G52MAL 2012/13: Lecture 15

Recursive-Descent Parsing: Predictive Parsing

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These lectures:

- The problem of choice revisited.
- Predictive Parsing and LL(1) grammars.
- Computation of First and Follow Sets.
- Left factoring

Recap: Recursive-Descent Parsing

Recursive-descent parsing is an example of the top-down parsing method:

- One ***parsing function*** associated with each nonterminal:

```
parseX :: [Token] -> Maybe [Token]
```

- Each function tries to derive a prefix of the current input according to the productions for the nonterminal in question.
- Function for other nonterminals are invoked recursively as needed.

Recap: Handling Choice

We also need a way to handle ***choice***, as in

$$S \rightarrow AB \mid CD$$

- Looking at the ***next input symbol*** is sometimes enough:

$$S \rightarrow aB \mid cD$$

- If not, ***all alternatives*** could be explored through ***backtracking***:

```
parseX :: [Token] -> [[Token]]
```


Predictive Parsing (1)

Today, we are going to look into exactly when the next input symbol, a one symbol ***lookahead***, can be used to make ***all*** parsing decisions.

We note that this can be the case even if the RHSs start with nonterminals:

$$\begin{aligned} S &\rightarrow AB \mid CD \\ A &\rightarrow a \mid b \\ C &\rightarrow c \mid d \end{aligned}$$

Predictive Parsing (2)

- ***Predictive parsing*** is an example of recursive descent parsing where ***no*** backtracking is needed.
- The grammar must be such that the next input symbol uniquely determines the next production to use.

Productions: $X \rightarrow \alpha \mid \beta$

```
parseX (t : ts) =
    | t ??        -> parse α
    | t ??        -> parse β
    | otherwise -> Nothing
```


Predictive Parsing (3)

How to make the choices? Idea:

- Compute the set of terminal symbols that can start strings derived from each alternative, the ***first set***.
- If there is a choice between two or more alternatives, insist that the first sets for those are ***disjoint***.
- The right choice can now be made simply by determining to which alternative's first set the next input symbol belongs.

Predictive Parsing (4)

Productions: $X \rightarrow \alpha \mid \beta$

```
parseX (t : ts) =
    | t ∈ first(α) -> parse α
    | t ∈ first(β) -> parse β
    | otherwise   -> Nothing
```


Predictive Parsing (5)

Again, consider: $X \rightarrow \alpha \mid \beta$
What if e.g. $\beta \stackrel{*}{\Rightarrow} \epsilon$?

Clearly, the next input symbol could be a terminal that can ***follow*** a string derivable form X!

```
parseX (t : ts) =
    | t ∈ first(α)              -> parse α
    | t ∈ first(β) ∪ follow(X) -> parse β
    | otherwise   -> Nothing
```

The branches must be mutually exclusive!

First and Follow Sets (1)

Following (roughly) “the Dragon Book” [ASU86]

For a CFG $G = (N,\ T,\ P,\ S)$:

$$\begin{aligned} \mathrm{first}(\alpha) &= \{a \in T \mid \alpha \overset{*}{\underset{G}{\Rightarrow}} a\beta\} \\ \mathrm{follow}(A) &= \{a \in T \mid S \overset{*}{\underset{G}{\Rightarrow}} \alpha A a \beta\} \\ &\quad \cup \{\$ \mid S \overset{*}{\underset{G}{\Rightarrow}} \alpha A\} \end{aligned}$$

where we assume $\alpha,\ \beta \in (N \cup T)^*$, $A \in N$, and where $\$$ is a special “end of input” marker.

First and Follow Sets (2)

Consider:

$$\begin{aligned} S &\rightarrow ABC & B &\rightarrow b \mid \epsilon \\ A &\rightarrow aA \mid \epsilon & C &\rightarrow c \mid d \end{aligned}$$

$$\begin{aligned} \mathrm{first}(C) &= \{c,\ d\} \\ \mathrm{first}(B) &= \{b\} \\ \mathrm{first}(A) &= \{a\} \\ \mathrm{first}(S) &= \mathrm{first}(ABC) \\ &= [\text{because } A \overset{*}{\Rightarrow} \epsilon \text{ and } B \overset{*}{\Rightarrow} \epsilon] \\ &\quad\ \mathrm{first}(A) \cup \mathrm{first}(B) \cup \mathrm{first}(C) \\ &= \{a,\ b,\ c,\ d\} \end{aligned}$$

First and Follow Sets (3)

Same grammar:

$$\begin{aligned} S &\rightarrow ABC & B &\rightarrow b \mid \epsilon \\ A &\rightarrow aA \mid \epsilon & C &\rightarrow c \mid d \end{aligned}$$

