. We use it daily to inform, enquire, demand, order, convince, sympathise and insult. *Linguistics* studies its inner working and its infinite range of expressions and forms. In this book we have a more modest purpose: we want to understand the part of language that is used in mathematics to formulate theorems and equations and to give proofs. Only a very limited portion of the grammatical and semantic wealth of natural language will be needed.

The kind of phrases we are interested in are *propositions*. A proposition is any statement that can be true or false. Examples of propositions are:

- *The sky is blue.*
- *Paris is the capital of Jamaica.*
- *All red cats have bushy tails.*
- *The moon is made of cheese.*

Not all expressions in natural language are propositions. Questions (*What is the moon made of?*), orders (*Don't step on my toes.*), interjections (*Good grief!*) are neither true nor false, but serve a purpose different from stating a fact.

For our aims, we are more interested in propositions that state properties of numbers and other mathematical objects:

- *30 is a multiple of 5.*
- *7 is a divisor of 23.*
- *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the two legs.*
- *61 is a prime number.*
- *There are infinitely many prime numbers.*
- *Every even integer larger than 2 can be written as the sum of two primes.*

Some of these properties are true and some are false. The last one is a mystery: as of today nobody has been able to prove it or to come up with an integer that doesn't satisfy it.

Several propositions can be combined in one by the use of special words called *connectives*: *and, or, if .. then .., not, only if, etc.* The following are examples of complex propositions:

- ✐ *The sky is blue and the grass is green.*
- ✐ *If the moon is made of cheese, then 7 is a divisor of 23.*
- ✐ *You will pass the exam only if you study hard.*
- ✐ *If Paris is not the capital of Jamaica or 61 is a prime number, then all red cats have bushy tails.*

Some propositions are about specific entities: *The moon is red during an eclipse* is about the moon.

Other propositions state a common property of a group of entities: *Every squirrel likes chestnuts* is about the group of all squirrels. Words like *all, every, each, etc.* which are used for this purpose are called *universal quantifiers*.

Still other propositions state that there is an entity that has a property, but without specifying which it is: *Someone stole my bicycle* states something about an individual without specifying who he or she is. Words like *some, something, someone, there is, there exists, for some, etc.* which are used for this purpose are called *existential quantifiers*.

1.1 Derivations

Suppose that we know the following facts:

- ✐ *I sing if Jenny plays drums and Edward plays guitar.*
- ✐ *If Jenny doesn't play drums, then Sue plays bongos.*
- ✐ *Edward plays guitar and I don't sing.*

Then we can certainly conclude that *Sue plays bongos*.

Now let's consider this other deduction, apparently of a completely different kind. Suppose that we know the following facts:

- ✐ *If the door was locked and the killer escaped, then he must have exited by the window.*
- ✐ *If the door wasn't locked, then the killer had a key.*
- ✐ *The killer escaped but he didn't exit by the window.*

Then we can conclude that *the killer had a key*.

The two inferences are about quite different situations: the first is about the composition of a rock band, the second about a murder mystery. However, to a logician they are exactly the same, because a logician only cares about the structure of an argument and not about its content.

An excessively precise logical derivation for the first inference is as follows:

1. *I sing if Jenny plays drums and Edward plays guitar.* ◁ Premise.
2. *If Jenny doesn't play drums, then Sue plays bongos.* ◁ Premise.
3. *Edward plays guitar and I don't sing.* ◁ Premise.
4. Suppose that *Jenny plays drums.* ◁ Assumption.
5. We know that *Edward plays guitar.* ◁ From line 3.
6. So *Jenny plays drums and Edward plays guitar.* ◁ Lines 4 and 5.
7. Therefore *I sing.* ◁ By lines 1 and 6.
8. But *I don't sing.* ◁ From line 3.
9. Impossible! ◁ Lines 7 and 8 are contradictory.
10. So *Jenny doesn't play drums.* ◁ Assumption 4 must be false because it produced the contradiction 9.
11. In conclusion *Sue plays bongos.* ◁ By lines 2 and 10.

On every line we wrote one proposition, followed by a justification explaining why that proposition is valid. The first three lines are just the premises from which we start, so they don't need any justification. In line number 4 we make an assumption, that is, we suppose that a proposition (*Jenny plays drums*) is true and we investigate what would then follow. To indicate that the following steps are not necessarily true, but only proceed from this assumption, we indented them a bit. The sequence of indented steps is called a *subderivation*. In this case, the subderivation leads to a blatant contradiction (*I sing* and *I don't sing*). Since these two propositions cannot be both true, our conclusion must be that the assumption that we made was actually false. We then can close the subderivation and conclude that the negation of its assumption is true (*Jenny doesn't play drums*).

