G51APS, Algorithmic Problem Solving
Coursework 2, 2011/2012

School of Computer Science
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Abstract

This document details the second coursework for the module G51APS in the academic year 2011/2012. The coursework does not count towards the final assessment but, if submitted by the deadline, will be marked and returned for feedback purposes. A substantial proportion of the (unseen 2-hour) examination will be based on the two courseworks. Marks shown are indicative of the marks that would be awarded in a written examination. A record will be taken of submitted work; non-submission may result in your being assumed to have withdrawn from the course.

Answer ALL questions.

1 Games

In these questions, we use the general rules for matchstick games. (A game is lost when a player cannot move, and, at each turn, matches may only be removed from one pile.)

(i) (5 marks) A game is played with a single pile of matches. In each move, 1, 2 or 6 matches may be removed. For each n, $0 \leq n < 21$, determine whether a pile of n matches is a winning or losing position.

(ii) (5 marks) For each n, $0 \leq n < 21$, determine the mex number of a pile of matches (using the rule stated in part (i)).

(iii) (2 marks) If the mex number of a position in a game is known, it is possible to determine whether or not it is a winning position. How is this done? Explain your answer: use the $\forall$-$\exists$ and $\exists$-$\forall$ characterisations of losing and winning positions.

(iii) (3 marks) A game is played with a single pile of matches. A move is to remove at least 1 and at most M matches. What is the mex number of the position where there are n matches in the pile?

(iv) (10 marks) Consider a game which is the sum of two games. In the left game, 1, 2, 3 or 4 matches may be removed at each turn. In the right game, 1, 2 or 6 matches may be removed. In the sum game, a move is made by choosing to play in the left game, or choosing to play in the right game.

The table below shows a number of different positions in this game. A position is given by a pair of numbers: the number of matches in the left pile, and the number of matches in the right pile.

Left Game	Right Game	"losing" or winning move
10	20	?
13	20	?
12	5	?
5	5	?
37	43	?

Table 1: Fill in entries marked "?"

For each position, state whether it is a winning or a losing position. For winning positions, give the winning move in the form Xm where "X" is one of "L" (for "left game") or "R" (for right game), and m is the number of matches to be removed. You may assume that the pattern you observe in your solution to (iii) is repeated for all larger numbers of matches.

2 Tower of Hanoi

(a) Construct the state-transition diagram for the Tower of Hanoi problem for the case that there are 3 disks. (See section 5 for the general method.)

(10)

(b) How many states are there in the state-transition diagram when the number of disks is n? Give an inductive proof of your answer. **(5)**

(c) As discussed in the lectures, an inductive solution to the Tower of Hanoi problem is as follows.

$$\begin{aligned} H_0(d) &= [] \\ H_{n+1}(d) &= H_n(\neg d)\,;\,[\langle n{+}1\ ,\ d\rangle]\,;\,H_n(\neg d) \end{aligned}$$

($H_n(d)$ prescribes how to move the n smallest disks one-by-one from one pole to its neighbour in the direction d, following the rule of never placing a larger disk on top of a smaller disk.)

Suppose $K(m,n,d)$ counts the number of times that disk m moves when the n smallest disks are moved one step in the direction d. Write down inductive equations for $K(m,0,d)$,

$K(n{+}1\,,n{+}1\,,d)$ and for $K(m\,,n{+}1\,,d)$ when $m \neq n{+}1$. What is the value of $K(m,n,d)$ when $m > n$? Use induction to show that $K(m,n,d) = 2^{n-m}$ when $1 \leq m \leq n$. The right side of this equation does not depend on d. Explain in words what this means. **(10)**

3 Bridge Problem

A group of n people wish to cross a bridge. It is dark, and it is necessary to use a torch when crossing the bridge, but they only have one torch between them. The bridge is narrow and only two people can be on it at any one time. Each person takes a different amount of time to cross the bridge; when two cross together they must proceed at the speed of the slowest. The people are all numbered from 1 to n , and person i takes t_i minutes to cross. The torch must be ferried back and forth across the bridge, so that it is always carried when the bridge is crossed.

