Diagonals and Block-Ordered Relations

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Abstract

More than 70 years ago, Jaques Riguet suggested the existence of an "analogie frappante" (striking analogy) between so-called "relations de Ferrers" and a class of difunctional relations, members of which we call "diagonals". Inspired by his suggestion, we formulate an "analogie frappante" linking the notion of a block-ordered relation and the notion of the diagonal of a relation. We formulate several novel properties of the core/index of a diagonal, and use these properties to rephrase our "analogie frappante". Loosely speaking, we show that a block-ordered relation is a provisional ordering up to isomorphism and reduction to its core. (Our theorems make this informal statement precise.) Unlike Riguet (and others who follow his example), we avoid almost entirely the use of nested complements to express and reason about properties of these notions: we use factors (aka residuals) instead. The only (and inevitable) exception to this is to show that our definition of a "staircase" relation is equivalent to Riguet's definition of a "relation de Ferrers". Our "analogie frappante" also makes it obvious that a "staircase" relation is not necessarily block-ordered, in spite of the mental picture of such a relation presented by Riguet.

1 Introduction

More than seventy years ago, in a series of publications [Rig48, Rig50, Rig51], Jacques Riguet introduced the notions of a "relation difonctionelle" and "relations de Ferrers". In [Rig51] he remarked on an "analogie frappante" between these two notions via what he referred to as the "différence" of a given relation. Riguet [Rig51] states the following theorem:

> Pour que R soit une relation de Ferrers, il faut et il suffit que R soit réunion de rectangles dont les projections de même nom sont totalement ordonnées par inclusion et tels que si la première projection de l'un des rectangles est contenue dans la première projection d'un autre rectangle, la seconde projection du second est contenue dans la seconde projection du premier.

For those unable to read French, the theorem states a necessary and sufficient condition for a relation to be "de Ferrers" in terms of totally ordered rectangles ("rectangles … totalement

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ordonnées"). The theorem clearly begs the question what is the definition of a "relation de Ferrers". We postpone answering this question until section 7. The reason for doing so is that Riguet gives both a formal definition and a mental picture —a picture like the one in fig. 1 of what we call a "staircase relation"— but it is far from obvious how Riguet's definition and the mental picture are related.

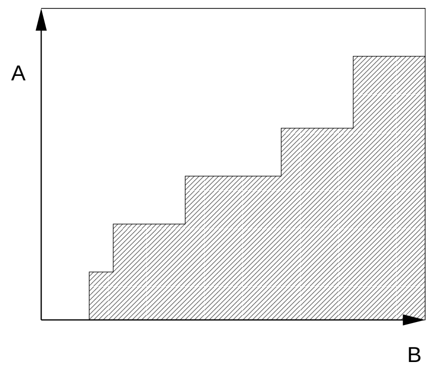

Figure 1: Mental Picture of a Staircase Relation

Riguet does not give a proof of the theorem and his "analogie frappante" between "relations de Ferrers" and difunctional relations is unclear. The proof of Riguet's theorem is quite straightforward: see [Bac21]. However, clarifying the "analogie frappante" is more difficult.

The work presented here initially began as an attempt to properly understand Riguet's work and to rectify errors in extant literature. We introduce, in section 5, the "diagonal" of a relation and, in section 6, the notion of a "block-ordered relation". The "diagonal" of a relation is what Riguet referred to as the "différence"; it is a difunctional relation. We formulate an "analogie frappante" (specifically, theorem 114) linking block-ordered relations and diagonals. We also formalise the notion of a "staircase" relation: as shown in [Bac21], our definition of a "staircase" relation and Riguet's definition of a "relation de Ferrers" are equivalent[1]. Ostensibly a "staircase" relation is "block-ordered" where the "blocks" are totally ordered. But, as we observed in our initial investigations, this is not the case: as we show in section 7, the less-than relation on real numbers is a "staircase" relation but is not "block-ordered".

In addition to our "analogie frappante", a major contribution of our work is the application of the notions of a core/index of a relation. These notions and their properties are briefly summarised in section 4. Informally, a core[2] of a relation captures its essential properties. For example, a core of a difunctional relation is an isomorphism. An index of a relation is a core of the relation that has the same type as the relation. A fuller account of their properties is given in [BV22, BV23]. In this paper, we state and prove a number of properties of the core/index of the diagonal of a relation. These properties are relevant to our analogie frappante: theorem 114 gives a method of testing whether or not a relation is block-ordered

[1]The name "staircase" is more informative than "relation de Ferrers" and our definition uses factors rather than complementation (from which Riguet's terminology "différence" is derived), which is why we prefer to deviate from Riguet's presentation

[2]We say "a" core because a relation typically has many cores; all cores of a relation are, however, isomorphic.

in terms of the diagonal of the relation, and theorem 116 reformulates the test in terms of the core of the relation.

During the course of our investigation, we became aware of very similar work by Winter [Win04]. Winter's notion of "relations of order shape" is identical to our notion of "block-ordered relation". Winter also introduces the notion of the "diagonal" of a relation — but does not give the notion a name and does not attribute the notion to Riguet [Rig51]. (He does cite [Rig51] and he denotes the "diagonal" of a relation R by R^d, his definition of which is identical to the definition of the "différence" of R given by Riguet.) Consequently, there is some overlap between our work and Winter's work. We believe the overlap is justified because we formulate the notion of a "diagonal" in terms of factors ("residuals" in the terminology used by Winter) and we avoid the use of complements altogether — with the exception of section 7 where we observe the equivalence of the notion of a "staircase" relation with Riguet's notion of a "relation de Ferrers". (Winter does express some properties in terms of factors but, following in Riguet's footsteps, his calculations invariably use complements, which he denotes by an overbar.) Winter also observes that not all "staircase" relations are "block-ordered" but does not give any concrete example. See the concluding section for further discussion of this aspect of Winter's paper.

In order to make this paper relatively self-contained, sections 2 and 3 summarise the axioms of (point-free) relation algebra and some basic concepts. Typically, proofs of properties stated in these sections are omitted. Exceptions to this rule concern properties that we deem less familiar to many readers (for example, factorisation properties of functional relations in section 3.4). Section 4 introduces the notions of a core/index of a relation. For proofs of properties stated in this section see [BV22, BV23].

2 Axioms of Point-free Relation Algebra

In this section, we give a brief summary of the axioms of point-free relation algebra. For full details of the axioms, see [BDGv22].

2.1 Summary

Point-free relation algebra comprises three layers with interfaces between the layers plus additional axioms peculiar to relations. The axiom system is typed. For types A and B, $A{\sim}B$ denotes a set; the elements of the set are called *(heterogeneous) relations of type* $A{\sim}B$. Elements of type $A{\sim}A$, for some type A, are called *homogeneous relations*.