Now let's repeat the process with our second inference and give a full derivation for it:

1. *If the door was locked and the killer escaped, then he must have exited by the window.* ◁ Premise.
2. *If the door wasn't locked, then the killer had a key.* ◁ Premise.
3. *The killer escaped but he didn't exit by the window.* ◁ Premise.
4. Suppose that *The door was locked.* ◁ Assumption.
5. We know that *The killer escaped.* ◁ From line 3.
6. So *the door was locked and the killer escaped.* ◁ Lines 4 and 5.
7. Therefore *The killer exited by the window.* ◁ By lines 1 and 6.
8. But *The killer didn't exit by the window.* ◁ From line 3.
9. Impossible! ◁ Lines 7 and 8 are contradictory.
10. So *The door wasn't locked.* ◁ Assumption 4 must be false because it produced the contradiction 9.

11. In conclusion *The killer had a key.* ◁ By lines 2 and 10.

You probably noticed that the two derivations have exactly the same structure. Logic is concerned just with this structure and not with the specific topics discussed. We can then summarise the structure by replacing the component phrases, which we call *atomic propositions*, with letters A, B, C, D standing for any possible sentence. The gist of the argument then goes like this. Suppose that we know the following facts:

- *If A and B then C.*
- *If not A then D.*
- *B and not C.*

Then we can conclude that D must be true. Our two examples are obtained by replacing the variables by atomic propositions as follows:

$A :=$ *Jenny plays drums.*	$A :=$ *The door was locked.*
$B :=$ *Edward plays guitar.*	$B :=$ *The killer escaped.*
$C :=$ *I sing.*	$C :=$ *The killer exited by the window.*
$D :=$ *Sue plays bongos.*	$D :=$ *The killer had a key.*

The common structure of the two derivations is the following:

1. *If A and B then C.* ◁ Premise.
2. *If not A then D.* ◁ Premise.
3. *B and not C.* ◁ Premise.
4.   Suppose that A is true. ◁ Assumption.
5.   We know B. ◁ From line 3.
6.   So *A and B.* ◁ Lines 4 and 5.
7.   Therefore C. ◁ By lines 1 and 6.
8.   But *not C.* ◁ From line 3.
9.   Impossible! ◁ Lines 7 and 8 are contradictory.
10. So *not A.* ◁ Assumption 4 lead to the contradiction 9.
11. In conclusion D must be true. ◁ By lines 2 and 10.

1.2 Propositional Formulas

Propositional logic is the discipline that studies the correct logical rules to make derivations using propositions. We ignore the meaning of the component propositions: we use variable names A, B, C, etc. to denote them. Instead we concentrate on the way the connectives affect the truth or falsity of complex propositions. We use symbols to denote the connectives: $\wedge$ for *'and'*, $\vee$ for *'or'*, $\Rightarrow$ for *'if .. then ..'* and $\neg$ for *'not'*.

For example, if we denote by A the atomic proposition *'Paris is the capital of Jamaica'*, by B the atomic proposition *'61 is a prime number'* and by C

the atomic proposition *'all red cats have bushy tails'*, then the last complex proposition at the end of the introduction of this chapter (*If Paris is not the capital of Jamaica or 61 is a prime number, then all red cats have bushy tails*) can be written symbolically as:

$$\neg A \vee B \Rightarrow C.$$

Each connective may be rendered in natural language with different expressions and it can be tricky to recognise it. Sometimes we may have the feeling that two logically equivalent expressions actually say quite different things.

The conjunction connective $\wedge$ states that both its arguments are true, so it formalises the conjunction *'and'*. However, it also translates the conjunction *'but'*! If I say *'I have a trombone but I don't know how to play it'*, I am stating that both atomic propositions *'I have a trombone'* and *'I don't know how to play the trombone'* are true. I use *'but'* to express that there is a psychological opposition between the two sentences, but there is no logical contradiction. From the logical point of view, I am just making a conjunction.

The disjunction connective $\vee$ states that at least one of its arguments is true. This is called *inclusive disjunction*, meaning that it is possible that they are both true. We use $\vee$ to translate *'or'*, but we must be careful that sometimes, in everyday language, we use it in an *exclusive* sense: only one of the two components can be true. *'You can have ice cream or cake'* doesn't mean that you can have both. But in formal logic, we always use $\vee$ in an inclusive way.