Follow sets:

$$\begin{aligned} \mathrm{follow}(C) &= \{\$\} \\ \mathrm{follow}(B) &= \mathrm{first}(C) = \{c,\ d\} \\ \mathrm{follow}(A) &= [\text{because } B \overset{*}{\Rightarrow} \epsilon] \\ &\quad\ \mathrm{first}(B) \cup \mathrm{first}(C) \\ &= \{b,\ c,\ d\} \end{aligned}$$

LL(1) Grammars (1)

Consider all productions for a nonterminal A in some grammar:

$$A \ \rightarrow\ \alpha_1 \mid \alpha_2 \mid \ldots \mid \alpha_n$$

In the parsing function for A, on input symbol t, we parse according to α_i if $t \in \mathrm{first}(\alpha_i)$.

If $\alpha_i \overset{*}{\Rightarrow} \epsilon$, we should parse according to α_i also if $t \in \mathrm{follow}(A)$!

LL(1) Grammars (2)

Thus, if:

- $\mathrm{first}(\alpha_i) \cap \mathrm{first}(\alpha_j) = \emptyset$ for $1 \leq i < j \leq n$, and
- if $\alpha_i \overset{*}{\Rightarrow} \epsilon$ for some i, then, for all $1 \leq j \leq n$, $j \neq i$,
  - $\alpha_j \overset{*}{\not\Rightarrow} \epsilon$, and
  - $\mathrm{follow}(A) \cap \mathrm{first}(\alpha_j) = \emptyset$

then it is always clear what do do!

A grammar satisfying these conditions is said to be an ***LL(1)*** grammar.

Nullable Nonterminals (1)

In order to compute the first and follow sets for a grammar $G = (N,\ T,\ P,\ S)$, we first need to know all nonterminals $A \in N$ such that $A \overset{*}{\Rightarrow} \epsilon$; i.e. the set $N_\epsilon \subseteq N$ of ***nullable*** nonterminals.

Let $\mathrm{syms}(\alpha)$ denote the ***set*** of symbols in a string α:

$$\begin{aligned} \mathrm{syms} &\in (N \cup T)^* \rightarrow \mathcal{P}(N \cup T) \\ \mathrm{syms}(\epsilon) &= \emptyset \\ \mathrm{syms}(X\alpha) &= \{X\} \cup \mathrm{syms}(\alpha) \end{aligned}$$

Nullable Nonterminals (2)

The set N_ϵ is the ***smallest*** solution to the equation

$$N_\epsilon \ = \ \{A \mid A \rightarrow \alpha \in P \ \wedge\ \forall X \in \mathrm{syms}(\alpha)\ .\ X \in N_\epsilon\}$$

(Note that $A \in N_\epsilon$ if $A \rightarrow \epsilon \in P$.)

We can then define a predicate $\mathrm{nullable}$ on ***strings*** of grammar symbols:

$$\begin{aligned} \mathrm{nullable} &\in (N \cup T)^* \rightarrow \mathrm{Bool} \\ \mathrm{nullable}(\epsilon) &= \mathrm{true} \\ \mathrm{nullable}(X\alpha) &= X \in N_\epsilon \wedge \mathrm{nullable}(\alpha) \end{aligned}$$

Nullable Nonterminals (3)

The equation for N_ϵ can be solved iteratively as follows:

1. Initialize N_ϵ to $\{A \mid A \rightarrow \epsilon \in P\}$.
2. If there is a production $A \rightarrow \alpha$ such that $\forall X \in \mathrm{syms}(\alpha)\ .\ X \in N_\epsilon$, then add A to N_ϵ.
3. Repeat step 2 until no further nullable nonterminals can be found.

Nullable Nonterminals (4)

Consider the following grammar:

$$\begin{aligned} S &\rightarrow ABC \mid AB & B &\rightarrow b \mid \epsilon \\ A &\rightarrow aA \mid BB & C &\rightarrow c \mid d \end{aligned}$$

- Because $B \rightarrow \epsilon$ is a production, $B \in N_\epsilon$.
- Because $A \rightarrow BB$ is a production and $B \in N_\epsilon$, additionally $A \in N_\epsilon$.
- Because $S \rightarrow AB$ is a production, and $A, B \in N_\epsilon$, additionally $S \in N_\epsilon$.
- No more production with nullable RHS. The set of nullable symbols $N_\epsilon = \{S, A, B\}$.