The first layer axiomatises the properties of a partially ordered set. We postulate that, for each pair of types A and B, $A{\sim}B$ forms a complete, universally distributive lattice. We use the symbol "$\subseteq$" for the ordering relation, and "$\cup$" and "$\cap$" for the supremum and infimum operators. We assume that this notation is familiar to the reader, allowing us to skip a more detailed account of its properties. However, we use $\perp\!\!\!\perp$ for the least element of the ordering (rather than the conventional $\emptyset$) and $\top\!\!\!\top$ for the greatest element. In keeping with the conventional practice of overloading the symbol "$\emptyset$", both these symbols are overloaded.

It is important to note that there is no axiom stating that a relation is a set, and there is no corresponding notion of membership. The lack of a notion of membership distinguishes point-free relation algebra from pointwise algebra.

The second layer adds a composition operator. If R is a relation of type $A{\sim}B$ and S is a relation of type $B{\sim}C$, the composition of R and S is a relation of type $A{\sim}C$ which we denote by $R{\circ}S$. Composition is associative and, for each type A, there is an identity relation which we denote by I_A. We often overload the notation for the identity relation, writing just I. Occasionally, for greater clarity, we do supply the type information.

The interface between the first and second layers defines a relation algebra to be an instance of a *regular algebra* [Bac06] (also called a *standard Kleene algebra*, or **S**-*algebra* [Con71]). For this paper, the most important aspect of this interface is the existence and properties of the factor operators. These are introduced in section 2.2. Also, $\perp\!\!\!\perp$ is a zero of composition: for all R, $\perp\!\!\!\perp{\circ}R = \perp\!\!\!\perp = R{\circ}\perp\!\!\!\perp$.

The completeness axiom in the first layer allows the reflexive-transitive closure R^* of each element R of type $A{\sim}A$, for some type A, to be defined.

The third layer is the introduction of a *converse* operator. If R is a relation of type $A{\sim}B$, the converse of R, which we denote by $R^{\cup}$ (pronounced R "wok") is a relation of type $B{\sim}A$. The interface with the first layer is that converse is simultaneously the lower and upper adjoint in a Galois connection, and thus a poset isomorphism (in particular, $\perp\!\!\!\perp^{\cup} = \perp\!\!\!\perp$ and $\top\!\!\!\top^{\cup} = \top\!\!\!\top$); the interface with the second layer is formed by the two rules $I^{\cup} = I$ and, for all relations R and S of appropriate type, $(R{\circ}S)^{\cup} = S^{\cup}\circ R^{\cup}$.

Additional axioms characterise properties peculiar to relations. The modularity rule (aka Dedekind's rule [Rig48]) is that, for all relations R, S and T,

(1) $\qquad R{\circ}S \cap T \subseteq R\circ(S \cap R^{\cup}\circ T)$.

The dual property, obtained by exploiting properties of the converse operator, is, for all relations R, S and T,

(2) $\qquad S{\circ}R \cap T \subseteq (S \cap T\circ R^{\cup})\circ R$.

The modularity rule is necessary to the derivation of some of the properties we state without proof (for example, the properties of the domain operators given in section 3.2). Another rule is the *cone rule*:

(3) $\qquad \langle \forall R :: \top\!\!\!\top{\circ}R{\circ}\top\!\!\!\top = \top\!\!\!\top \;\equiv\; R \neq \perp\!\!\!\perp \rangle$.

The cone rule limits consideration to "unary" relation algebras: constructing the cartesian product of two relation algebras to form a relation algebra (whereby the operators are defined pointwise) does not yield an algebra satisfying the cone rule.

2.2 Factors

If R is a relation of type $A{\sim}B$ and S is a relation of type $A{\sim}C$, the relation $R\backslash S$ of type $B{\sim}C$ is defined by the Galois connection, for all T (of type $B{\sim}C$),

(4) $\qquad T \subseteq R\backslash S \;\equiv\; R{\circ}T \subseteq S$.

Similarly, if R is a relation of type $A{\sim}B$ and S is a relation of type $C{\sim}B$, the relation R/S of type $A{\sim}C$ is defined by the Galois connection, for all T,

$$(5)\qquad T \subseteq R/S \;\equiv\; T{\circ}S \subseteq R \ .$$

The existence of factors is a property of a regular algebra; in relation algebra, factors are also known as "residuals". Factors enjoy a rich theory which underlies many of our calculations.

The relations $R\backslash R$ (of type $B{\sim}B$ if R has type $A{\sim}B$) and R/R (of type $A{\sim}A$ if R has type $A{\sim}B$) play a central rôle in section 7. As is easily verified, both are *preorders*. That is, both are *transitive*:

$$R\backslash R\circ R\backslash R \;\subseteq\; R\backslash R \quad\wedge\quad R/R\circ R/R \;\subseteq\; R/R$$

and both are *reflexive*:

$$I \subseteq R\backslash R \quad\wedge\quad I \subseteq R/R \ .$$

(The notation "I" is overloaded in the above equation. In the left conjunct, it denotes the identity relation of type $B{\sim}B$ and, in the right conjunct, it denotes the identity relation of type $A{\sim}A$, assuming R has type $A{\sim}B$. We often overload constants in this way. Note, however, that we do not attempt to combine the two inclusions into one.) In addition, for all R,

$$(6)\qquad R\circ R\backslash R \;=\; R \;=\; R/R\circ R \ ,$$

$$(7)\qquad R/(R\backslash R) \;=\; R \;=\; (R/R)\backslash R \ ,$$

$$(8)\qquad (R\backslash R)/(R\backslash R) \;=\; R\backslash R \;=\; (R\backslash R)\backslash(R\backslash R) \ \text{ and}$$

$$(9)\qquad (R/R)\backslash(R/R) \;=\; R/R \;=\; (R/R)/(R/R) \ .$$

In fact, we don't use (7) directly; its relevance is as the initial step in proving the leftmost equations of (8) and (9). We choose not to exploit the associativity of the over and under operators in (8) and (9) —by writing, for example, $(R\backslash R)/(R\backslash R)$ as $R\backslash R/(R\backslash R)$— in order to emphasise their rôle as properties of the preorders $R\backslash R$ and R/R.

In relation algebra (as opposed to regular algebra) it is possible to eliminate the factor operators altogether because they can be expressed in terms of complements and converse. The rules for doing so are as follows: for all R, S and T,

$$(10)\qquad R\backslash S \;=\; \neg(R^{\cup}\circ\neg S) \quad\wedge\quad S/T \;=\; \neg(\neg S\circ T^{\cup}) \ .$$

$$(11)\qquad R\backslash S/T \;=\; \neg(R^{\cup}\circ\neg S\circ T^{\cup}) \ .$$

Although the elimination of factors is highly undesirable, we are obliged to introduce complements in order to compare our work with that of Riguet in section 7 on staircase relations.

