Raster Display Algorithms

A *raster display* is a two-dimensional grid of *pixels*. The pixels are all squares of equal size. A black-and-white *drawing* is an assignment of booleans to each of the pixels.

A drawing of a curve on a raster display involves approximating functions from reals to reals by functions from integers to integers. Bresenham's algorithms are efficient ways of drawing straight lines, circles and ellipses on a raster display.

This chapter is about the construction of Bresenham's algorithms for drawing straight lines and for drawing circles. The construction provides a good example of calculations with the ceiling and floor functions (see chapter 6). It is also a good, and relatively simple, example of the laws of programming summarised in the Appendix. The principles are made sufficiently clear that additional exercises can easily be set — for example, in rough order of increasing difficulty, drawing a hyperbola, a parabola or an ellipse.

Drawing Straight Lines

Consider the straight line:

$$x,y:: \quad ax - by = 0 \ .$$

For a given real value x, the integer "closest to" x is

$$\lceil x - \tfrac{1}{2} \rceil \ .$$

(When x is $n+\frac{1}{2}$, for some integer n, this chooses n as the "closest" value. Otherwise, it is the integer n that minimises $|n-x|$.)

We approximate the straight line by the function g from integers to integers given by

$$g.m = \left\lceil \frac{am}{b} - \frac{1}{2} \right\rceil \ .$$

Thus, the problem we consider is to compute $g.m$ successively for m equal to 0, 1, 2, $\cdots$. To this end, we construct a loop. We ignore termination of the loop itself. (The loop body must, however, be guaranteed to terminate.) Under these conditions, the following is a valid implementation:

$\{\ \ \mathsf{true}\ \ \}$

$m,n := 0,0\ \ ;$

$\{\ \ \textbf{Invariant:}\qquad n = g.m\ \ \}$

$\mathsf{do\ true} \rightarrow\qquad plot.(m,n)\ \ ;$

$\qquad\qquad\qquad m,n\ :=\ m{+}1\,,g.(m{+}1)$

$\mathsf{od}\quad .$

A major inefficiency is, however, the recomputation of g at each iteration.

Suppose that a and b are integers. If $0 \le a \le b$, g increases by at most one when m is increased by one. Formally,

(5) $\quad 0 \le g.(m{+}1) - g.m \le 1\ \ \Leftarrow\ \ 0 \le a \le b\ \ .$

(We recommend that the reader verify this property before continuing.)

From now on, we assume that $0 \le a \le b$. (This is without loss of generality. If $0 \le b \le a$, simply interchange a and b, and interchange m and n. If one of a or b is negative, use the derived algorithm to compute $-(g.m)$.)

From (5), we have:

$$g.(m{+}1) = g.m\ \ \vee\ \ g.(m{+}1) = g.m + 1\ \ .$$

This means that it is valid to replace the assignment $m,n\ :=\ m{+}1\,,g.(m{+}1)$, in the body of the loop, by the sequential composition:

$\mathsf{if}\quad g.(m{+}1) = g.m+1\ \rightarrow\ n\ :=\ n{+}1$

$\Box\quad g.(m{+}1) = g.m\ \rightarrow\ \mathsf{skip}$

$\mathsf{fi}\ \ ;$

$\{\ \ n = g.(m{+}1)\ \ \}$

$m\ :=\ m{+}1\ \ .$

We now investigate the circumstances in which $g.(m{+}1) = g.m+1$. Since g is defined in terms of the ceiling function, it is easiest to calculate with the equivalent: $g.m < g.(m{+}1)$.

$g.m < g.(m{+}1)$

$=\qquad \{\qquad n = g.m$ is an invariant of the algorithm,

so we may replace the left side by n,

definition of g on the right side $\quad\}$

$$
\begin{array}{ll}
 & n \;<\; \left\lceil \dfrac{a(m{+}1)}{b} - \dfrac{1}{2} \right\rceil \\
= & \quad \{ \quad \text{properties of ceiling} \quad \} \\
 & n \;<\; \dfrac{a(m{+}1)}{b} - \dfrac{1}{2} \\
= & \quad \{ \quad \text{arithmetic} \quad \} \\
 & \dfrac{b}{2} \;<\; a(m{+}1) - bn \\
= & \quad \{ \quad \text{floor function, arithmetic} \quad \} \\
 & 0 \;<\; a(m{+}1) - bn - \left\lfloor \dfrac{b}{2} \right\rfloor \\
= & \quad \{ \quad \text{assume } h \;=\; a(m{+}1) - bn - \left\lfloor \dfrac{b}{2} \right\rfloor \quad \} \\
 & 0 < h \quad .
\end{array}
$$