The implication connective $\Rightarrow$ is the trickiest one when it comes to translating from natural language. There are several ways of expressing it: $A \Rightarrow B$ can be rendered by *'A implies B'*, *'if A then B'*, *'B if A'* (mind the inversion), *'B is a consequence of A'*, *'A only if B'*. This last one is very deceptive, let me repeat it: *'A only if B'* means that A logically implies B. Be careful not to confuse logical consequence with *causal relation*, where the state of affairs described by the first proposition is the cause of the events expressed by the second. *'It will snow only if the temperature drops below zero'* tells us that a freezing weather is a cause (among others) of snow. The logical connection goes in the opposite direction: if it snows, we can deduce that the temperature must have been below zero. But, since the temperature might fall below zero without it snowing, the implication in the other direction is not valid.

Let's summarise in a table the uses of the logical connectives and some of

their translations: $A \wedge B$: *the conjunction of A and B,*
A and B,
A but B,
both A and B are true;
$A \vee B$: *the disjunction of A and B,*
A or B,
either A or B (or both),
one of A and B is true;
$A \Rightarrow B$: *the implication from A to B,*
A implies B,
if A then B,
B if A,
A only if B,
B is a consequence of A,
whenever A is true also B is true;
$\neg A$: *the negation of A,*
not A,
A is false.

It is useful to have also two special symbols denoting a true and false proposition, respectively. We use $\top$ for a proposition that is undoubtedly true, without the need for a proof, for example $0 = 0$. We use $\bot$ for a proposition that is undoubtedly false, for example $0 = 1$.

Let us now be more formal and give an exact definition of the kind of entities that form the object of our study.

Definition 1 *A* propositional formula *is any expression obtained by applying some of the following construction rules a finite number of times:*

- *Propositional variables $A, B, C, \ldots$ are propositional formulas;*
- *The symbols $\top$ and $\bot$ are propositional formulas;*
- *If A and B are propositional formulas, then $(A \wedge B), (A \vee B), (A \Rightarrow B)$ are propositional formulas;*
- *If A is a propositional formula, then $(\neg A)$ is a propositional formula.*

This definition is the first example we see of an *inductively defined set*. This means that the collection of propositional formulas is constructed in stages: first we have the *atomic* formulas, which are just variables represented as uppercase letters and the two symbols $\top$ and $\bot$; all other formulas are obtained by applying the connectives again and again, constructing increasingly more complex expressions. Examples of propositional formulas are the following:

$$A$$
$$(A \wedge B)$$
$$(A \Rightarrow (B \vee C))$$
$$((A \vee (\neg B)) \Rightarrow (C \Rightarrow (\neg (B \wedge A)))).$$

As the formulas become more complex, all the parentheses start to make them very difficult to read. For this reason there are *precedence* conventions that allow us to drop some of the parentheses. In fact, the first propositional formula that we wrote, $\neg A \vee B \Rightarrow C$, doesn't have any. Without any convention, this would be ambiguous: there are several different ways of restoring parentheses:

$$(((\neg A) \vee B) \Rightarrow C), \quad ((\neg A) \vee (B \Rightarrow C)), \quad ((\neg(A \vee B)) \Rightarrow C),$$
$$(\neg((A \vee B) \Rightarrow C)), \quad (\neg(A \vee (B \Rightarrow C)));$$

each with a different meaning. The one that we really mean is the first one. To enforce this, we decide that there is an order of precedence between connectives: $\neg$ has the highest precedence, followed by $\wedge$, then $\vee$ and finally $\Rightarrow$. This means that in restoring parentheses, when there is some ambiguity, we should put them first around subformulas containing the higher connective.

1.3 Rules of Propositional Logic

Now that we have defined formally what a proposition is, we want to give exact rules to make logical derivations. We are inspired by the semi-formal proofs that we wrote down in Section 1.1: A derivation is a sequence of numbered propositions, starting with the premises, each one given with a justification showing how it follows from the previous ones.

These *justifications* must be mathematically precise. We give exact rules to derive propositions. A proof is a kind of game played according to these rules. In the justifications we must say what rule we used and to what previous propositions we applied it.

We organise these derivations as a series of numbered steps displayed along a vertical line: each step consists of a proposition and a justification of how it follows logically from the steps that came before. At the beginning we put some propositions that we assume to be true, these are the initial *hypotheses.* So here is how a specific derivation looks like:

1	$A \wedge B \Rightarrow C$	
2	$\neg C \wedge B$	
⋮	⋮	justifications
9	$\neg A$	justification of the last step

In this example we start with two hypotheses $A \wedge B \Rightarrow C$ and $\neg C \wedge B$; they are separated from the actual steps of the derivation by a horizontal line.