Property (6) exemplifies how much easier calculations with factors can be compared to calculations that combine complements with converses. The property is very easy to spot and apply. Expressed using (11), it is equivalent to

$$R\circ\neg(R^{\cup}\circ\neg R) \;=\; R \;=\; \neg(\neg R\circ R^{\cup})\circ R \ .$$

In this form, the property is difficult to spot and its correct application is difficult to check.

It is useful to record the distributivity properties of converse over the factor operators:

Lemma 12 For all R and S,

(13) $R^{\cup}\backslash S^{\cup} \;=\; (S/R)^{\cup} \;=\; \neg R/\neg S$.

Symmetrically,

(14) $R^{\cup}/S^{\cup} \;=\; (S\backslash R)^{\cup} \;=\; \neg R\backslash\neg S$.

Also,

(15) $(R\backslash S/T)^{\cup} \;=\; T^{\cup}\backslash S^{\cup}/R^{\cup}$.

□

In (13) and (14), the inclusion of terms involving complements is only relevant in section 7.

3 Some Definitions

This section introduces a number of concepts which have been studied in detail elsewhere. For the most part, their properties are stated without proof. An exception to this is in section 3.4 where we combine the Galois connections defining factors with the Galois connection defining functionality.

3.1 Coreflexives

In point-free relation algebra, "coreflexives" of a given type represent sets of elements of that type. A *coreflexive of type* A is a relation p such that $p \subseteq I_A$. Frequently used properties are that, for all coreflexives p,

$$p = p^{\cup} = p{\circ}p$$

and, for all coreflexives p and q,

$$p{\circ}q \;=\; p \cap q \;=\; q{\circ}p \;\; .$$

(The proof of these properties relies on the modularity rule.) In the literature, coreflexives have several different names, usually depending on the application area in question. Examples are "monotype", "pid" (short for "partial identity") and "test".

3.2 The Domain Operators

The "domain operators" (see eg. [BH93]) play a dominant and unavoidable role. We exploit their properties frequently in calculations, so much so that we assume great familiarity with them.

Definition 16 (Domain Operators) Given relation R of type $A{\sim}B$, the *left domain* $R{<}$ of R is a relation of type A defined by the equation

$$R{<} \;=\; I_A \cap R \circ R^{\cup}$$

and the *right domain* $R{>}$ of R is a relation of type B is defined by the equation

$$R{>} \;=\; I_B \cap R^{\cup} \circ R \;\;.$$

□

The name "domain operator" is chosen because of the fundamental properties: for all R and all coreflexives p,

(17) $\quad R = R \circ p \;\;\equiv\;\; R{>} = R{>} \circ p$

and

(18) $\quad R = p \circ R \;\;\equiv\;\; R{<} = p \circ R{<} \;\;.$

A simple, often-used consequence of (17) and (18) is the property:

(19) $\quad R{<} \circ R \;=\; R \;=\; R \circ R{>} \;\;.$

As is the case for factors, the domain operators enjoy a rich theory which underlies many of our calculations but we omit the details here.

3.3 Pers and Per Domains

Given relations R of type $A{\sim}B$ and S of type $A{\sim}C$, the symmetric *right-division* is the relation $R\backslash\!\!\backslash S$ of type $B{\sim}C$ defined in terms of *right* factors as

(20) $\quad R\backslash\!\!\backslash S \;=\; R\backslash S \cap (S\backslash R)^{\cup} \;\;.$

Dually, given relations R of type $B{\sim}A$ and S of type $C{\sim}A$, the symmetric *left-division* is the relation $R/\!\!/S$ of type $B{\sim}C$ defined in terms of left factors as

(21) $\quad R/\!\!/S \;=\; R/S \cap (S/R)^{\cup} \;\;.$

The relation $R\backslash\!\!\backslash R$ is an equivalence relation. Voermans [Voe99] calls it the "greatest right domain" of R. Riguet [Rig48] calls $R\backslash\!\!\backslash R$ the "noyau" of R (but defines it using nested complements). Others (see [Oli18] for references) call it the "kernel" of R.

As remarked elsewhere [Oli18], the *symmetric left-division* inherits a number of (left) cancellation properties from the properties of factorisation in terms of which it is defined. In this section the focus is on the left and right "per domains" introduced by Voermans [Voe99].

Definition 22 (Right and Left Per Domains) The *right per domain* of relation R, denoted $R{\succ}$, is defined by the equation

(23) $\quad R{\succ} \;=\; R{>} \circ R\backslash\!\!\backslash R \;\;.$

Dually, the *left per domain* of relation R, denoted $R{\prec}$, is defined by the equation

(24) $\quad R{\prec} \;=\; R/\!\!/R \circ R{<} \;\;.$

□

In order to prove additional properties, it is useful to record the left and right domains of the relation $R\backslash\!\!\backslash R\circ R{>}$.

Lemma 25 For all R,

$$(R\backslash\!\!\backslash R\circ R{>}){>} \;=\; R{>} \;=\; (R{>}\circ R\backslash\!\!\backslash R){<} \quad ,$$

$$(R\backslash\!\!\backslash R\circ R{>}){<} \;=\; R{>} \;=\; (R{>}\circ R\backslash\!\!\backslash R){>} \quad ,$$

$$R\backslash\!\!\backslash R\circ R{>} \;=\; R{>}\circ R\backslash\!\!\backslash R\circ R{>} \;=\; R{>}\circ R\backslash\!\!\backslash R \quad .$$

□

The left and right per domains are "pers" where "per" is an abbreviation of "partial equivalence relation".

Definition 26 (Partial Equivalence Relation (per)) A relation is a *partial equivalence relation* iff it is symmetric and transitive. That is, R is a partial equivalence relation iff

$$R = R^{\cup} \;\wedge\; R\circ R\subseteq R \quad .$$

Henceforth we abbreviate partial equivalence relation to *per*.
□

That $R{\prec}$ and $R{\succ}$ are indeed pers is a direct consequence of the symmetry and transitivity of $R\backslash\!\!\backslash R$.

The left and right per domains are called "domains" because, like the coreflexive domains, we have the properties: for all relations R and pers P,

(27) $\quad R = R\circ P \;\equiv\; R{\succ} = R{\succ}\circ P$, and

(28) $\quad R = P\circ R \;\equiv\; R{\prec} = P\circ R{\prec}$.

The right per domain $R{\succ}$ can be defined equivalently by the equation

(29) $\quad R{\succ} \;=\; R\backslash\!\!\backslash R\circ R{>}$.

Moreover,

(30) $\quad (R{\succ}){<} \;=\; R{>} \;=\; (R{\succ}){>}$.

(See [Bac21] for the proofs of these properties.) Symmetrical properties hold of $R{\prec}$.

For further properties of pers and per domains, see [Voe99].

