(Impartial two-person) Games

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Outline

We use impartial, two-person games to illustrate the notions of *least* and *greatest* fixed points of a monotonic function.

The winning and losing positions in a game satisfy fixed point equations.

If the equations do not have unique solutions, stalemate is possible.

The positions from which a win is guaranteed are characterised by the least fixed point of a certain equation. The positions from which losing is inevitable (against a perfect player) are also characterised by the least fixed point of a certain equation. The stalemate positions are characterised by greatest fixed points.

Two-person, Impartial Games

A two-person, impartial game is given by a set of *positions* and a *move* relation on the positions.

We use G, H, K, etc. to denote positions. The move relation is denoted by the symbol " $\mapsto$". So, $\mathsf{G} \mapsto \mathsf{H}$ means there is a move from position G to position H.

The players take it in turns to move. The game is lost when no move can be made.

Such games are called *impartial* because the same moves are available to both players. (Games like chess are *partisan*. Players can only move their own pieces.)

Winning and Losing

Let $W.G$ mean that G is a position from which a perfect player is guaranteed to win.

Let $L.G$ mean that G is a position from which losing is inevitable (against a perfect player).

The predicates W and L satisfy the fixed point equations:

$$W = \langle G :: \langle \exists H : G \mapsto H : L.H \rangle \rangle$$

$$L = \langle G :: \langle \forall H : G \mapsto H : W.H \rangle \rangle$$

In words, a winning position is one from which it is always possible to move to a losing position, and a losing position is one from which every move is to a winning position.

These equations need not have unique solutions.

Examples

Matchsticks.

Lollipop.

Conjugate Predicate Transformers

Consider the *predicate transformers* f and g given by

$$\mathsf{f.X} \;=\; \langle \mathsf{G} :: \langle \exists \mathsf{H} : \mathsf{G} \mapsto \mathsf{H} : \mathsf{X.H} \rangle\rangle$$

$$\mathsf{g.X} \;=\; \langle \mathsf{G} :: \langle \forall \mathsf{H} : \mathsf{G} \mapsto \mathsf{H} : \mathsf{X.H} \rangle\rangle$$

Note that

$$W \;=\; (\mathsf{f} \circ \mathsf{g}).W$$

and

$$\mathsf{L} \;=\; (\mathsf{g} \circ \mathsf{f}).\mathsf{L}$$

Predicates are ordered by implication, and f and g are monotonic with respect to this ordering, as are $(\mathsf{f} \circ \mathsf{g})$ and $(\mathsf{g} \circ \mathsf{f})$.

f and g are *conjugates*. That is, for all predicates X,

$$\neg(\mathsf{f.X}) \;=\; \mathsf{g}.(\neg \mathsf{X}) \quad \wedge \quad \neg(\mathsf{g.X}) \;=\; \mathsf{f}.(\neg \mathsf{X})$$

As a consequence, so are $(\mathsf{f} \circ \mathsf{g})$ and $(\mathsf{g} \circ \mathsf{f})$: for all predicates X,

$$\neg((\mathsf{f} \circ \mathsf{g}).\mathsf{X}) \;=\; (\mathsf{g} \circ \mathsf{f}).(\neg \mathsf{X}) \quad \wedge \quad \neg((\mathsf{g} \circ \mathsf{f}).\mathsf{X}) \;=\; (\mathsf{f} \circ \mathsf{g}).(\neg \mathsf{X})$$

Fixed Points of Conjugates

Suppose f and g are monotonic, conjugate predicate transformers.

Let $\mu\mathsf{f}$ denote the least fixed point of f. Let $\nu\mathsf{f}$ denote the greatest fixed point of f. Similarly, for g.

Then,

$$\neg(\mu\mathsf{f}) = \nu\mathsf{g}$$

This has corollary

$$\begin{aligned}
\mu\mathsf{f} \wedge \mu\mathsf{g} &= \mathsf{false} \\
\mu\mathsf{f} \wedge (\nu\mathsf{f} \wedge \nu\mathsf{g}) &= \mathsf{false} \\
\mu\mathsf{g} \wedge (\nu\mathsf{f} \wedge \nu\mathsf{g}) &= \mathsf{false} \\
\mu\mathsf{f} \vee \mu\mathsf{g} \vee (\nu\mathsf{f} \wedge \nu\mathsf{g}) &= \mathsf{true}
\end{aligned}$$

The set of states is thus divided into three disjoint sets, $\mu\mathsf{f}$, $\mu\mathsf{g}$ and $\nu\mathsf{f} \wedge \nu\mathsf{g}$.

Winning, Losing and Stalemate

We can apply the theory above to the predicate transformers

$$f \;=\; \langle X :: \langle G :: \langle \exists H : G \mapsto H : X.H \rangle\rangle\rangle$$

and

$$g \;=\; \langle X :: \langle G :: \langle \forall H : G \mapsto H : X.H \rangle\rangle\rangle$$

defined by an impartial game.

The predicates $\mu(f \bullet g)$, $\mu(g \bullet f)$ and $\nu(f \bullet g) \wedge \nu(g \bullet f)$ are mutually distinct and together cover all positions.

$\mu(f \bullet g)$ characterises the positions from which a win is guaranteed.

$\mu(g \bullet f)$ characterises the positions from which losing is inevitable.

$\nu(f \bullet g) \wedge \nu(g \bullet f)$ characterises stalemate positions.

(All these assume perfect players.)