Next we explain how the logical steps work for each of the connectives. Each of them has two kinds of rule: the first, called an *introduction rule* tells us how we can derive a combined proposition containing that connective; the second, called an *elimination rule* tells us how we can use it to prove other propositions.

How to Use Variables Remember that propositional variables stand for any possible statement. The rules of logic are expressed using variables. When we apply them in a concrete derivation, they can be replaced with any proposition. Consider the example derivation sketched above. If we instantiate the variables so: $A :=$ *'Jenny plays drums,'* $B :=$ *'Edward plays guitar,'* $C :=$ *'I sing;'* then the two hypotheses and the conclusion become:

- (*Jenny plays drums*) $\wedge$ (*Edward plays guitar*) $\Rightarrow$ (*I sing*);
- $\neg$(*I sing*) $\wedge$ (*Edward plays guitar*);
- $\neg$(*Jenny plays drums*).

I left the connectives as symbols, rather than translating them in words: Let's get used to mathematical notation.

Propositional variables can be instantiated not only by atomic formulas written in natural language, but also by other propositional formulas. For example, we could choose the instantiations: $A := Q \vee R$, $B := \neg R$, $C := R \Rightarrow Q$. Then our hypotheses and conclusion are instantiated so:

$$(Q \vee R) \wedge (\neg R) \Rightarrow (R \Rightarrow Q);$$
$$\neg(R \Rightarrow Q) \wedge (\neg R);$$
$$\neg(Q \vee R).$$

Be careful to use parentheses appropriately when you make the substitutions: Always put them around the formula that you are substituting and only afterwards you may delete those that are not necessary.

The propositional formulas by which you replace the variables might also contain those same variables. Be careful not to confuse the occurrences of those variables before and after the substitution. For example, if we choose the instantiations: $A := A \Rightarrow B$, $B := C \vee \neg B$, $C := A$; the three propositions become:

$$(A \Rightarrow B) \wedge (C \vee \neg B) \Rightarrow A;$$
$$\neg A \wedge (C \vee \neg B);$$
$$\neg(A \Rightarrow B).$$

Conjunction The *conjunction* of two propositions A and B is the combined proposition that uses the "and" connective, $A \wedge B$. The introduction rule states that to prove $A \wedge B$, we must have already proved separately both A and B:

m	A	
$\vdots$	$\vdots$	
n	B	
$\vdots$	$\vdots$	
p	$A \wedge B$	$\wedge$I, m, n

m	B	
$\vdots$	$\vdots$	
n	A	
$\vdots$	$\vdots$	
p	$A \wedge B$	$\wedge$I, n, m

The elimination rule states that if we have proved $A \wedge B$, we can use it to prove A and to prove B:

m	$A \wedge B$	
$\vdots$	$\vdots$	
p	A	$\wedge$E, m

m	$A \wedge B$	
$\vdots$	$\vdots$	
p	B	$\wedge$E, m

Implication When we want to prove a proposition in the form $A \Rightarrow B$, what we have to do is to assume that A is true and show that using this assumption we can derive that B must also be true. So we need to add A to our stock of hypotheses, but only temporarily until we demonstrate B. We indicate this by drawing a new vertical line on the right of the present position. The part of the derivation spanned by this line shows that we can derive B from A. The new hypothesis A that we introduce at the top of the line can be used only inside this *subderivation*:

m	│ A	
$\vdots$	│ $\vdots$	
n	│ B	
$n+1$	$A \Rightarrow B$	$\Rightarrow$I, m–n

The elimination rule for implication simply states that if we know that A implies B and we also know that A is true, then B must be true as well:

m	$A \Rightarrow B$	
$\vdots$	$\vdots$	
n	A	
$\vdots$	$\vdots$	
p	B	$\Rightarrow$E, m, n

m	A	
$\vdots$	$\vdots$	
n	$A \Rightarrow B$	
$\vdots$	$\vdots$	
p	B	$\Rightarrow$E, n, m

This rule is so important that it also has a Latin name: *modus ponens*.