3.4 Functionality

In this section, we present a number of lesser-known properties of "functional" relations. A relation R of type $A{\sim}B$ is said to be *left-functional* iff $R \circ R^{\cup} = R{<}$. Equivalently, R is *left-functional* iff $R \circ R^{\cup} \subseteq I_A$. It is said to be *right-functional* iff $R^{\cup} \circ R = R{>}$ (equivalently, $R^{\cup} \circ R \subseteq I_B$). A relation R is said to be a *bijection* iff it is both left- and right-functional.

Rather than left- and right-functional, the more common terminology is "functional" and "injective" but publications differ on which of left- or right-functional is "functional" or "injective". We choose to abbreviate "left-functional" to *functional* and to use the term *injective* instead of right-functional. Typically, we use f and g to denote functional relations, and Greek letters to denote bijections (although the latter is not always the case). Other authors make the opposite choice.

The properties we present here stem from the observation that functionality can be defined via a Galois connection. Specifically, the relation f is (left-)functional iff, for all relations R and S (of appropriate type),

$$(31) \qquad f \circ R \subseteq S \quad \equiv \quad f{>} \circ R \subseteq f^{\cup} \circ S \ .$$

It is a simple exercise to show that (31) is equivalent to the property $f \circ f^{\cup} \subseteq I$. (Although (31) doesn't immediately fit the standard definition of a Galois connection, it can be turned into standard form by restricting the range of the dummy R to relations that satisfy $f{>} \circ R = R$, i.e. relations R such that $R{<} \subseteq f{>}$.)

The converse-dual of (31) is also used frequently: g is functional iff, for all R and S,

$$(32) \qquad R \circ g^{\cup} \subseteq S \quad \equiv \quad R \circ g{>} \subseteq S \circ g \ .$$

Comparing the Galois connections defining the over and under operators with the Galois connection defining functionality (see (31)) suggests a formal relationship between "division" by a functional relation and composition with the relation's converse. The precise form of this relationship is given by the following lemma.

Lemma 33 For all R and all functional relations f,

$$f{>} \circ f \backslash R \ = \ f^{\cup} \circ R \ .$$

Proof We use the anti-symmetry of the subset relation. First,

$$
\begin{array}{ll}
 & f^{\cup} \circ R \ \subseteq \ f{>} \circ f \backslash R \\
= & \quad \{ \quad \text{domains} \quad \} \\
 & f{>} \circ f^{\cup} \circ R \ \subseteq \ f{>} \circ f \backslash R \\
\Leftarrow & \quad \{ \quad \text{monotonicity} \quad \} \\
 & f^{\cup} \circ R \ \subseteq \ f \backslash R \\
= & \quad \{ \quad \text{factors} \quad \} \\
 & f \circ f^{\cup} \circ R \ \subseteq \ R \\
\Leftarrow & \quad \{ \quad \text{definition and monotonicity} \quad \} \\
 & f \text{ is functional} \quad .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & f{>}\circ f\backslash R \;\subseteq\; f^{\cup}\circ R \\
\Leftarrow & \{ \quad f{>} \subseteq f^{\cup}\circ f \,;\text{ monotonicity and transitivity} \quad \} \\
 & f^{\cup}\circ f\circ f\backslash R \;\subseteq\; f^{\cup}\circ R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & f\circ f\backslash R \;\subseteq\; R \\
= & \{ \quad \text{cancellation} \quad \} \\
 & \text{true} \;\; .
\end{array}
$$

□

Two lemmas that will be needed later now follow. Lemma 34 allows the converse of a functional relation (i.e. an injective relation) to be cancelled, whilst lemma 35 expresses a distributivity property.

Lemma 34 For all R and all functional relations f,

$$
f{<}\circ f^{\cup}\backslash(f^{\cup}\circ R) \;=\; f{<}\circ R \;\; .
$$

Proof

$$
\begin{array}{ll}
 & f{<}\circ f^{\cup}\backslash(f^{\cup}\circ R) \\
= & \{ \quad \text{assumption: } f \text{ is functional} \quad \} \\
 & f\circ f^{\cup}\circ f^{\cup}\backslash(f^{\cup}\circ R) \\
\subseteq & \{ \quad \text{cancellation} \quad \} \\
 & f\circ f^{\cup}\circ R \\
= & \{ \quad \text{assumption: } f \text{ is functional} \quad \} \\
 & f{<}\circ R \;\; .
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & f{<}\circ R \subseteq f{<}\circ f^{\cup}\backslash(f^{\cup}\circ R) \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R \subseteq f^{\cup}\backslash(f^{\cup}\circ R) \\
= & \{ \quad \text{factors} \quad \} \\
 & \text{true} \;\; .
\end{array}
$$

The lemma follows by anti-symmetry of the subset relation.

□

Lemma 35 For all R and S and all functional relations f ,

$$
R\backslash(S{\circ}f)\circ f{>} \;=\; R\backslash S\circ f \;\; .
$$

Proof

$$
\begin{array}{ll}
 & R\backslash(S{\circ}f)\circ f{>} \;\subseteq\; R\backslash S\circ f \\
\Leftarrow & \{ \quad f{>} \subseteq f^{\cup}\circ f \,,\text{ monotonicity} \quad \} \\
 & R\backslash(S{\circ}f)\circ f^{\cup} \;\subseteq\; R\backslash S \\
= & \{ \quad \text{factors} \quad \} \\
 & R\circ R\backslash(S{\circ}f)\circ f^{\cup} \;\subseteq\; S
\end{array}
$$

$$\Leftarrow \quad \{ \quad \text{cancellation} \quad \}$$
$$S \circ f \circ f^{\cup} \subseteq S$$
$$= \quad \{ \quad \text{assumption: } f \text{ is functional} \quad \}$$
$$\text{true} \quad .$$

Also,

$$R \backslash S \circ f \subseteq R \backslash (S \circ f) \circ f{>}$$
$$\Leftarrow \quad \{ \quad \text{monotonicity, } f = f \circ f{>} \quad \}$$
$$R \backslash S \circ f \subseteq R \backslash (S \circ f)$$
$$= \quad \{ \quad \text{factors and cancellation} \quad \}$$
$$\text{true} \quad .$$

The lemma follows by anti-symmetry of the subset relation.

□

The following lemma is crucial to fully understanding Riguet's "analogie frappante". (The lemma is complicated by the fact that it has five free variables. Simpler, possibly better known, instances can be obtained by instantiating one or more of f, g, U and W to the identity relation.)

