Iteration (Kleene Star)

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October 15, 2002

Outline

- Axioms
- Calculational Properties
- Graphs and Matrices

Iteration ("Kleene star")

$a^* \cdot b$ is a prefix point of the function mapping x to $b + a \cdot x$:

$$b + a \cdot (a^* \cdot b) \ \leq \ a^* \cdot b \quad ,$$

and is the least among all such prefix points:

$$a^* \cdot b \leq x \quad \Leftarrow \quad b + a \cdot x \ \leq \ x \quad .$$

$b \cdot a^*$ is a prefix point of the function mapping x to $b + x \cdot a$:

$$b + (b \cdot a^*) \cdot a \ \leq \ b \cdot a^* \quad , \tag{1}$$

and is the least among all such prefix points:

$$b \cdot a^* \leq x \ \Leftarrow \ b + x \cdot a \ \leq \ x \quad . \tag{2}$$

Interpretations

Languages a, b, c, $\ldots$ x, y and z are sets of words.

a^* is the set of all words formed by repeated concatenation of words in the language a.

Relations a, b, c, $\ldots$ x, y and z are binary relations on some set A. (That is, a set of pairs of elements of A.)

a^* is the reflexive, transitive closure of a. That is, a^* is the smallest relation that contains a and is reflexive and transitive. (Thus a^* is the smallest preorder containing a.)

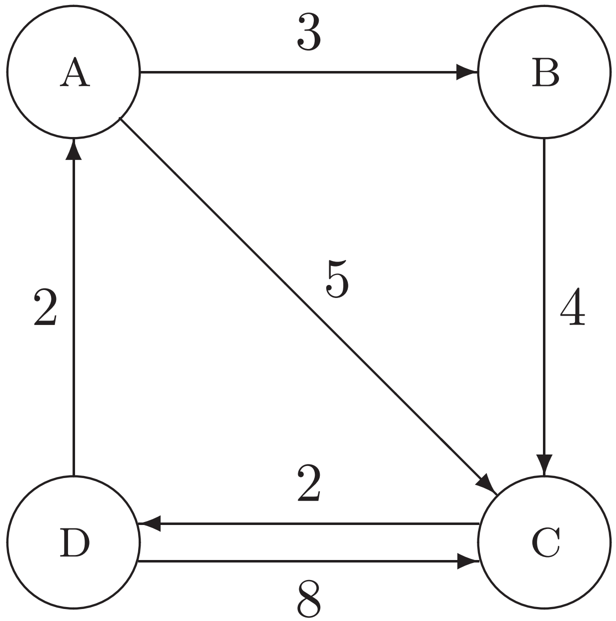
Booleans (The interpretation of $+$ is disjunction, the interpretation of $\cdot$ is conjunction.)

a^* is true for all booleans a.

Costs (The interpretation of $+$ is minimum, the interpretation of $\cdot$ is (real) addition.)

a^* is $\mathsf{0}$ for all nonnegative a. a^* is $-\infty$ for negative a.

Extremal Path Problems



Edge labels are used to "weight" paths, and the problem is to find the "extreme" weight of paths between given pairs of nodes.

Interpretations — Matrices

Suppose $\mathbf{A}$ is a square matrix representing the edges in a labelled graph. Suppose the edge labels are elements of a Kleene algebra.

$\mathbf{A}^*$ represents paths through the graph $\mathbf{A}$ of arbitrary (finite) edge length.

The (i,j)th element of $\mathbf{A}^*$ is the Kleene sum over all finite-length paths p from node i to node j of the weight of path p (the Kleene product of the path's edge labels).

Interpretations

- Boolean matrices. (Kleene addition is "or", multiplication is "and".

  Assume that the (i,j)th element of $\mathbf{A}$ is true exactly when there is an edge in the graph represented by $\mathbf{A}$ from node i to node j.

  The (i,j)th element of $\mathbf{A}^*$ is true exactly when there is a path of arbitrary edge-length from node i to node j.

- Cost matrices. (Kleene addition is "minimum", Kleene multiplication is (real) addition.)

  Assume that the (i,j)th element of $\mathbf{A}$ is the cost of the edge from node i to node j.

  The (i,j)th element of $\mathbf{A}^*$ is the least cost of going from node i to node j.

- Height matrices. (Kleene addition is "maximum", Kleene multiplication is "minimum".)

  Assume that the (i,j)th element of $\mathbf{A}$ is the height of an underpass on the road from node i to node j.

  The (i,j)th element of $\mathbf{A}^*$ is the height of the lowest underpass on the best route from node i to node j.

Properties

reflexivity

$$1 \leq a^* \quad ,$$

transitivity

$$a^* = a^* \cdot a^* \quad ,$$

closure operator

$$a \leq b^* \equiv a^* \leq b^* \quad .$$

Further Properties

leapfrog

$$a \cdot b^* \leq c^* \cdot a \;\Leftarrow\; a{\cdot}b \leq c{\cdot}a \quad ,$$

$$c^* \cdot a \leq a \cdot b^* \;\Leftarrow\; c{\cdot}a \leq a{\cdot}b \quad ,$$

$$a \cdot b^* = c^* \cdot a \;\Leftarrow\; a{\cdot}b = c{\cdot}a \quad ,$$

mirror

$$a \cdot (b{\cdot}a)^* \;=\; (a{\cdot}b)^* \cdot a \quad ,$$

decomposition

$$(a{+}b)^* \;=\; b^* \cdot (a \cdot b^*)^* \;=\; (b^* \cdot a)^* \cdot b^* \quad ,$$

idempotency

$$(a^*)^* \;=\; a^* \quad .$$

Exercise: Prove the properties not proved in the lectures.