Disjunction The *disjunction* of two propositions A and B is the combined proposition that uses the "or" connective, $A \vee B$. We understand it as a *non-exclusive or*, meaning that $A \vee B$ is true in the case that only A is true, in the case that only B is true, and also in the case that they're both true. It is false only in the case that both A and B are false. To prove $A \vee B$ it is enough to prove one or the other of the disjuncts:

$$
\begin{array}{l|ll}
m & A & \\
\vdots & \vdots & \\
p & A \lor B & \lor\text{I}, m
\end{array}
\qquad
\begin{array}{l|ll}
m & B & \\
\vdots & \vdots & \\
p & A \lor B & \lor\text{I}, m
\end{array}
$$

The elimination rule is a bit more complex: suppose we have proved $A \lor B$; how can we use it to prove some other proposition C? We know that either A or B must be true, but we don't know which. To be sure that the conclusion is valid, we must show that C can be derived both from A and from B. Therefore we make two subderivations. As in the case of implication, the new assumptions A and B can be used only locally in those subderivations:

$$
\begin{array}{l|lll}
m & A \lor B & & \\
\vdots & \vdots & & \\
h & & \begin{array}{|l} A \\ \hline \end{array} & \\
\vdots & & \begin{array}{|l} \vdots \end{array} & \\
i & & \begin{array}{|l} C \end{array} & \\
j & & \begin{array}{|l} B \\ \hline \end{array} & \\
\vdots & & \begin{array}{|l} \vdots \end{array} & \\
k & & \begin{array}{|l} C \end{array} & \\
p & C & & \lor\text{E}, m, h\text{–}i, j\text{–}k
\end{array}
$$

Truth and Falsity The two atomic propositions $\top$ and $\bot$ have especially simple rules. $\top$ is a certainly true proposition, it doesn't need a proof; its introduction rule just states that it is valid. On the other hand, it doesn't tell us anything useful, that is, it won't have an elimination rule. The false proposition $\bot$ should not be provable, so it doesn't have an introduction rule. On the other hand, if we manage, by some miracle, to prove it, then we can prove anything: if falsity is true, then everything is true. In conclusion, $\top$ only has an introduction rule, without elimination; $\bot$ only has an elimination rule, without introduction:

$$
\begin{array}{l|ll}
\vdots & \vdots & \\
p & \top & \top\text{I}
\end{array}
\qquad
\begin{array}{l|ll}
m & \bot & \\
\vdots & \vdots & \\
p & C & \bot\text{E}, m
\end{array}
$$

Negation The negation of a proposition, $\neg A$, is actually equivalent to saying that A implies something false, that is, $A \Rightarrow \bot$. And indeed the rules for negation are exactly those for implication when the conclusion is the false proposition:

m		A	
$\vdots$		$\vdots$	
n		$\bot$	
$n+1$	$\neg A$		$\neg$I, m–n

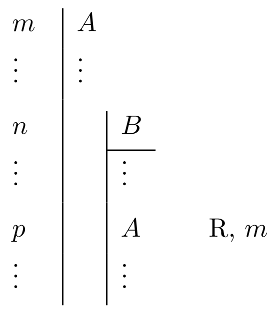
m	$\neg A$	
$\vdots$	$\vdots$	
n	A	
$\vdots$	$\vdots$	
p	$\bot$	$\neg$E, m, n

m	A	
$\vdots$	$\vdots$	
n	$\neg A$	
$\vdots$	$\vdots$	
p	$\bot$	$\neg$E, n, m

Reiteration Sometimes we just want to copy a proposition that we proved earlier in order to use it again. We are free to do it, but be careful: we can copy only propositions that are *in scope* at the place where we want to copy them. This means that the vertical line immediately to the left of the proposition that we want to copy must extend down to the left of the place where we want to copy it:

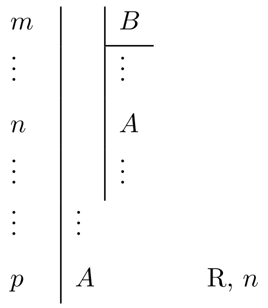
m	A	
$\vdots$	$\vdots$	
p	A	R, m

We are allowed to copy a proposition to a more internal subderivation, that is, to a vertical line on the right; but we cannot copy a proposition from a subderivation to a more external place, that is, to the left of its vertical line; also, we cannot copy a proposition after there has been a break of its line:

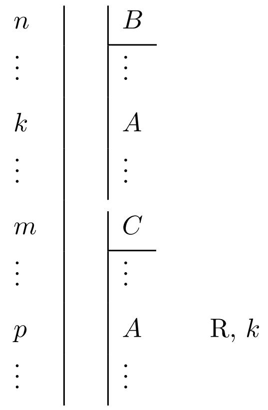


OK

WRONG:



WRONG:



1.4 Example Derivations

Here are a couple of examples of full derivations. In the first, we have three atomic propositions A, B and C. We assume to know that $A \Rightarrow B$, $B \Rightarrow C$ and $\neg C$ are true and we want to prove that $\neg A$ follows from them. Here is the complete formal proof:

1	$A \Rightarrow B$	
2	$B \Rightarrow C$	
3	$\neg C$	
4	$\quad$ A	
5	$\quad$ B	$\Rightarrow$E, 1, 4
6	$\quad$ C	$\Rightarrow$E, 2, 5
7	$\quad$ $\bot$	$\neg$E, 3, 6
8	$\neg A$	$\neg$I, 4–7

Let us see how this formal proof corresponds to an argument in English by using some actual sentences for our propositional letters. Let A mean *the universe is a fairy land*, B mean *the moon is made of cheese* and C mean *mice live in the moon.* Then the formal proof above corresponds to the following argument:

> Suppose that if the universe were a fairy land, then the moon would be made of cheese. Suppose also that if the moon were made of cheese, then mice would live in it. Finally, suppose that mice don't live in the moon. Assume now that the universe were a fairy land. Then, by the first supposition, we would know that the moon is made of cheese. But then, by the second supposition, we would also have that mice live in the moon. But we know, by the third supposition, that mice don't live in the moon. We reached a contradiction: mice live and don't live in the moon. This means that our assumption that the universe is a fairy land was wrong. Then we can conclude that the universe is not a fairy land.

In my opinion the English version is more difficult to understand. Symbols and mathematical rules make thinking easier.

In our second example we make the assumptions $A \Rightarrow C$, $B \Rightarrow D$ and $A \vee B$

and we conclude that $C \vee D$ must hold:

1	$A \Rightarrow C$	
2	$B \Rightarrow D$	
3	$A \vee B$	
4	$\quad A$	
5	$\quad C$	$\Rightarrow$E, 1, 4
6	$\quad C \vee D$	$\vee$I, 5
7	$\quad B$	
8	$\quad D$	$\Rightarrow$E, 2, 7
9	$\quad C \vee D$	$\vee$I, 8
10	$C \vee D$	$\vee$E, 3, 4–6, 7–9

Also in this case we can show an English argument corresponding to this formal proof. Let's assign to A the meaning *I eat spinach*, to B the meaning *I eat carrots*, to C the meaning *I can jump over a mountain* and to D the meaning *I can jump across a river*. Then the formal proof above corresponds to the following argument:

> Suppose that, if I eat spinach, then I can jump over a mountain. Suppose also that, if I eat carrots, then I can jump across a river. Finally, suppose that I eat spinach or carrots. In the first case, spinach, the first supposition tells me that I can jump over a mountain; therefore I can jump over a mountain or across a river (the first being the case). In the second case, carrots, the second supposition tells me that I can jump across a river; therefore I can jump over a mountain or across a river (the second being the case). Since I reached the same conclusion in both cases, I have proved that I can jump over a mountain or across a river.

Let's go back to the informal examples that we discussed in Section 1.1. We noted that the two derivations, about the composition of a rock band and about a murder mystery, had the same structure. We also formulated the argument in a general way using variables in place of atomic sentences. Here is the completely formalised proof:

1	$A \wedge B \Rightarrow C$	
2	$\neg A \Rightarrow D$	
3	$B \wedge \neg C$	
4	$\quad A$	
5	$\quad B$	$\wedge$E, 3
6	$\quad A \wedge B$	$\wedge$I, 4, 5
7	$\quad C$	$\Rightarrow$E, 1, 6
8	$\quad \neg C$	$\wedge$E, 3
9	$\quad \bot$	$\neg$E, 8, 7
10	$\neg A$	$\neg$I, 4–9
11	D	$\Rightarrow$E, 2, 10

1.5 More Examples

Here are a few more examples of formal derivations for you to study. After looking at them, you should be able to write them yourselves.

1	$A \wedge (B \wedge C)$	
2	A	$\wedge$E, 1
3	$B \wedge C$	$\wedge$E, 1
4	B	$\wedge$E, 3
5	C	$\wedge$E, 3
6	$A \wedge B$	$\wedge$I, 2, 4
7	$(A \wedge B) \wedge C$	$\wedge$I, 6, 5

1	$\neg(A \Rightarrow B)$	
2	$\quad B$	
3	$\quad\quad A$	
4	$\quad\quad B$	R, 2
5	$\quad A \Rightarrow B$	$\Rightarrow$I, 3–4
6	$\quad \bot$	$\neg$E, 1, 5
7	$\neg B$	$\neg$I, 2–6

1	$A \vee B$	
2	$\neg A$	
3	A	
4	$\bot$	$\neg$E, 2, 3
5	B	$\bot$E, 4
6	B	
7	B	R, 6
8	B	$\vee$E, 1, 3–5, 6–7
9	$\neg A \Rightarrow B$	$\Rightarrow$I, 2–8