Lemma 36 Suppose f and g are functional. Then, for all U, V and W,

$$f^{\cup} \circ (g{<} \circ U) \backslash V / (W \circ f{<}) \circ g$$
$$= \quad f{>} \circ (g^{\cup} \circ U \circ f) \backslash (g^{\cup} \circ V \circ f) / (g^{\cup} \circ W \circ f) \circ g{>} \quad .$$

Proof Guided by the assumed functionality of f and g, we use the rule of indirect equality. Specifically, we have, for all R, U, V and W,

$$f{>} \circ R \circ g{>} \subseteq f^{\cup} \circ (g{<} \circ U) \backslash V / (W \circ f{<}) \circ g$$
$$= \quad \{ \quad \text{assumption: } f \text{ and } g \text{ are functional, (31) and (32)} \quad \}$$
$$f \circ R \circ g^{\cup} \subseteq (g{<} \circ U) \backslash V / (W \circ f{<})$$
$$= \quad \{ \quad \text{factors} \quad \}$$
$$g{<} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f{<} \subseteq V$$
$$= \quad \{ \quad \text{assumption: } f \text{ and } g \text{ are functional}$$
$$\text{i.e.} \quad f \circ f^{\cup} = f{<} \wedge g \circ g^{\cup} = g{<} \quad \}$$
$$g \circ g^{\cup} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f \circ f^{\cup} \subseteq V$$
$$= \quad \{ \quad \text{assumption: } f \text{ and } g \text{ are functional, (31) and (32)} \quad \}$$
$$g{>} \circ g^{\cup} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f \circ f{>} \subseteq g^{\cup} \circ V \circ f$$
$$= \quad \{ \quad \text{domains (four times)} \quad \}$$
$$g^{\cup} \circ U \circ f \circ f{>} \circ R \circ g{>} \circ g^{\cup} \circ W \circ f \subseteq g^{\cup} \circ V \circ f$$
$$= \quad \{ \quad \text{factors} \quad \}$$
$$f{>} \circ R \circ g{>} \subseteq (g^{\cup} \circ U \circ f) \backslash (g^{\cup} \circ V \circ f) / (g^{\cup} \circ W \circ f)$$
$$= \quad \{ \quad f{>} \text{ and } g{>} \text{ are coreflexives} \quad \}$$
$$f{>} \circ R \circ g{>} \subseteq f{>} \circ (g^{\cup} \circ U \circ f) \backslash (g^{\cup} \circ V \circ f) / (g^{\cup} \circ W \circ f) \circ g{>}$$

The lemma follows by instantiating R to the left and right sides of the claimed equation, simplifying using domain calculus, and then applying the reflexivity and anti-symmetry of the subset relation.

□

The final lemma in this section anticipates the discussion of per domains in section 3.8.

Lemma 37 Suppose relations R, f and g are such that

$$f\circ f^{\cup} \;=\; f< \;=\; R< \quad\wedge\quad g< \;=\; g\circ g^{\cup} \quad .$$

Then, for all S,

(38) $\quad g>\circ(f^{\cup}\circ R\circ g)\backslash(f^{\cup}\circ S) \;=\; g^{\cup}\circ R\backslash S \quad .$

It follows that

(39) $\quad g>\circ(f^{\cup}\circ R\circ g)\backslash(f^{\cup}\circ R\circ g)\circ g> \;=\; g^{\cup}\circ R\backslash R\circ g \quad .$

Proof The proof of (38) is as follows.

$$\begin{array}{ll}
 & g>\circ(f^{\cup}\circ R\circ g)\backslash(f^{\cup}\circ S) \\
= & \{ \quad \text{factors:} \quad \} \\
 & g>\circ g\backslash((f^{\cup}\circ R)\backslash(f^{\cup}\circ S)) \\
= & \{ \quad \text{lemma 33 with } f,R := g\,,(f^{\cup}\circ R)\backslash(f^{\cup}\circ S) \quad \} \\
 & g^{\cup}\circ(f^{\cup}\circ R)\backslash(f^{\cup}\circ S) \\
= & \{ \quad \text{factors} \quad \} \\
 & g^{\cup}\circ R\backslash(f^{\cup}\backslash(f^{\cup}\circ S)) \\
= & \{ \quad [\,R\backslash S = R\backslash(R<\circ S)\,] \text{ with } R,S := R\,,f^{\cup}\backslash(f^{\cup}\circ S) \\
 & \quad\;\; \text{assumption: } f< = R< \quad \} \\
 & g^{\cup}\circ R\backslash(f<\circ f^{\cup}\backslash(f^{\cup}\circ S)) \\
= & \{ \quad \text{lemma 34 with } f,R := f,S \quad \} \\
 & g^{\cup}\circ R\backslash(f<\circ S) \\
= & \{ \quad \text{assumption: } f< = R<\,,\; [\,R\backslash S = R\backslash(R<\circ S)\,] \quad \} \\
 & g^{\cup}\circ R\backslash S \quad .
\end{array}$$

Now we prove (39).

$$\begin{array}{ll}
 & g>\circ(f^{\cup}\circ R\circ g)\backslash(f^{\cup}\circ R\circ g)\circ g> \\
= & \{ \quad (38) \text{ with } S := R\circ g \quad \} \\
 & g^{\cup}\circ R\backslash(R\circ g)\circ g> \\
= & \{ \quad \text{lemma 35} \quad \} \\
 & g^{\cup}\circ R\backslash R\circ g \quad .
\end{array}$$

□

3.5 Difunctions

Formally, relation R is *difunctional* iff

(40) $\quad R\circ R^{\cup}\circ R \subseteq R \quad .$

As for pers, there are several equivalent definitions of "difunctional", as formulated below.

Theorem 41 For all R, the following statements are all equivalent.

(i) R is difunctional (i.e. $R \circ R^{\cup} \circ R \subseteq R$) ,

(ii) $R = R \circ R^{\cup} \circ R$,

(iii) $R{>} \circ R \backslash R = R^{\cup} \circ R$,

(iv) $R{\succ} = R^{\cup} \circ R$,

(v) $R/R \circ R{<} = R \circ R^{\cup}$,

(vi) $R{\prec} = R \circ R^{\cup}$,

(vii) $R = R \cap (R \backslash R/R)^{\cup}$.

□

3.6 Provisional Orderings

There are various well-known notions of ordering: preorder, partial and linear (aka total) ordering. For our purposes all of these are too strict — the fact is that, in practice, relations are rarely "total" (for example, not everyone has a sibling). So, in this section, we introduce the notion of a "provisional ordering". The adjective "provisional" has been chosen because the notion "provides" just what we need. For later use, we state a number of properties but without proof. All proofs can be found in the companion working document [BV22].

The standard definition of an ordering is an anti-symmetric preorder whereby a preorder is required to be reflexive and transitive. It is the reflexivity requirement that is too strict for our purposes. So, with the intention of weakening the standard definition of a preorder to requiring reflexivity of a relation over some superset of its left and right domains, we propose the following definition.

