University of Nottingham

SCHOOL OF COMPUTER SCIENCE
AND INFORMATION TECHNOLOGY

A LEVEL 1 MODULE, SPRING SEMESTER 2000-2001

PROGRAMMING ALGEBRA
(Course G53PAL)
Solutions)

Question 1:

a)

$$L \;=\; \mu\langle X\!::\,\{a\}XXX \cup \{b\}\rangle$$

b)

$$\#_b w \;=\; 2\times\#_a w + 1$$

c) Let M be the set of all words w such that $\#_b w \;=\; 2\times\#_a w + 1$. We have to show that $L\subseteq M$. We use fixed point induction:

$$
\begin{array}{ll}
 & L\subseteq M \\
\Leftarrow & \{\quad \text{by definition } L=\mu f \text{ where } f=\langle X\!::\,\{a\}XXX\cup\{b\}\rangle, \\
 & \quad\;\; \text{induction: (??)} \quad \} \\
 & \{a\}MMM\cup\{b\} \subseteq M \\
\equiv & \{\quad \text{set theory, definition of concatenation} \quad \} \\
 & \forall\langle x,y,z\!:\, x\in M\wedge y\in M\wedge z\in M\!:\, axyz\in M\rangle \;\wedge\; b\in M \\
\equiv & \{\quad \text{definition of } M \quad \} \\
 & \quad \forall\;\langle\quad x,y,z \\
 & \quad\quad :\quad \#_b x = 2\times\#_a x + 1 \;\wedge\; \#_b y = 2\times\#_a y + 1 \;\wedge\; \#_b z = 2\times\#_a z + 1 \\
 & \quad\quad :\quad \#_b(axyz) = 2\times\#_a(axyz) + 1 \\
 & \quad\quad \rangle \\
 & \wedge\; \#_b b \;=\; 2\times\#_a b + 1 \\
\equiv & \{\quad \text{definition of } \#_a \text{ and } \#_b \quad \} \\
 & \quad \mathsf{true} \\
 & \wedge\; \forall\;\langle\quad x,y,z \\
 & \quad\quad :\quad \#_b x = 2\times\#_a x + 1 \;\wedge\; \#_b y = 2\times\#_a y + 1 \;\wedge\; \#_b z = 2\times\#_a z + 1 \\
 & \quad\quad :\quad \#_b x+\#_b y+\#_b z = 2\times(\#_a x+\#_a y+\#_a z+1)+1 \\
 & \quad\quad \rangle \\
\equiv & \{\quad \text{arithmetic} \quad \}
\end{array}
$$

$$\mathsf{true} \quad .$$

d) Generalise the problem to $w{\in}L^k$. Then,

$$w{\in}L^0 \;\equiv\; w = \varepsilon \quad .$$

Also,

$$w{\in}L^{k+1} \;\equiv\; (\mathsf{fst}.w = b \wedge \mathsf{rest}.w \in L^k) \vee (\mathsf{fst}.w = a \wedge \mathsf{rest}.w \in L^{k+3})$$

So we get the program

$$w\,,k \;:=\; W\,,1$$

$\{$ Invariant: $W \in L \;\equiv\; w \in L^k$.
Bound function: length of w $\}$

; $\mathsf{do}\;\; \neg(w = \varepsilon \vee k = 0) \;\rightarrow\;\; \mathsf{if}\;\; \mathsf{fst}.w = b \;\rightarrow\; w\,,k \;:=\; \mathsf{rest}.w\,,k{-}1$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \Box\;\; \mathsf{fst}.w = a \;\rightarrow\; w\,,k \;:=\; \mathsf{rest}.w\,,k{+}2$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{fi}$
od

$\{\;\; (w = \varepsilon \vee k = 0) \wedge (W \in L \;\equiv\; w \in L^k) \;\;\}$

$\{\;\; W \in L \;\equiv\; w = \varepsilon \wedge k = 0 \;\;\}$

Question 2:

a)

$$\mathsf{root}.n \geq m \;\equiv\; m^2 \leq n \quad .$$

b) We have, for all m,

$$\mathsf{root}.\lfloor x \rfloor \geq m$$

$\equiv$ $\{$ definition of root $\}$

$$m^2 \leq \lfloor x \rfloor$$

$\equiv$ $\{$ definition of $\lfloor x \rfloor$ $\}$

$$m^2 \leq x$$

$\equiv$ $\{$ property of $\sqrt{x}$ $\}$

$$m \leq \sqrt{x}$$

$\equiv$ $\{$ definition of $\lfloor x \rfloor$ $\}$

$$m \leq \lfloor \sqrt{x} \rfloor$$

The property follows by indirect equality.

c) The property is immediate from $\lfloor n \rfloor = n$ for all natural numbers n.

d) We have, for all m,

$$
\begin{array}{ll}
 & \mathsf{root}.(n^2) \geq m \\
\equiv & \{ \quad \text{definition of } \mathsf{root} \quad \} \\
 & m^2 \leq n^2 \\
\equiv & \{ \quad \text{property of squaring} \quad \} \\
 & m \leq n \\
\equiv & \{ \quad \text{converse of } \leq \quad \} \\
 & n \geq m
\end{array}
$$

The property follows by indirect equality.

The property $\mathsf{root}.(m \times n) \;=\; \mathsf{root}.m \times \mathsf{root}.n$ is not true. Take $m = n = 2$.

We have, for all k,

$$
\begin{array}{ll}
 & \mathsf{root}.(m \downarrow n) \geq k \\
\equiv & \{ \quad \text{definition of } \mathsf{root} \quad \} \\
 & k^2 \leq m \downarrow n \\
\equiv & \{ \quad \text{property of minimum} \quad \} \\
 & k^2 \leq m \wedge k^2 \leq n \\
\equiv & \{ \quad \text{first two steps reversed} \quad \} \\
 & \mathsf{root}.m \downarrow \mathsf{root}.n \;\geq\; k
\end{array}
$$

The property follows by indirect equality.

We have, for all k,

$$
\begin{array}{ll}
 & \mathsf{root}.(m \uparrow n) \geq k \\
\equiv & \{ \quad \text{definition of } \mathsf{root} \quad \} \\
 & k^2 \leq m \uparrow n \\
\equiv & \{ \quad \text{property of maximum} \quad \} \\
 & k^2 \leq m \vee k^2 \leq n \\
\equiv & \{ \quad \text{first two steps reversed} \quad \} \\
 & \mathsf{root}.m \uparrow \mathsf{root}.n \;\geq\; k
\end{array}
$$

The property follows by indirect equality.

(Note difference in these two proofs is in the second step — conjunction replaced by disjunction.)

Question 3:

a) The generalisation is from a regular grammar to a context-free grammar. The invariant property is that, for nodes x in D, μx is the shortest distance to x. (Algorithm omitted here.)

b) Each superior function g must satisfy

$$g(x_1, \ldots, x_k) \;\leq\; x_1 \downarrow \ldots \downarrow x_k \quad .$$

Also each must be monotonic in each of its arguments. ("Turning around" the ordering in Knuth's requirement doesn't change the requirement.)

The productions take the form

$$C_i \to w_{ij} \downarrow C_j$$

(Algorithm omitted again.)

c) The superior functions map a (distance,width) pair to a pair, pairs being ordered lexicographically. The productions thus take the form

$$C_i \to g(C_j)$$

where

$$g(d,w) = d_{ij}+d \,, w_{ij} \downarrow w \ .$$

d) The monotonicity requirement is not satisfied. Optimal routes are not necessarily composed of optimal subroutes. Once a road of width w has been traversed, a suboptimal route can be followed so long as the minimum road width is no more than w.