This calculation suggests that we introduce the integer variable h with the invariant property

(6) $\quad h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \quad .$

The decision whether or not to increment n is determined by the test $0<h$:

$$
\begin{array}{l}
\{ \;\; \textbf{Global Invariants:} \quad \text{integer}.a \wedge \text{integer}.b \wedge 0 \le a \le b \\
\quad \langle \forall m :: \; g.m = \left\lceil \dfrac{am}{b} - \dfrac{1}{2} \right\rceil \;\wedge\; 0 \le g.(m{+}1) - g.m \le 1 \rangle \quad \} \\
m,n,h \;:=\; 0\,,\,0\,,\,a - \lfloor b/2 \rfloor \;\; ; \\
\{ \;\; \textbf{Invariant:} \\
\qquad\quad n = g.m \\
\quad \wedge \quad h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \;\wedge\; (0<h \equiv g.(m{+}1) - g.m = 1) \quad \} \\
\text{do true} \rightarrow \quad plot.(m,n); \\
\qquad\qquad\quad \text{if} \quad 0<h \;\rightarrow \quad \{ \;\; g.(m{+}1) = g.m{+}1 \;\; \} \\
\qquad\qquad\qquad\qquad\qquad\quad n\,,h \;:=\; n{+}1\,,\,p \\
\qquad\qquad\quad \Box \;\; \neg(0<h) \rightarrow \quad \{ \;\; g.(m{+}1) = g.m \;\; \} \\
\qquad\qquad\qquad\qquad\qquad\quad \text{skip} \\
\qquad\qquad\quad \text{fi} \;\; ; \\
\qquad\qquad\quad \{ \;\; n = g.(m{+}1) \;\wedge\; h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \;\; \} \\
\qquad\qquad\quad m\,,h \;:=\; m{+}1\,,\,q \\
\text{od} \quad .
\end{array}
$$

The unknowns in the algorithm are p and q. The requirements on p and q are that they should maintain property (6) invariant. That is, p is required to satisfy

$$\begin{array}{l} \{ \quad h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \quad \} \\ n\,,h \;\;:=\;\; n{+}1\;,\,p \\ \{ \quad h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \quad \} \quad , \end{array}$$

whilst q must satisfy

$$\begin{array}{l} \{ \quad h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \quad \} \\ m\,,h \;\;:=\;\; m{+}1\;,\,q \\ \{ \quad h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \quad \} \quad . \end{array}$$

Applying the assignment axiom, we calculate p to satisfy

$$p \;=\; a(m{+}1) - b(n{+}1) - \lfloor b/2 \rfloor \quad ,$$

under the assumption

$$h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \quad .$$

Clearly,

$$p \;=\; h{-}b \quad .$$

Also, applying the assignment axiom, we calculate q to satisfy

$$q \;=\; a((m{+}1){+}1) - bn - \lfloor b/2 \rfloor \quad ,$$

under the assumption

$$h \;=\; a(m{+}1) - bn + \lfloor b/2 \rfloor \quad .$$

Clearly,

$$q \;=\; h{+}a \quad .$$

Substituting these values for p and q completes the derivation of the algorithm:

$$\begin{array}{l} \{ \;\; \textbf{Global Invariants:} \qquad \mathsf{integer}.a \wedge \mathsf{integer}.b \wedge 0 \leq a \leq b \\ \qquad \langle \forall m \;::\; g.m \;=\; \left\lceil \dfrac{am}{b} - \dfrac{1}{2} \right\rceil \wedge 0 \;\leq\; g.(m{+}1) - g.m \;\leq\; 1 \rangle \quad \} \\ m,n,h \;\;:=\;\; 0\;,\,0\;,\,a - \lfloor b/2 \rfloor \quad ; \\ \{ \;\; \textbf{Invariant:} \end{array}$$