Definition 42 Suppose T is a homogeneous relation. Then T is said to be a *provisional preorder* if

$$T{<} \subseteq T \;\wedge\; T{>} \subseteq T \;\wedge\; T \circ T \subseteq T \;.$$

□

Fig. 2 depicts a provisional preorder on a set of eight elements as a directed graph. The blue squares should be ignored for the moment. (See the discussion following lemma 48.) Note that the relation depicted is not a preorder because it is not reflexive: the top-right node depicts an element that is not in the left or right domain of the relation.

An immediate consequence of the definition is:

Lemma 43 If T is a provisional preorder then $T{<} = T{>}$.

□

A trivial property that is nevertheless used frequently:

Lemma 44 T is a provisional preorder equivales $T^{\cup}$ is a provisional preorder.

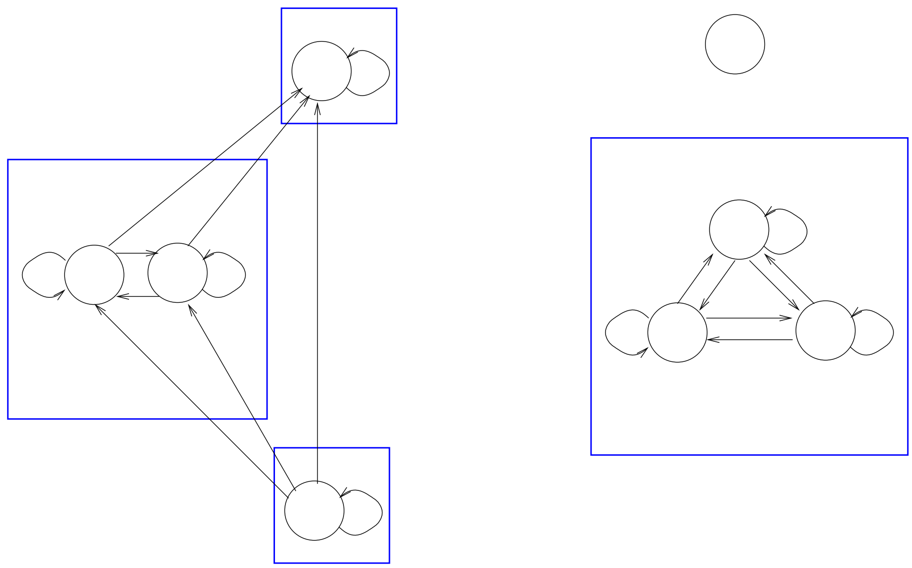

Figure 2: A Provisional Preorder

□

A preorder is a provisional preorder with left (equally right) domain equal to the identity relation. In other words, a preorder is a total provisional preorder. It is easy to show that, for any relation R, the relations $R\backslash R$ and R/R are preorders. It is also easy to show that R is a preorder if and only if $R = R\backslash R$ (or equivalently if and only if $R = R/R$). These properties generalise to provisional preorders.

Lemma 45 For all relations R, the relations $R_{>} \circ R\backslash R$ and $R/R \circ R_{<}$ are provisional preorders.

□

Lemma 46 T is a provisional preorder equivales

$$T \;=\; T_{<} \circ T\backslash T \;=\; T/T \circ T_{>} \;=\; T_{<} \circ T\backslash T/T \circ T_{>} \;\; .$$

□

Lemma 46 is sometimes used in a form where the domains are replaced by per domains.

Lemma 47 Suppose T is a provisional preorder. Then

$$T \;=\; T_{\prec} \circ T\backslash T \;=\; T/T \circ T_{\succ} \;=\; T_{\prec} \circ T\backslash T/T \circ T_{\succ} \;\; .$$

□

Lemma 48 Suppose T is a provisional preorder. Then

$$T_{\prec} \;=\; T \cap T^{\cup} \;=\; T_{\succ} \;\; .$$

Hence $T \cap T^{\cup}$ is a per.

□

Referring back to fig. 2, the blue squares depict the equivalence classes of the symmetric closure of a provisional preorder. As remarked earlier, the depicted relation is not a preorder; correspondingly, the blue squares depict a truly *partial* equivalence relation.

We assume the reader is familiar with the notions of an ordering and a linear (or total) ordering. We now extend these notions to provisional orderings. (The at-most relation on the integers is both anti-symmetric and linear. The at-most relation restricted to some arbitrary subset of the integers is an example of a linear provisional ordering according to the definition below.)

Definition 49 Suppose T is a homogeneous relation of type $A{\sim}A$, for some A. Then T is said to be *provisionally anti-symmetric* if

$$\mathsf{T}\cap\mathsf{T}^{\cup} \subseteq I_{\mathsf{A}} \quad .$$

Also, T is said to be a *provisional ordering* if T is provisionally anti-symmetric and T is a provisional preorder. Finally, T is said to be a *linear provisional ordering* if T is a provisional ordering and

$$\mathsf{T}\cup\mathsf{T}^{\cup} \;=\; (\mathsf{T}\cap\mathsf{T}^{\cup})\circ\mathbb{T}\circ(\mathsf{T}\cap\mathsf{T}^{\cup}) \quad .$$

□

Definition 49 weakens the equality in the standard notion of anti-symmetry to an inclusion. The standard definition of a partial ordering —an anti-symmetric preorder— is weakened accordingly (as mentioned earlier, in order to "provide" for our needs).

The following lemma anticipates the use of provisional preorders/orderings in examples presented later.

Lemma 50 Suppose T is a provisional ordering. Then

$$\mathsf{T}{<} \;=\; \mathsf{T}\cap\mathsf{T}^{\cup} \;=\; \mathsf{T}{>} \quad .$$

□

3.7 Squares and Rectangles

We now introduce the notions of a "rectangle" and a "square"; rectangles are typically heterogeneous whilst squares are, by definition, homogeneous relations. Squares are rectangles; properties of squares are typically obtained by specialising properties of rectangles.

Definition 51 (Rectangle and Square) A relation R is a *rectangle* iff $\mathsf{R}=\mathsf{R}\circ\mathbb{T}\circ\mathsf{R}$. A relation R is a *square* iff R is a symmetric rectangle.

□

It is easily shown that a rectangle is a difunction and a square is a per.

3.8 Isomorphic Relations

The (yet-to-be-defined) cores and indexes of a given relation are not unique; in common mathematical jargon, they are unique "up to isomorphism". In order to make this precise, we need to define the notion of isomorphic relation and establish a number of properties.

Definition 52 Suppose R and S are two relations (not necessarily of the same type). Then we say that R and S are *isomorphic* and write $R\cong S$ iff

$$
\begin{array}{lll}
\langle\exists\,\phi,\psi & & \\
 & : & \phi\circ\phi^{\cup} = R{<} \;\wedge\; \phi^{\cup}\circ\phi = S{<} \;\wedge\; \psi\circ\psi^{\cup} = R{>} \;\wedge\; \psi^{\cup}\circ\psi = S{>} \\
 & : & R = \phi\circ S\circ\psi^{\cup} \\
\rangle\quad . & &
\end{array}
$$

□

4 Indexes and Core Relations

This section introduces the notions of "index" and "core" of a relation and records some of their properties. An "index" is a special case of a "core" of a relation but, in general, it is more useful. For a detailed account of their properties (including proofs) see [BV22, BV23].