Assume that the people all start at the left bank (so they wish to cross to the right bank). Below, we use the term *trip* to mean a crossing in either direction (from left to right or from right to left); a *forward* trip is a trip from left to right and a *return* trip is a trip from right to left. Assume that the minimal number of trips is used. That is, assume that there are two people in each forward trip and one person in each return trip.

(a) Suppose f and r count the number of forward and return trips, respectively. Suppose k counts the number of people on the right bank. Write an assignment statement that shows how k and f change when a forward trip is made; write an assignment statement that shows how k and r change when a return trip is made. **(4)**

(b) Show that $k + r - 2 \times f$ is an invariant of both assignment statements. (Give full details of the calculation showing explicitly how the assignment axiom is used.) What is its initial value? **(4)**

(c) What is the relation between f and r when the torch is on the right bank? What conclusion can you draw about the number of people that have not made a return trip when the torch is on the right bank? **(2)**

(d) When 3 people and the torch are at the right bank, how many people have made a return trip? Who would you choose to return in order to optimise the total travel time? What is the optimal value of the total travel time? Assume that the travel time of person i is $t.i$ and $t.1 \leq t.2 \leq t.3$. Prove that the value you give is indeed optimal. **(5)**

(e) Suppose we use (d) as the basis of an inductive algorithm to get $2n{+}3$ people across the bridge as fast as possible. For the inductive step, suppose there are $2(n{+}1){+}3$ people and consider the two slowest people. What does (c) suggest? State two ways to get the two slowest people across the bridge and then return the torch to the left bank. Give a criterion for choosing between the two ways. (Assume that person i has travel time $t.i$ minutes and the people are numbered in increasing order of travel time.) **(5)**

(f) Suppose there are 7 people with crossing times 1, 4, 5, 6, 7, 8, 9. Use your solution to (e) to construct a sequence of trips that gets all across as fast as possible. Explain your solution. **(5)**

4 What to Submit and When

Your solutions should be submitted to the School Office by ***3.00pm on Friday, 2nd December***. Feedback on this coursework will be given before the end of the term.

5 Tower of Hanoi Transition Diagram

This section gives an inductive construction of the state-transition diagram for the Tower of Hanoi problem. Additional explanation is given in the solution to exercise 8.2 in the textbook.

The state of an n-disk puzzle is specified by a sequence of n pole names. The kth name in the sequence is the location (pole name) of disk k. Since each disk may be on one of three poles we conclude that there are 3^n different states in the n-disk problem.

Now we consider the transitions between states.

When there are no disks there is exactly one state: the state when there are no disks on any of the poles. There are no transitions.

We now explain how to construct the state-transition diagram for the $(n+1)$-disk problem, for an arbitrary n, given that we have constructed the diagram for the n-disk problem. (See fig. 1.) Each state is a sequence of $n+1$ pole names. The first n names specify the location of the smallest n disks and the $(n+1)$th specifies the location of the largest disk. Thus, each state in the state-transition diagram for the n-disk problem gives rise to 3 states in the state-transition diagram for the $(n+1)$-disk problem. That is, a state in the state-transition diagram for the $(n+1)$-disk problem is specified by a sequence of n pole numbers followed by the pole name A, B or C. We split the permissible moves into two sets: those where the largest disk (the disk numbered $n+1$) is moved and those where a disk other than the largest disk is moved.

Consider first moving a disk other than the largest disk. When doing so, the largest disk may be on pole A, B or C. But its position doesn't affect the permissibility or otherwise of a move of a smaller disk. That means that every transition from state s to state t in the n-disk problem is also a valid transition from state sp to state tp in the $(n+1)$-disk problem, where the pole name p is either A, B or C. The first step in the construction of the state-transition diagram for the $(n+1)$-disk problem given the state-transition diagram for the n-disk problem is to make three copies of the latter. The pth copy is then modified by simply adding p at the end of each sequence of pole numbers labelling the nodes.