$$n = g.m$$
$$\wedge \quad h \;=\; a(m{+}1) - bn - \lfloor b/2 \rfloor \;\wedge\; (0 < h \equiv g.(m{+}1) - g.m = 1) \quad \}$$

```
do true →   plot.(m,n);
            if  0<h  →     {  g.(m+1) = g.m+1  }
                           n,h := n+1 , h−b
            □  ¬(0<h) →    {  g.(m+1) = g.m  }
                           skip
            fi ;
            {  n=g.(m+1)  ∧  h = a(m+1) − bn − ⌊b/2⌋  }
            m,h := m+1 , h+a
od  .
```

Drawing Circles

The equation of a circle of radius r is

$$x,y{::}\quad x^2 + y^2 \;=\; r^2 \quad .$$

We assume that r is an integer, and consider the computation of the approximating function g, where

$$g.n \;=\; \left\lceil \sqrt{(r^2 - n^2)} \;-\; \tfrac{1}{2} \right\rceil \quad .$$

It is convenient to divide the (x,y) plane into 8 sectors. In the sector given by

$$x,y{::}\quad 0 \leq y \leq x$$

it is the case that the gradient of a circle centred at the origin is at least -1. That is, an increase of y by 1 decreases x by at most 1. Similar properties hold in the other seven sectors. We consider this one sector only, leaving the reader to fill in the details for the other sectors.

Formally, the corresponding property of g is that:

$$0 \leq g.n - g.(n{+}1) \leq 1 \;\Leftarrow\; 0 \leq n < g.n \quad .$$

This suggests an algorithm of the following structure.

```
{   integer.m ∧ integer.n   }
m,n  :=  r,0  ;
{ Invariant:      m = g.n  ∧  (0 ≤ g.n − g.(n+1) ≤ 1 ∨ m = n)
  Bound function:     m−n   }
do n ≤ m  →        plot.(m,n) ;
                   if  g.n − g.(n+1) = 0  →  skip
                   □ g.n − g.(n+1) = 1  →  m := m−1
                   fi  ;
                   n := n+1
od   .
```

Note that it is now n that is continually incremented, whilst m decreases or remains constant at each iteration.

As for drawing a straight line, we aim to replace the tests in the conditional statement by a test on the sign of a variable h, which is incrementally updated. The calculation of the invariant property of h goes as follows.

$$g.(n{+}1) < g.n$$

$=$ { $m = g.n$ is an invariant of the algorithm, so we may replace the right side by m, definition of g on the left side }

$$\left\lceil \sqrt{(r^2-(n{+}1)^2)} - \tfrac{1}{2} \right\rceil < m$$

$=$ { in order to apply the definition of the ceiling function, we need "at most", not "less than". }

$$\left\lceil \sqrt{(r^2-(n{+}1)^2)} - \tfrac{1}{2} \right\rceil \leq m-1$$

$=$ { definition of ceiling }

$$\sqrt{(r^2-(n{+}1)^2)} - \tfrac{1}{2} \leq m-1$$

$=$ { arithmetic (mainly squaring) }

$$r^2 - n^2 - 2n - 1 \leq m^2 - m + \tfrac{1}{4}$$

$=$ { arithmetic }

$$r^2 - n^2 - 2n - m^2 + m \leq \tfrac{5}{4}$$

$=$ { r, m and n are all integers, $\lfloor 5/4 \rfloor = 1$ }

$$r^2 - n^2 - 2n - m^2 + m - 1 \;\le\; 0 \;.$$

This calculation suggests that we introduce the integer variable h with the invariant property