4.1 Definitions

The definition of an "index" of a relation is as follows.

Definition 53 (Index) An *index* of a relation R is a relation J that has the following properties:

(a) $J\subseteq R$,

(b) $R{\prec}\circ J\circ R{\succ} \;=\; R$,

(c) $J{<}\circ R{\prec}\circ J{<} \;=\; J{<}$,

(d) $J{>}\circ R{\succ}\circ J{>} \;=\; J{>}$.

□

Indexes are a special case of what we call "core" relations.

Definition 54 (Core) Suppose R is an arbitrary relation and suppose C is a relation such that

$$C \;=\; \lambda\circ R\circ\rho^{\cup}$$

for some relations λ and ρ satisfying

$$R{\prec} \;=\; \lambda^{\cup}\circ\lambda \quad\wedge\quad \lambda{<} = \lambda\circ\lambda^{\cup} \quad\wedge\quad R{\succ} \;=\; \rho^{\cup}\circ\rho \quad\wedge\quad \rho{<} = \rho\circ\rho^{\cup} \quad .$$

Then C is said to be a *core of* R *as witnessed by* λ *and* ρ .

□

Note particularly requirement 53(a). A consequence of this requirement is that an index of a relation has the same type as the relation, which is not necessarily the case for cores.

An index J of a relation R is a core of the relation as witnessed by $J{<}\circ R{\prec}$ and $J{>}\circ R{\succ}$.

The property that is common to cores and indexes is captured by the following definition.

Definition 55 (Core Relation) A relation R is a *core relation* iff $R{<} = R{\prec}$ and $R{>} = R{\succ}$.
□

We exploit the fact that both indexes and cores satisfy definition 55 later. (The proof of this fact in [BV22] assumes that R has an index. The —much less straightforward— proof without this assumption is given in [Bac21].)

A number of properties of cores and indexes are needed below. Suppose C is a core of R as witnessed by λ and ρ. Then

(56) $$R \;=\; \lambda^{\cup}\circ C\circ\rho \quad .$$

(57) $$R{<} = \lambda{>} \;\;\wedge\;\; C{<} = \lambda{<} \;\;\wedge\;\; R{>} = \rho{>} \;\;\wedge\;\; C{>} = \rho{<} \quad .$$

Suppose J is an index of R. Then

(58) $$R{\prec}\circ J{<}\circ R{\prec} \;=\; R{\prec} \quad\wedge\quad R{\succ}\circ J{>}\circ R{\succ} \;=\; R{\succ} \quad .$$

4.2 Indexes of Pers

A relation R is a per iff $R = R{\prec} = R{\succ}$. Using this property, the definition of index can be simplified for pers.

Definition 59 (Index of a Per) Suppose P is a per. Then a (*coreflexive*) *index* of P is a relation J such that

(a) $J \subseteq P{<}$,

(b) $J\circ P\circ J = J$,

(c) $P\circ J\circ P = P$.

□

To our axiom system we add the postulate that every per has a coreflexive index. We call this the *axiom of choice*.

Axiom 60 (Axiom of Choice) Every per has a coreflexive index.
□

Assuming our axiom of choice, it follows that every relation has an index. Specficially, we have:

Theorem 61 Suppose J and K are (coreflexive) indices of $R{\prec}$ and $R{\succ}$, respectively. Then $J\circ R\circ K$ is an index of R.
□

It is also the case that Freyd and Ščedrov's [Fv90] so-called "splittings" of pers always exist.

Theorem 62 If per P has a coreflexive index J, then

$$P \;=\; (J\circ P)^{\cup}\circ(J\circ P) \quad\wedge\quad J \;=\; (J\circ P)\circ(J\circ P)^{\cup} \;\;.$$

Thus, assuming the axiom of choice, for all relations R,

$$\mathsf{per}.R \;\equiv\; \langle \exists f : f\circ f^{\cup} \;=\; f< \;:\; R = f^{\cup}\circ f\rangle \;\;.$$

□

The property that R is a difunction is equivalent to $R\prec = R\circ R^{\cup}$ (and symmetrically to $R\succ = R^{\cup}\circ R$). Also, since $R = R\circ R^{\cup}\circ R$, the definition of an index of a difunction can be restated as follows.

Definition 63 (Difunction Index) An index of a difunction R is a relation J that has the following properties:

(a) $J\subseteq R$,

(b) $R\circ J^{\cup}\circ R \;=\; R$.

(c) $J<\circ R\circ R^{\cup}\circ J< \;=\; J<$,

(d) $J>\circ R^{\cup}\circ R\circ J> \;=\; J>$.

□

In the same way that pers are characterised by a single function f —see theorem 62— difunctions are characterised by a pair of functions f and g:

Theorem 64 Assuming the axiom of choice (axiom 60), for all relations R,

$$\mathsf{difunction}.R \;\equiv\; \langle \exists f,g : f\circ f^{\cup} \;=\; f< \;=\; g\circ g^{\cup} \;=\; g< \;:\; R = f^{\cup}\circ g\rangle \;\;.$$

□

5 The Diagonal

This section anticipates the study of block-ordered relations We introduce the notion of the "diagonal" of a relation in section 5.1 and formulate some basic properties in section 5.2.

5.1 Definition and Examples

Straightforwardly from the definition of factors, properties of converse and set intersection,

(65) $\quad R \text{ is difunctional} \quad\equiv\quad R \;=\; R\cap(R\backslash R/R)^{\cup}$.

More generally, we have:

Lemma 66 For all R, $R\cap(R\backslash R/R)^{\cup}$ is difunctional.