1	$A \Rightarrow B$	
2	$\neg B$	
3	A	
4	B	$\Rightarrow$E, 1, 3
5	$\bot$	$\neg$E, 2, 4
6	$\neg A$	$\neg$I, 3–5
7	$\neg B \Rightarrow \neg A$	$\Rightarrow$I, 2–6

1	$\neg(A \vee B)$	
2	A	
3	$A \vee B$	$\vee$I, 2
4	$\bot$	$\neg$E, 1, 3
5	$\neg A$	$\neg$I, 2–4
6	B	
7	$A \vee B$	$\vee$I, 6
8	$\bot$	$\neg$E, 1, 7
9	$\neg B$	$\neg$I, 6–8
10	$\neg A \wedge \neg B$	$\wedge$I, 5, 9

1	$A \wedge (B \vee C)$	
2	A	$\wedge$E, 1
3	$B \vee C$	$\wedge$E, 1
4	B	
5	$A \wedge B$	$\wedge$I, 2, 4
6	$(A \wedge B) \vee (A \wedge C)$	$\vee$I, 5
7	C	
8	$A \wedge C$	$\wedge$I, 2, 7
9	$(A \wedge B) \vee (A \wedge C)$	$\vee$I, 8
10	$(A \wedge B) \vee (A \wedge C)$	$\vee$E, 3, 4–6, 7–9

1.6 WRONG Derivations

So you are aware of the pitfalls of formal logic, let us look at some derivations that may look correct at first, but contain subtle mistakes.

1	$\neg A$		
2	$(A \Rightarrow B) \Rightarrow C$		
3	$\quad A$		
4	$\quad \bot$	$\neg$E, 1, 3	**WRONG**
5	$\quad B$	$\bot$E, 4	
6	$A \Rightarrow B$	$\Rightarrow$I, 3–5	
7	C	$\Rightarrow$E, 2, 6	
8	$A \wedge C$	$\wedge$I, 3, 7	

The last step of this derivation is incorrect: It uses the hypothesis A that was made at step 3 in a subderivation. That hypothesis can be used only in the subderivation going from step 3 to step 5, so it is not allowed in the conclusion.

1	$A \vee B$		
2	$A \Rightarrow C$		
3	$\quad A$		
4	$\quad C$	$\Rightarrow$E, 2, 3	**WRONG**
5	$\quad B$		
6	$\quad C$	R, 4	
7	C	$\vee$E, 1, 3–4, 5–6	

The mistake here is similar to the previous one: the proposition C derived in step 4 belongs to the subderivation 3-4, so we can't copy it in step 6 because the vertical line has been broken.

1	$A \Rightarrow C$		
2	$B \Rightarrow D$		
3	$A \vee B$		
4	$\quad A$		
5	$\quad C$	$\Rightarrow$E, 1, 4	**WRONG**
6	$\quad B$		
7	$\quad D$	$\Rightarrow$E, 2, 6	
8	$C \wedge D$	$\vee$E, 3, 4–5, 6–7	

Here the rule for disjunction elimination has been applied erroneously: when we eliminate a disjunction $A \lor B$ we must make two distinct subderivations with the assumption A and B, respectively; the conclusion of the two subderivations must be the same. Here we have two different conclusions, so nothing can be inferred from them.

1.7 Forward and Backward Thinking

Finding a valid derivation from a set of hypotheses to a conclusion can be a challenging puzzle. We'll see later that there is a way to determine for sure if it is possible or not, at least for propositional logic. But when we add quantifiers to our logical operators, the problem can't be solved automatically anymore. So proving is a sort of art.

In general you have to guess what rules to apply and you may need to try several possibilities. To help you play the right moves in this game, I suggest two possible strategies, called *forward reasoning* and *backward reasoning*.

In forward reasoning we start with the hypotheses and we try to derive all the interesting consequences that follow from them with simple applications of the elimination rules. Here is an example:

1	$A \Rightarrow (B \Rightarrow C)$	
2	$A \land B$	
3	A	$\land$E, 2
4	$B \Rightarrow C$	$\Rightarrow$E, 1, 3
5	B	$\land$E, 2
6	C	$\Rightarrow$E, 4, 5
7	...	

If we're lucky, sooner or later we'll hit the conclusion that we're looking for. But often there are many possible rules that we can apply and we can't see a clear path towards the conclusion.

