Kleene Algebra
Graphs and Matrices

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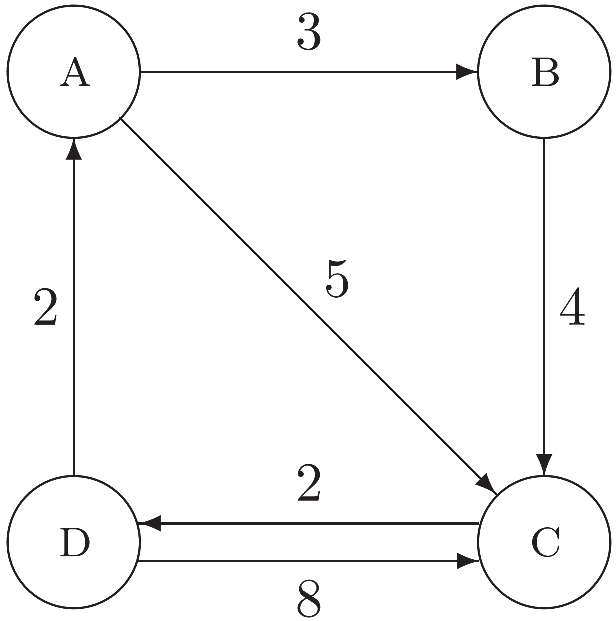
Outline

- Matrices represent graphs
- Addition as choice
- Matrix multiplication as edge concatenation
- Powers as paths of given edge length

Extremal Path Problems



A *graph* consists of a finite set of *nodes*, V, a finite set of *edges*, E and two functions source and target, each with domain E and range V.

A *path* through the graph from node s to node t of *edge length* n is a finite sequence of nodes $x_0, x_1, \ldots, x_n$ such that $s = x_0$ and $t = x_n$ and, for each i, $0 \leq i < n$, there is an edge in the graph from x_i to x_{i+1}. A graph is *labelled* if it is supplied with a function label whose domain is the set of edges, E.

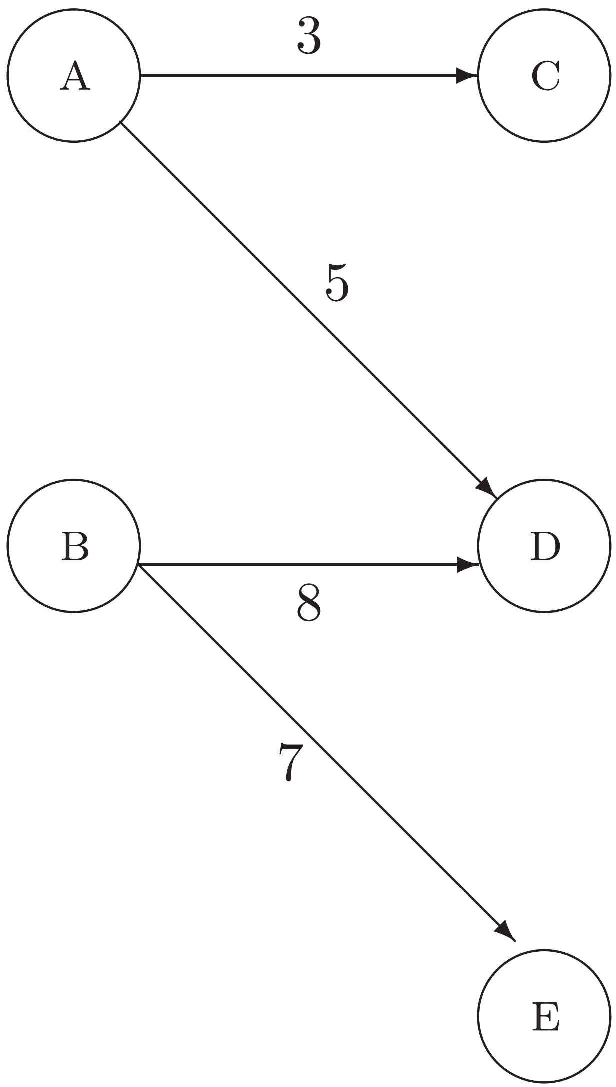
Extremal Path Problems

Edge labels are used to “weight” paths, and the problem is to find the “extreme” weight of paths between given pairs of nodes.