Now consider moving the largest disk, the disk numbered $n+1$. Being the largest disk it may only be moved if all the other disks are on one and the same pole different to the pole that the largest disk is on. This gives six possibilities for moving disk $n+1$, or three edges in the undirected state-transition diagram: an edge connecting the states A^nB and A^nC, an edge connecting the states B^nC and B^nA and an edge connecting the states C^nA and C^nB. The construction is shown schematically in fig. 1, the three inner triangles representing the set of all moves that do not move disk $n+1$.

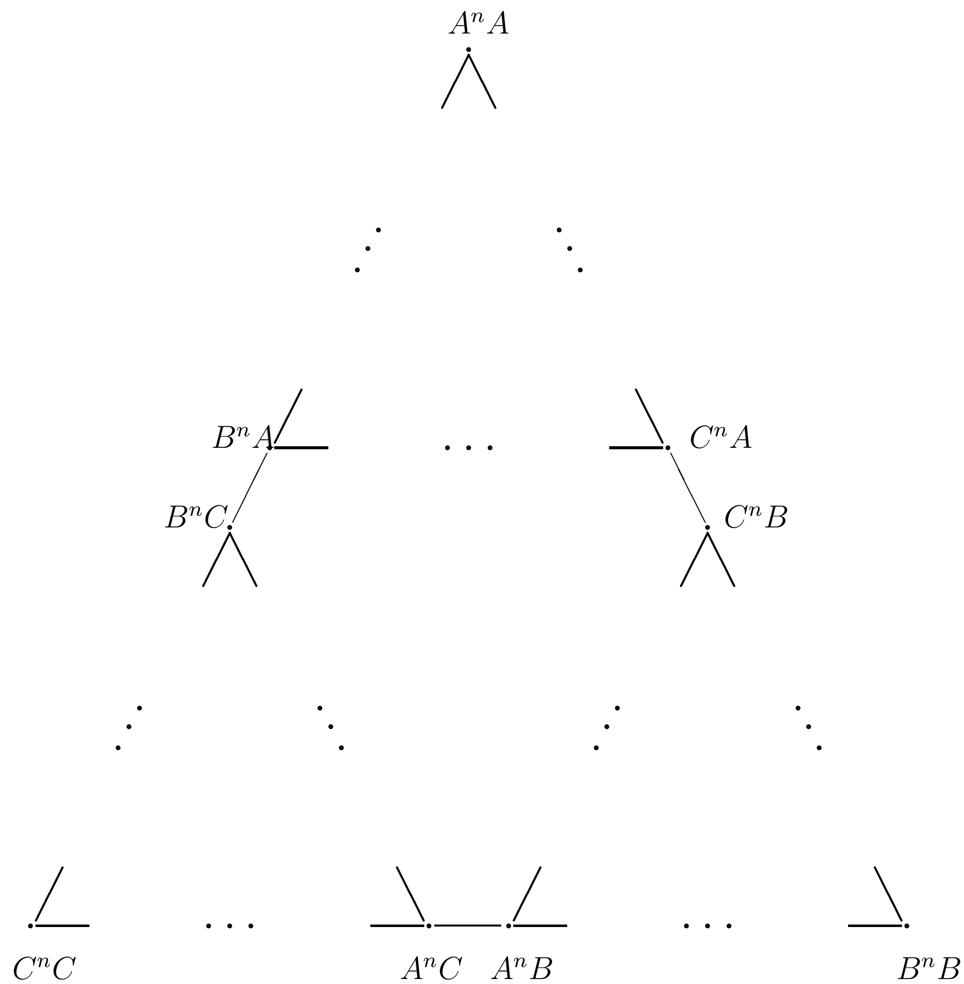

Figure 1: Construction of the state-transition diagram for the $(n+1)$-disk problem