That's when backward reasoning becomes useful. This time we look at the conclusion first and we try to see what introduction rules could lead to it. To apply those rules we need to have their hypotheses: we write them in the derivation and then apply backward reasoning to them as well. For example, suppose we're trying to prove the conclusion $C \land ((A \Rightarrow D) \Rightarrow D)$. Since the main connective is a conjunction, we try to apply the conjunction introduction rule. This will require that the formulas C and $(A \Rightarrow D) \Rightarrow D$ have been proved. So we write them in the derivation like so (we use letters n and m for the line numbers, since we still don't know how long the proof is going to be):

$\vdots$	$\dots$	
n	C	
$\vdots$	$\dots$	
m	$(A \Rightarrow D) \Rightarrow D$	
$m+1$	$C \wedge ((A \Rightarrow D) \Rightarrow D)$	$\wedge$I, n, m

We must now fill in the blanks by applying backward reasoning to the line m. Since it is an implication, we'll use the implication introduction rule.

$\vdots$	$\dots$	
n	C	
p	$A \Rightarrow D$ (assumption)	
$\vdots$	$\dots$	
q	D	
m	$(A \Rightarrow D) \Rightarrow D$	$\Rightarrow$I, p–q
$m+1$	$C \wedge ((A \Rightarrow D) \Rightarrow D)$	$\wedge$I, n, m

At this point, the goals that still need to be solved are at lines n and q. They're atomic formulas, so we can't deconstruct them further using backwards reasoning. We'll have to use forward reasoning to complete the proof.

Now you should be able to put together the two fragments of derivation, backward and forward, to obtain a full proof of the conclusion from the hypotheses.

In general, the best strategy consists in starting with backward reasoning from the conclusion; deconstruct it as much as possible using introduction rules; once we reduced it to atomic formulas, start forward reasoning from the hypotheses to prove those.

1.8 Cats and Gorillas

Our goal is to solve, using propositional logic, the following puzzle from Ian Stewart, *Professor Stewart's Cabinet of Mathematical Curiosities*. It asks to determine whether we can logically derive a certain conclusion from a set of five assumptions.

Suppose you know that the following sentences are true:

- *No cat that wears a heron suit is unsociable.*

- ✐ *No cat without a tail will play with a gorilla.*
- ✐ *Cats with whiskers always wear heron suits.*
- ✐ *No sociable cat has blunt claws.*
- ✐ *No cats have tails unless they have whiskers.*

Can you then logically conclude that *No cat with blunt claws will play with a gorilla?*

To solve it, we first assign letters to the atomic propositions used as components of the sentences. Suppose we are talking about some generic cat. Each of the atomic propositions states some fact that may be true or false of the cat:

$A :=$ *The cat wears a heron suit.*

$B :=$ *The cat is sociable.*

$C :=$ *The cat has a tail.*

$D :=$ *The cat will play with a gorilla.*

$E :=$ *The cat has whiskers.*

$F :=$ *The cat has blunt claws.*

Then the five assumptions can be translated to propositional formulas:

$\neg(A \wedge \neg B) =$ *No cat that wears a heron suit is unsociable.*

$\neg(\neg C \wedge D) =$ *No cat without a tail will play with a gorilla.*

$E \Rightarrow A =$ *Cats with whiskers always wear heron suits.*

$\neg(B \wedge F) =$ *No sociable cat has blunt claws.*

$\neg C \vee E =$ *No cats have tails unless they have whiskers.*

The conclusion can in turn be formulated as the following propositional formula:

$$\neg(F \wedge D) = \textit{No cat with blunt claws will play with a gorilla.}$$

Now that we have translated our problem to that of finding a formal derivation from five hypotheses to a conclusion, we can use the rules of propositional logic to solve it:

1	$\neg(A \land \neg B)$	
2	$\neg(\neg C \land D)$	
3	$E \Rightarrow A$	
4	$\neg(B \land F)$	
5	$\neg C \lor E$	
6	$F \land D$	
7	F	$\land$E, 6
8	D	$\land$E, 6
9	$\neg C$	
10	$\neg C$	R, 9
11	E	
12	A	$\Rightarrow$E, 3, 11
13	B	
14	$B \land F$	$\land$I, 13, 7
15	$\bot$	$\neg$E, 4, 14
16	$\neg B$	$\neg$I, 13–15
17	$A \land \neg B$	$\land$I, 12, 16
18	$\bot$	$\neg$E, 1, 17
19	$\neg C$	$\bot$E, 18
20	$\neg C$	$\lor$E, 5, 9–10, 11–19
21	$\neg C \land D$	$\land$I, 20, 8
22	$\bot$	$\neg$E, 2, 21
23	$\neg(F \land D)$	$\neg$I, 6–22