(7) $\quad h \;=\; r^2 - n^2 - 2n - m^2 + m - 1 \;.$

The decision whether or not to decrement m is determined by the test $h \le 0$:

$$
\begin{array}{l}
\{\;\; \textbf{Global Invariant:} \quad \langle \forall n :: g.n \;=\; \left\lceil \sqrt{(r^2 - n^2)} - \tfrac{1}{2} \right\rceil \rangle \;\; \} \\
m,n,h \;:=\; r\,,\,0\,,\,r-1 \;\; ; \\
\{\;\; \textbf{Invariant:} \\
\qquad\qquad m = g.n \;\wedge\; (0 \le g.n - g.(n+1) \le 1 \;\vee\; m = n) \\
\quad \wedge \quad h \;=\; r^2 - n^2 - 2n - m^2 + m - 1 \;\wedge\; (h \le 0 \;\equiv\; g.n - g.(n+1) = 1) \\
\quad \textbf{Bound function:} \quad m - n \;\; \} \\
\mathsf{do}\; n \le m \;\to\; \quad \mathrm{plot}.(m,n) \;\; ; \\
\qquad\qquad\qquad\; \mathsf{if}\;\; h \le 0 \;\to\; \quad \{\;\; g.(n+1) = g.n - 1 \;\; \} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\; m,h \;:=\; m-1\,,u \\
\qquad\qquad\qquad\; \Box\; \neg(h \le 0) \to \quad \{\;\; g.(n+1) = g.n \;\; \} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; \mathsf{skip} \\
\qquad\qquad\qquad\; \mathsf{fi} \;\; ; \\
\qquad\qquad\qquad\; n,h \;:=\; n+1\,,\,v \\
\mathsf{od} \;\; .
\end{array}
$$

Note the decision to change h in two steps, the first as a consequence of decreasing m, and the second as a consequence of increasing n.

The unknowns in this algorithm are u and v. The requirements on u and v are that they should maintain property (7) invariant. That is, u is required to satisfy

$$
\begin{array}{l}
\{\;\; h \;=\; r^2 - n^2 - 2n - m^2 + m - 1 \;\; \} \\
m,h \;:=\; m-1\,,u \\
\{\;\; h \;=\; r^2 - n^2 - 2n - m^2 + m - 1 \;\; \} \;\; ,
\end{array}
$$

whilst v must satisfy

$$
\begin{array}{l}
\{\;\; h \;=\; r^2 - n^2 - 2n - m^2 + m - 1 \;\; \} \\
n,h \;:=\; n+1\,,\,v \\
\{\;\; h \;=\; r^2 - n^2 - 2n - m^2 + m - 1 \;\; \} \;\; .
\end{array}
$$

In order to calculate the appropriate value of u, we note that only the subterm "$-m^2+m$" is changed by an assignment to m. Accordingly, we calculate that, for any k,

$$k-(m-1)^2+(m-1) \;=\; (k-m^2+m)+2(m-1) \quad .$$

Thus, applying the assignment axiom,

$$\{\;\; h \,=\, k-m^2+m \;\;\}$$
$$m,h \;:=\; m-1 \,,\, h+2(m-1)$$
$$\{\;\; h \,=\, k-m^2+m \;\;\}$$

A similar argument is used to calculate v. We have, for all k,

$$k-(n+1)^2-2(n+1) \;=\; (k-n^2-2n)-(2n+3) \quad .$$

So,

$$\{\;\; h \,=\, k-n^2-2n \;\;\}$$
$$n,h \;:=\; n+1 \,,\, h-(2n+3)$$
$$\{\;\; h \,=\, k-n^2-2n \;\;\}$$

Substituting these values for u and v completes the derivation of the algorithm:

```
{ Global Invariant:   ⟨∀n :: g.n = ⌈√(r²−n²) − ½⌉⟩ }
m,n,h := r , 0 , r−1 ;
{ Invariant:
        m = g.n ∧ (0 ≤ g.n − g.(n+1) ≤ 1 ∨ m = n)
   ∧    h = r² − n² − 2n − m² + m − 1 ∧ (h ≤ 0 ≡ g.n − g.(n+1) = 1) }
do n ≤ m →    plot.(m,n) ;
              if h ≤ 0 →       { g.(n+1) = g.n − 1 }
                               m,h := m−1 , h+2(m−1)
              □ ¬(h ≤ 0) →     { g.(n+1) = g.n }
                               skip
              fi ;
              n,h := n+1 , h−(2n+3)
od .
```

Bibliographic Remarks

The development is based on "Drawing Lines, Circles and Ellipses in a Raster" by N.Wirth. In "Beauty Is Our Business", W.H.J.Feijen, A.J.M.van Gasteren, D.Gries, J.Misra (Eds.), Springer-Verlag, 1990, pp.427–434.