# The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN SEMESTER 2010-2011

**MATHEMATICS FOR COMPUTER SCIENTISTS**

Time allowed 1 hour and 30 minutes

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Candidates must NOT start writing their answers until told to do so

**Answer Question 1 and TWO questions of the remaining 4 (Questions 2-5)**

Appendices A and B on pg. 8 and pg. 9 give the rules of propositional logic and Boolean algebra

Marks available for sections of questions are shown in brackets in the right-hand margin.

No calculators are permitted in this examination.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

**DO NOT turn examination paper over until instructed to do so**

**Question 1:** (25)

For the following questions, each correct answer gives 1 mark. Every incorrect or blank answer receives a negative mark of -1.

(a) Given the following three propositional variables, defined as statements in English: (5)

$A =$ *Too many cooks spoil the broth.*
$B =$ *A little learning is a dangerous thing.*
$C =$ *A leopard can change its spots.*

Translate the following two English sentences into propositional formulas:

(i) *If a leopard cannot change its spots, then too many cooks spoil the broth.*

(ii) *A little learning is a dangerous thing only if a leopard can change its spots or too many cooks don't spoil the broth*

In the following two formulas, put parentheses around all the subformulas, according to the precedence rules for the connectives:

(iii) $A \vee \neg B \Rightarrow C \wedge D$.

(iv) $A \Rightarrow B \vee \neg C \Rightarrow \neg A$.

(v) Is the propositional formula $(A \vee B) \wedge B \Leftrightarrow B$ a tautology?

(b) Compute the values of the following expressions: (5)

(i) $|\lceil -4.8 \rceil|$

(ii) $\lceil |-4.8| \rceil$

(iii) $\lfloor \lceil 8/3 \rceil / 2 \rfloor$

For each of the following propositions, write whether it is true or false:

(iv) For every real number $x$, $\lfloor x \rfloor < x \Rightarrow x = |x|$.

(v) For every real number $x$, $\lfloor |x| \rfloor = |\lfloor x \rfloor|$.

(c) For each of the following propositions, write if it is true or false (all variables denote natural numbers): (5)

(i) Every natural number divides itself.

(ii) Divisibility is a symmetric relation.

(iii) $(n \mid m) \wedge (h \mid k) \Rightarrow (n + h \mid m + k)$.

(iv) $(i \geq j) \wedge (h < k) \Rightarrow (i + k > j + h)$.

(v) The "greater or equal" relation, $\geq$, is transitive.

(d) Consider the following two sets: (5)

$$A = \{\text{Cairo}, \text{Beijing}, \text{London}, \text{Rome}\}$$
$$B = \{\text{Washington}, \text{London}, \text{Beijing}, \text{Delhi}\}$$

List the elements of the following sets:

(i) $(B \setminus A) \cup (A \setminus B)$

(ii) $\mathcal{P}(A \cap B)$

For each of the following propositions, write if it is true or false for all sets $X$ and $Y$:

(iii) $(X \cap Y) \setminus (X \cup Y) = \varnothing$

(iv) $(Y = Y \setminus X) \Rightarrow (X \setminus Y) = X$

(v) $((Y \setminus X) \cup X) = Y$

(e) Compute the following: (5)

(i) $\sum_{i=0}^{2}(i + 2i^2)$

(ii) $\sum_{i=2}^{0}(i^9)$

(iii) The number of non-empty subsets of $\mathbb{Z}_4$.

(iv) The number of subsets of $\mathbb{Z}_6$ of cardinality 5.

(v) The multiplicative inverse of 4 in $\mathbb{Z}_7$.

**Question 2:**

This question is about propositional logic and Boolean algebra. (25)

(a) Write the truth table for the following formula and state whether it is a tautology or not: (5)

$$(A \Rightarrow B) \vee (\neg A \Rightarrow \neg B).$$

(b) Complete the following derivation: (10)

$$\begin{array}{r|ll}
1 & A \Rightarrow B \wedge C & \\
2 & \underline{A \wedge C \Rightarrow \neg B} & \\
\vdots & \vdots & \\
\cdots & \neg A & \qquad\qquad \cdots
\end{array}$$