Computing First Sets (1)

For a CFG $G = (N, T, P, S)$, the sets $\mathrm{first}(A)$ for $A \in N$ are the smallest sets satisfying:

$$\begin{aligned} \mathrm{first}(A) &\subseteq T \\ \mathrm{first}(A) &= \bigcup_{A \to \alpha \in P} \mathrm{first}(\alpha) \end{aligned}$$

Computing First Sets (2)

For strings, first is defined as (note the ***overloaded*** notation):

$$\begin{aligned} \mathrm{first} &\in (N \cup T)^* \to \mathcal{P}(T) \\ \mathrm{first}(\epsilon) &= \emptyset \\ \mathrm{first}(a\alpha) &= \{a\} \\ \mathrm{first}(A\alpha) &= \mathrm{first}(A) \cup \begin{cases} \mathrm{first}(\alpha), & \text{if } A \in N_\epsilon \\ \emptyset, & \text{if } A \notin N_\epsilon \end{cases} \end{aligned}$$

where $a \in T$, $A \in N$, and $\alpha \in (N \cup T)^*$.

Computing First Sets (3)

The solutions can often be obtained directly by expanding out all definitions.

If necessary, the equations can be solved by iteration in a similar way to how N_ϵ is computed.

However, note that the smallest solution to set equations of the type

$$A = A \cup B$$

is simply

$$A = B$$

Computing First Sets (4)

Consider (again):

$$\begin{aligned} S &\to ABC & \quad B &\to b \mid \epsilon \\ A &\to aA \mid \epsilon & \quad C &\to c \mid d \end{aligned}$$

First compute the nullable nonterminals: $N_\epsilon = \{A, B\}$.

Then compute first sets:

$$\begin{aligned} \mathrm{first}(A) &= \mathrm{first}(aA) \cup \mathrm{first}(\epsilon) \\ &= \{a\} \cup \emptyset = \{a\} \end{aligned}$$

Computing First Sets (5)

$$\begin{aligned} S &\to ABC & \quad B &\to b \mid \epsilon \\ A &\to aA \mid \epsilon & \quad C &\to c \mid d \end{aligned}$$

$$\begin{aligned} \mathrm{first}(B) &= \mathrm{first}(b) \cup \mathrm{first}(\epsilon) \\ &= \{b\} \cup \emptyset = \{b\} \end{aligned}$$

$$\begin{aligned} \mathrm{first}(C) &= \mathrm{first}(c) \cup \mathrm{first}(d) \\ &= \{c\} \cup \{d\} = \{c, d\} \end{aligned}$$

Computing First Sets (6)

$$\begin{aligned} S &\to ABC & \quad B &\to b \mid \epsilon \\ A &\to aA \mid \epsilon & \quad C &\to c \mid d \end{aligned}$$

$$\begin{aligned} \mathrm{first}(S) &= \mathrm{first}(ABC) \\ &= [A \in N_\epsilon] \\ &\quad\ \mathrm{first}(A) \cup \mathrm{first}(BC) \\ &= [B \in N_\epsilon \wedge C \notin N_\epsilon] \\ &\quad\ \mathrm{first}(A) \cup \mathrm{first}(B) \cup \mathrm{first}(C) \cup \emptyset \\ &= \{a\} \cup \{b\} \cup \{c, d\} = \{a, b, c, d\} \end{aligned}$$

Computing Follow Sets (1)

For a CFG $G = (N, T, P, S)$, the sets $\mathrm{follow}(A)$ are the smallest sets satisfying:

- $\{\$\} \subseteq \mathrm{follow}(S)$
- If $A \to \alpha B \beta \in P$, then $\mathrm{first}(\beta) \subseteq \mathrm{follow}(B)$
- If $A \to \alpha B \beta \in P$, and $\mathrm{nullable}(\beta)$ then $\mathrm{follow}(A) \subseteq \mathrm{follow}(B)$

$A, B \in N$, and $\alpha, \beta \in (N \cup T)^*$.

(It is assumed that there are no ***useless*** symbols; i.e., all symbols can appear in the derivation of some sentence.)

Computing Follow Sets (2)

$$\begin{aligned} S &\to ABC & \quad B &\to b \mid \epsilon \\ A &\to aA \mid \epsilon & \quad C &\to c \mid d \end{aligned}$$

Constraints for $\mathrm{follow}(S)$:

$$\{\$\} \subseteq \mathrm{follow}(S)$$

Constraints for $\mathrm{follow}(A)$ (note: $\neg\mathrm{nullable}(BC)$):

$$\begin{aligned} \mathrm{first}(BC) &\subseteq \mathrm{follow}(A) \\ \mathrm{first}(\epsilon) &\subseteq \mathrm{follow}(A) \\ \mathrm{follow}(A) &\subseteq \mathrm{follow}(A) \end{aligned}$$

Computing Follow Sets (3)

$$\begin{aligned} S &\to ABC & \quad B &\to b \mid \epsilon \\ A &\to aA \mid \epsilon & \quad C &\to c \mid d \end{aligned}$$

