Fixed Points and Prefix Points

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Examples

$$\langle\text{Expression}\rangle \;::=\; \langle\text{Expression}\rangle + \langle\text{Expression}\rangle \;\mid\; (\,\langle\text{Expression}\rangle\,) \;\mid\; \langle\text{Variable}\rangle$$

$$\begin{aligned} \text{fac}.0 &= 1 \\ \text{fac}.n &= n * \text{fac}.(n-1), \text{ for } n > 0. \end{aligned}$$

$$\mathsf{List}\ a \;=\; \mathsf{Nil} \;\mid\; \mathsf{Cons}\ a\ (\mathsf{List}\ a)$$

$$\begin{aligned} &\mathsf{List}([\,]) \;. \\ &\mathsf{List}([X|Ys]) \;:-\; \mathsf{List}(Ys) \;. \end{aligned}$$

Fixed Points

A *fixed point* of an endofunction f is a value x such that

$$x = f.x \quad .$$

(An endofunction is a function whose domain and range are the same.)

Examples

$$\langle \mathsf{Expression} \rangle \quad ::= \quad \langle \mathsf{Expression} \rangle + \langle \mathsf{Expression} \rangle \quad | \quad (\ \langle \mathsf{Expression} \rangle \ ) \quad | \quad \langle \mathsf{Variable} \rangle$$

$\langle \mathsf{Expression} \rangle$ is a fixed point of the function f mapping languages to languages defined by

$$\mathsf{f.x} \quad = \quad \mathsf{x} \cdot \{+\} \cdot \mathsf{x} \cup \{(\} \cdot \mathsf{x} \cdot \{)\} \cup \langle \mathsf{Variable} \rangle \quad .$$

Examples (Continued)

$$\begin{aligned} \mathsf{fac}.0 &= 1 \\ \mathsf{fac}.n &= n * \mathsf{fac}.(n-1), \text{ for } n>0. \end{aligned}$$

Using the notation $\langle x{:}\, x\in\mathsf{Type}{:}\, \mathsf{Exp}\rangle$ for a function that maps a value x of type Type to the value given by expression Exp, we have:

$$\mathsf{fac} = \langle n{:}\ n\in\mathbb{N}{:}\ \textbf{if}\ \ n=0\ \textbf{then}\ \ 1\ \ \textbf{else}\ \ n*\mathsf{fac}.(n-1)\rangle\ \ .$$

Now, abstracting from fac on the right side of this equation, define the function $\mathcal{F}$ by

$$\mathcal{F} = \langle f{:}\ f\in\mathbb{N}\leftarrow\mathbb{N}{:}\ \langle n{:}\, n\in\mathbb{N}{:}\,\textbf{if}\ \ n=0\ \textbf{then}\ \ 1\ \ \textbf{else}\ \ n*f.(n-1)\rangle\rangle\ \ .$$

Then

$$\mathsf{fac} = \mathcal{F}.\mathsf{fac}\ \ .$$

Prefix Points

$$\mathsf{List}([\,])\quad .$$
$$\mathsf{List}([X|Ys])\quad :-\quad \mathsf{List}(Ys)\quad .$$

means, for all Xs,

$$\mathsf{List}.Xs \;\Leftarrow\; Xs = [\,] \;\vee\; \langle \exists X,Ys :: Xs = [X|Ys] \wedge \mathsf{List}.Ys \rangle \quad .$$

This is an “if” not an “is”. I.e. not an equality.

Let f be an endofunction on a partially ordered set $(\mathcal{A}, \leq)$.

A *prefix point* of f is a value $x \in \mathcal{A}$ such that

$$f.x \leq x \quad .$$

Prefix Points — Example

$$\mathsf{List.Xs} \;\Leftarrow\; \mathsf{Xs} = [\,] \;\vee\; \langle \exists \mathsf{X,Ys} :: \mathsf{Xs} = [\mathsf{X}|\mathsf{Ys}] \;\wedge\; \mathsf{List.Ys} \rangle \quad .$$

List is a prefix point of the function f mapping predicates to predicates and defined by

$$(\mathsf{f.p}).\mathsf{Xs} \;=\; (\mathsf{Xs} = [\,] \;\vee\; \langle \exists \mathsf{X,Ys} :: \mathsf{Xs} = [\mathsf{X}|\mathsf{Ys}] \;\wedge\; \mathsf{p.Ys} \rangle) \quad .$$

The ordering relation on predicates is $\dot{\Rightarrow}$ where

$$p \mathrel{\dot{\Rightarrow}} q \;=\; \forall \langle x :: p.x \Rightarrow q.x \rangle \quad .$$

Note: this is a so-called *pointwise lifting* of the $\Rightarrow$ ordering on booleans. It is common to omit the dot and write $p \Rightarrow q$. The other boolean operators are lifted to predicates in the same way. In each case, it is common to "overload" the operator symbol by omitting the dot.