- Reachability – is there a path?
- Shortest or least cost paths.
- Bottleneck problems.
- All paths (considered in a later lecture).

Graphs

An $m \times n$ matrix is the same as an $m \times n$ "bipartite" graph. Eg. a 2×3 matrix:



Matrix Algebra — Addition

Let $\mathbf{A}$ and $\mathbf{B}$ denote two matrices both of dimension $m \times n$. Then the *sum* $\mathbf{A}+\mathbf{B}$ is a matrix of dimension $m \times n$ defined by

$$(\mathbf{A}+\mathbf{B})_{ij} = a_{ij} + b_{ij} \quad .$$

Matrix Algebra — Multiplication

Let $\mathbf{A}$ denote a real matrix of dimension $m \times n$ and let $\mathbf{B}$ denote a real matrix of dimension $n \times p$. Then the *product* $\mathbf{A} \cdot \mathbf{B}$ of the two matrices is a matrix of dimension $m \times p$ where

$$(\mathbf{A} \cdot \mathbf{B})_{ij} \quad = \quad \langle \Sigma k : 0 \leq k < n : a_{ik} \cdot b_{kj} \rangle \quad .$$

Properties

For all matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of appropriate dimensions,

$$\begin{aligned}
\mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A} \\
\mathbf{A}+(\mathbf{B}+\mathbf{C}) &= (\mathbf{A}+\mathbf{B})+\mathbf{C} \\
(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} &= \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C}) \\
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} \\
(\mathbf{B}+\mathbf{C})\cdot\mathbf{A} &= \mathbf{B}\cdot\mathbf{A}+\mathbf{C}\cdot\mathbf{A}
\end{aligned}$$

These properties are inherited from the corresponding properties of the elements.

Note that product is not symmetric.

Zero and Unit Matrices

For each pair of natural numbers m and n there is a *zero* matrix of dimension $m \times n$ whose (i,j)th entry is 0 for all i and j. Denote zero matrices by $\mathbf{0}$ leaving the dimension to be deduced from the context.

$$\mathbf{A} + \mathbf{0} = \mathbf{A} = \mathbf{0} + \mathbf{A}$$

$$\mathbf{A} \cdot \mathbf{0} = \mathbf{0} = \mathbf{0} \cdot \mathbf{A}$$

For each natural number m there is a *unit* matrix of dimension $m \times m$ whose (i,j)th entry is 0 whenever $i \neq j$ and is 1 whenever $i = j$. Denote unit matrices by $\mathbf{1}$, again leaving the dimension to be deduced from the context.

$$\mathbf{A} \cdot \mathbf{1} = \mathbf{A} = \mathbf{1} \cdot \mathbf{A}$$

Idempotent Addition

If addition at the element level is idempotent, then addition of matrices is idempotent.

$$\langle \forall a :: a+a=a \rangle \quad \Rightarrow \quad \langle \forall \mathbf{A} :: \mathbf{A}+\mathbf{A}=\mathbf{A} \rangle \quad .$$

The inherited ordering relation on matrices is the so-called *pointwise* ordering.

$$\mathbf{A} \mathrel{\dot{\leq}} \mathbf{B} \quad \equiv \quad \langle \forall i,j :: \mathbf{A}_{ij} \leq \mathbf{B}_{ij} \rangle$$

Powers

For $m \times m$ matrices, powers are well-defined.

$$\mathbf{A}^0 = \mathbf{1}$$

$$\mathbf{A}^{n+1} = \mathbf{A} \cdot \mathbf{A}^n$$

$\mathbf{A}^n$ represents paths through the graph $\mathbf{A}$ of edge-length n.

The (i,j)th element of $\mathbf{A}^n$ is the sum over all paths p of edge-length n from node i to node j of the weight of path p.