(c) Using Boolean algebra, prove the following propositional equality, justifying every step by one of the rules: (10)

$$\neg((C \wedge A) \wedge \neg B) = (A \Rightarrow (C \Rightarrow B)).$$

**Question 3:**

This question is about recursion and induction. (25)
Consider the recursive function defined as follows:

$$\begin{aligned} &\mathsf{oldFun}(0) = 0 \\ &\mathsf{oldFun}(n) = \mathsf{oldFun}(n-1) + n(n+1) \qquad \text{if } n > 0 \end{aligned}$$

(a) Compute the following values of $\mathsf{oldFun}$: (5)

$$\begin{gathered} \mathsf{oldFun}(1) \\ \mathsf{oldFun}(2) \\ \mathsf{oldFun}(3) \\ \mathsf{oldFun}(4) \end{gathered}$$

(b) Prove by induction that the following property holds for every natural number $n$: (10)

$$P(n): \qquad \mathsf{oldFun}(n) = \frac{n(n+1)(n+2)}{3}.$$

(c) Complete the following recursive definition (replace the question marks with two natural numbers): (10)

$$\begin{aligned} &\mathsf{newFun}(0) = 3 \\ &\mathsf{newFun}(1) = 4 \\ &\mathsf{newFun}(n) = ? \cdot \mathsf{newFun}(n-1) - ? \cdot \mathsf{newFun}(n-2) \qquad \text{if } n > 1 \end{aligned}$$

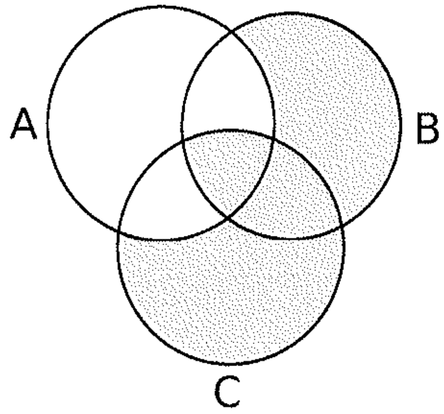
knowing the following values of the function:

$$\begin{aligned} \mathsf{newFun}(2) &= 6 \\ \mathsf{newFun}(3) &= 10 \\ \mathsf{newFun}(4) &= 18 \\ \mathsf{newFun}(5) &= 34 \end{aligned}$$

**Question 4:**

This question is about sets and functions. (25)

(a) Let $A$, $B$ and $C$ be three sets. Write a set expression, using the union, intersection and difference operators, that describes the shaded area in the following Venn diagram: (10)




(b) Now take the three sets $A$, $B$ and $C$ to be defined as follows: (5)

$$A = \{n \in \mathbb{N} \mid n^2 \leq 100\}$$
$$B = \{n \in \mathbb{N} \mid 5 \text{ divides } n\}$$
$$C = \{n \in \mathbb{N} \mid n \text{ divides } 60\}$$

Which of the following numbers belong to the set that you wrote down in part (a)?

$$3, 5, 7, 10, 12, 13, 25, 30, 31, 100$$

(c) Consider the function: (10)

$$f : \mathbb{Z}_3 \rightarrow \mathbb{Z}_3$$
$$f(x) = x^3 + x + 2$$

(i) Is the function a bijection?

(ii) If the answer to (i) is 'yes', write down the inverse of $f$ by giving its values on every element of $\mathbb{Z}_3$.

If the answer to (i) is 'no', give a counterexample (either an element of $\mathbb{Z}_3$ that is not in the image of $f$, or two elements that have the same image through $f$).

**Question 5:**

This question is about combinatorics and modular arithmetic. (25)
Consider the following bijective function:

$$
\begin{array}{ll}
f : \mathbb{Z}_6 \to \mathbb{Z}_6 & \\
f(0) = 2 & f(3) = 1 \\
f(1) = 4 & f(4) = 3 \\
f(2) = 5 & f(5) = 0
\end{array}
$$

(a) Write down the inverse of $f$, $f^{-1}$. (4)

(b) Write down a function $g$ such that, for every $x \in \mathbb{Z}_6$, (8)

$$g(x) \oplus f(x) = x^2.$$

(c) What is the smallest number $n$ such that $f^n = \mathsf{id}$? (5)

(d) The Quack Health Centre has three medical doctors: Dr Asclepius, Dr Hippocrates and Dr Galenus. 10 people are in the waiting room. The nurse has to choose 4 persons to send to Dr Asclepius and 3 persons to send to Dr Hippocrates; the remaining 3 people will see Dr Galenus. How many different ways does the nurse have to make the choice? (8)