Constraints for $\mathrm{follow}(B)$ (note: $\neg\mathrm{nullable}(C)$):

$$\mathrm{first}(C) \subseteq \mathrm{follow}(B)$$

Constraints for $\mathrm{follow}(C)$ (note: $\mathrm{nullable}(\epsilon)$):

$$\begin{aligned} \mathrm{first}(\epsilon) &\subseteq \mathrm{follow}(C) \\ \mathrm{follow}(S) &\subseteq \mathrm{follow}(C) \end{aligned}$$

Computing Follow Sets (4)

In general:

$$A \subseteq C \ \wedge \ B \subseteq C \quad \Longleftrightarrow \quad A \cup B \subseteq C$$

Also, constraints like $A \subseteq A$ are trivially satisfied and can be omitted.
The constraints can thus be written as:

$$\begin{aligned} \{\$\} &\subseteq \mathrm{follow}(S) \\ \mathrm{first}(BC) \cup \mathrm{first}(\epsilon) &\subseteq \mathrm{follow}(A) \\ \mathrm{first}(C) &\subseteq \mathrm{follow}(B) \\ \mathrm{first}(\epsilon) \cup \mathrm{follow}(S) &\subseteq \mathrm{follow}(C) \end{aligned}$$

Computing Follow Sets (5)

Using

$$\begin{aligned} \mathrm{first}(\epsilon) &= \emptyset \\ \mathrm{first}(C) &= \{c, d\} \\ \mathrm{first}(BC) &= \mathrm{first}(B) \cup \mathrm{first}(C) \cup \emptyset \\ &= \{b\} \cup \{c, d\} = \{b, c, d\} \end{aligned}$$

the constraints can be simplified further:

$$\begin{aligned} \{\$\} &\subseteq \mathrm{follow}(S) \\ \{b, c, d\} &\subseteq \mathrm{follow}(A) \\ \{c, d\} &\subseteq \mathrm{follow}(B) \\ \mathrm{follow}(S) &\subseteq \mathrm{follow}(C) \end{aligned}$$

Computing Follow Sets (6)

Looking for the smallest sets satisfying these constraints, we get:

$$\begin{aligned} \mathrm{follow}(S) &= \{\$\} \\ \mathrm{follow}(A) &= \{b, c, d\} \\ \mathrm{follow}(B) &= \{c, d\} \\ \mathrm{follow}(C) &= \mathrm{follow}(S) = \{\$\} \end{aligned}$$

LL(1), Left-Recursion, Ambiguity (1)

No left-recursive or ambiguous grammar can be LL(1)! For example, consider:

$$A \to Aa \mid \beta$$

First assume $\mathrm{first}(\beta) \neq \emptyset$.

Note that

- $\mathrm{first}(\beta) \subseteq \mathrm{first}(A)$
- $\mathrm{first}(A) \subseteq \mathrm{first}(Aa)$
  ($\mathrm{first}(A) = \mathrm{first}(Aa)$ if $A \not\overset{*}{\Rightarrow} \epsilon$)
- Thus $\mathrm{first}(Aa) \cap \mathrm{first}(\beta) \neq \emptyset$. Not LL(1)!

LL(1), Left-Recursion, Ambiguity (2)

Now assume $\mathrm{first}(\beta) = \emptyset$

This can only be the case if $\beta \overset{*}{\Rightarrow} \epsilon$ and nothing else.

Assuming $S \overset{*}{\Rightarrow} \alpha A \gamma$, we note

- $a \in \mathrm{first}(Aa)$ because $A \Rightarrow \beta \overset{*}{\Rightarrow} \epsilon$, and
- $a \in \mathrm{follow}(A)$ because $A \overset{*}{\Rightarrow} \alpha A \gamma \Rightarrow \alpha A a \gamma$
- Because $\beta \overset{*}{\Rightarrow} \epsilon$, the LL(1) conditions require that $\mathrm{first}(Aa)$ and $\mathrm{follow}(A)$ be disjoint. But that is clearly not the case!

Left Factoring (1)

Left factoring means factoring out a common prefix among a group of productions. This can help making a grammar suitable for predictive recursive descent parsing.

Example:

$$S \ \to \ aXbY \mid aXbYcZ$$

Not suitable for predictive parsing!

But note common prefix! Let's try to postpone the choice!

Left Factoring (2)

Before left factoring:

$$S \ \to \ aXbY \mid aXbYcZ$$

After left factoring:

$$\begin{aligned} S &\to aXbYS' \\ S' &\to \epsilon \mid cZ \end{aligned}$$

Now suitable for predictive parsing!