Ordering Relations

A binary relation R on a set $\mathcal{A}$ is a *partial ordering* if it is

reflexive: for all $x \in \mathcal{A}$,

$$x \; R \; x \quad ,$$

transitive: for all $x, y, z \in \mathcal{A}$,

$$x \; R \; y \wedge y \; R \; z \;\Rightarrow\; x \; R \; z \quad ,$$

anti-symmetric: for all $x, y \in \mathcal{A}$,

$$x \; R \; y \wedge y \; R \; x \;\Rightarrow\; x = x \quad .$$

A *preordering* is a reflexive and transitive binary relation.

Symbols like $\preceq$, $\leq$ and $\sqsubseteq$ will be used to denote partial orderings.

Least Fixed Points

Assume f is an endofunction on the partially ordered set $(\mathcal{A}\,,\ \leq)$.

A least fixed point of f, denoted $\mathsf{lfp.f}$, is a value x that is a fixed point of f and is least among all fixed points of f. Formally, $\mathsf{lfp.f}$ is characterised by:

$$\mathsf{lfp.f} \;=\; \mathsf{f.(lfp.f)}$$

($\mathsf{lfp.f}$ is a fixed point of f) and, for all $\mathsf{x}{\in}\mathcal{A}$,

$$\mathsf{lfp.f} \leq \mathsf{x} \;\;\Leftarrow\;\; \mathsf{x} = \mathsf{f.x}$$

($\mathsf{lfp.f}$ is at most any fixed point of f).

Least Prefix Points

Assume f is an endofunction on the partially ordered set $(\mathcal{A}, \leq)$.

A least prefix point of f, denoted $lpp.f$, is a value x that is a prefix point of f and is least among all prefix points of f. Formally, $lpp.f$ is characterised by:

$$f.(lpp.f) \leq lpp.f$$

($lfp.f$ is a prefix point of f) and, for all $x \in \mathcal{A}$,

$$lpp.f \leq x \;\Leftarrow\; f.x \leq x$$

($lfp.f$ is at most any prefix point of f).

Tarski's Theorem

Assume f is a *monotonic* endofunction on the partially ordered set $(\mathcal{A}\,,\,\leq)$.

Then

$$\mathsf{lpp.f} \;=\; \mathsf{lfp.f} \quad .$$

It is customary to denote the least fixed point of monotonic function f by $\mu\mathsf{f}$ (or $\mu_{\leq}\mathsf{f}$ if we want to be explicit about the ordering).

$\mu\mathsf{f}$ is characterized by the rules:

computation rule

$$\mu\mathsf{f} = \mathsf{f}.\mu\mathsf{f}$$

induction rule: for all $\mathsf{x}{\in}\mathcal{A}$,

$$\mu\mathsf{f} \leq \mathsf{x} \;\Leftarrow\; \mathsf{f.x} \leq \mathsf{x} \quad .$$

Using (Fixed Point) Induction

Let L be the least solution of the equation

$$\mathsf{X}{::}\ \{\mathsf{a}\}\cup\{\mathsf{b}\}\cdot\mathsf{X}\cdot\mathsf{X} \subseteq \mathsf{X} \quad .$$

Let M be the set of all words w such that the number of a's in w is one more than the number of b's in w. Let $\#_{\mathsf{a}}w$ denote the number of a's in w, and $\#_{\mathsf{b}}w$ denote the number of b's in w.

Then

$$\mathsf{L} \subseteq \mathsf{M} \quad .$$

Proof

$$L \subseteq M$$

$\Leftarrow$ { by definition $L = \mu f$ where $f = \langle X :: \{a\} \cup \{b\} \cdot X \cdot X \rangle$,
induction }

$$\{a\} \cup \{b\} \cdot M \cdot M \ \subseteq \ M$$

$=$ { set theory, definition of concatenation }

$$a \in M \ \wedge \ \langle \forall x,y : x \in M \wedge y \in M : bxy \in M \rangle$$

Proof (Continued)

$$a \in M \ \wedge \ \langle \forall x,y : x \in M \wedge y \in M : bxy \in M \rangle$$

$=$ { definition of M }

$$\#_a a = \#_b a + 1$$

$$\wedge \langle \forall x,y$$

$$: \quad \#_a x = \#_b x + 1 \quad \wedge \quad \#_a y = \#_b y + 1$$

$$: \quad \#_a (bxy) = \#_b (bxy) + 1$$

$$\rangle$$

$=$ { definition of $\#_a$ and $\#_b$ }

$$\mathsf{true}$$

$$\wedge \langle \forall x,y$$

$$: \quad \#_a x = \#_b x + 1 \quad \wedge \quad \#_a y = \#_b y + 1$$

$$: \quad \#_a x + \#_a y = 1 + \#_b x + \#_b y + 1$$

$$\rangle$$

$= \quad \{ \quad \text{arithmetic} \quad \}$

$\quad \mathsf{true} \quad .$