Proof Let S denote $R\cap(R\backslash R/R)^{\cup}$. We have to prove that S is difunctional. That is, by definition,

$$S\circ S^{\cup}\circ S \subseteq S \ .$$

Since the right side is an intersection, this is equivalent to

$$S\circ S^{\cup}\circ S \subseteq R \ \wedge \ S\circ S^{\cup}\circ S \subseteq (R\backslash R/R)^{\cup} \ .$$

The first is (almost) trivial:

$$\begin{array}{ll} & S\circ S^{\cup}\circ S \\ \subseteq & \{ \quad S\subseteq R,\ S\subseteq(R\backslash R/R)^{\cup}, \\ & \quad \text{converse, monotonicity} \quad \} \\ & R\circ R\backslash R/R\circ R \\ \subseteq & \{ \quad \text{cancellation} \quad \} \\ & R \ . \end{array}$$

In the above calculation, the trick was to replace the outer occurrences of S on the left side by R and the middle occurrence by $(R\backslash R/R)^{\cup}$. The replacement is done the opposite way around in the second calculation.

$$\begin{array}{ll} & S\circ S^{\cup}\circ S \subseteq (R\backslash R/R)^{\cup} \\ \Leftarrow & \{ \quad S\subseteq(R\backslash R/R)^{\cup},\ S\subseteq R, \text{ monotonicity and transitivity} \quad \} \\ & (R\backslash R/R)^{\cup}\circ R^{\cup}\circ(R\backslash R/R)^{\cup} \subseteq (R\backslash R/R)^{\cup} \\ = & \{ \quad \text{converse} \quad \} \\ & R\backslash R/R\circ R\circ R\backslash R/R \subseteq R\backslash R/R \\ = & \{ \quad \text{Galois connection} \quad \} \\ & R\circ R\backslash R/R\circ R\circ R\backslash R/R\circ R \subseteq R \\ = & \{ \quad \text{cancellation, monotonicity and transitivity} \quad \} \\ & \text{true} \ . \end{array}$$

□

We call the relation $R\cap(R\backslash R/R)^{\cup}$ the *diagonal* of R; Riguet [Rig51] calls it the "différence". (Riguet's definition does not use factors but is equivalent.; indeed, rewriting the definition using (11) the diagonal of R is the "différence" of R and $R^{\cup}\circ\neg R\circ R^{\cup}$.)

Definition 67 (Diagonal) The *diagonal* of relation R is the relation $R\cap(R\backslash R/R)^{\cup}$. For brevity, $R\cap(R\backslash R/R)^{\cup}$ will be denoted by ΔR.
□

Many readers will be familiar with the decomposition of a preorder into a partial ordering on a set of equivalence classes. The diagonal of a preorder T is the equivalence relation $T\cap T^{\cup}$. More generally:

Example 68 The diagonal of a provisional preorder T is $T\cap T^{\cup}$. This is because, for an arbitrary relation T,

$$T\cap(T\backslash T/T)^{\cup} \quad = \quad T \ \cap \ T{<}\circ(T\backslash T/T)^{\cup}\circ T{>} \ .$$

But, if T is a provisional preorder,

$$T{<}\circ(T\backslash T/T)^{\cup}\circ T{>} \;=\; T^{\cup} \;\; .$$

(See lemmas 43 and 46.)
□

Example 69 A particular instance of example 68 is if G is the edge relation of a finite graph. Then $\Delta(G^*)$ is $G^*\cap(G^{\cup})^*$, the relation that holds between nodes a and b if there is a path from a to b and a path from b to a in the graph. Thus $\Delta(G^*)$ is the equivalence relation that holds between nodes that are in the same strongly connected component of G.
□

Example 70 In this example, we consider three versions of the less-than relation: the homogeneous less-than relation on integers, which we denote by $<_{\mathbb{Z}}$, the homogeneous less-than relation on real numbers, which we denote by $<_{\mathbb{R}}$, and the heterogeneous less-than relation on integers and real numbers, which we denote by ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$. Specifically, the relation ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$ relates integer m to real number x when $m<x$ (using the conventional over-loaded notation). We also subscript the at-most symbol $\leq$ in the same way in order to indicate the type of the relation in question.

The diagonal of the less-than relation on integers is the predecessor relation (i.e. it relates integer m to integer n exactly when $n=m+1$). This is because $<_{\mathbb{Z}}\backslash<_{\mathbb{Z}} = \leq_{\mathbb{Z}}$, and $\leq_{\mathbb{Z}}/<_{\mathbb{Z}}$ relates integer m to integer n exactly when $m\leq_{\mathbb{Z}} n+1$ (where the subscript $\mathbb{Z}$ indicates the type of the ordering). The diagonal is thus functional and injective.

The diagonal of the less-than relation on real numbers is the empty relation. This is because $<_{\mathbb{R}}\backslash<_{\mathbb{R}} = \leq_{\mathbb{R}}$, $\leq_{\mathbb{R}}/<_{\mathbb{R}} = \leq_{\mathbb{R}}$ and $<_{\mathbb{R}}\cap\geq_{\mathbb{R}} = \perp\!\!\!\perp_{\mathbb{R}}$. (Again, the subscript indicates the type of the ordering.)

The diagonal of the heterogeneous less-than relation ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$ relates integer m to real number x when $m<x\leq m+1$. This is equivalent to $\lceil x\rceil = m+1$. The diagonal is thus a difunctional relation characterised by —in the sense of theorem 64— the ceiling function $\langle x :: \lceil x\rceil\rangle$ and the successor function $\langle m :: m+1\rangle$.
□

The following example introduces a general mechanism for constructing illustrative examples of the concepts introduced later. The example exploits the observation that ΔR is injective if the preorder $R\backslash R$ is anti-symmetric; that is, ΔR is injective if $R\backslash R$ is a partial ordering. (Equivalently, ΔR is functional if R/R is a partial ordering.) We leave the straightforward proof to the reader.

Example 71 Suppose $\mathcal{X}$ is a finite type. We use dummy x to range over elements of type $\mathcal{X}$. Let $\mathcal{S}$ denote a subset of $2^{\mathcal{X}}$. Let in denote the membership relation of type $\mathcal{X}{\sim}\mathcal{S}$. That is, if S is a subset of $\mathcal{S}$, $x{\circ}\mathbb{T}{\circ}S\subseteq in$ exactly when x is an element of the set S. The relation $in\backslash in$ is the subset relation of type $\mathcal{S}{\sim}\mathcal{S}$.

(Conventionally, in is denoted by the symbol "$\in$" and one writes $x{\in}S$ instead of $x{\circ}\mathbb{T}{\circ}S\subseteq in$. Also, the relation $in\backslash in$ is conventionally denoted by the symbol "$\subseteq$". That is, if S and S' are both elements of $\mathcal{S}$, $S{\circ}\mathbb{T}{\circ}S'\subseteq in\backslash in$ exactly when $S\subseteq S'$. Were we to adopt conventional practice, the overloading of the notation that occurs is likely to cause confusion, so we choose to avoid it.)

The relation $in\backslash in$ is anti-symmetric. As a consequence, Δin is injective. (Equivalently, $(\Delta in)^{\cup}$ is functional.) Specifically, for all x of type $\mathcal{X}$ and S of type $\mathcal{S}$,

$$x\circ\mathbb{T}\circ S \subseteq \Delta in \quad\equiv\quad x\circ\mathbb{T}\circ S \subseteq in \wedge \langle\forall S' : x\circ\mathbb{T}\circ S' \subseteq in : S\circ\mathbb{T}\circ S' \subseteq in\backslash in\rangle \;\;,$$

where dummy S' ranges over elements of $\mathcal{S}$. Using conventional notation, the right side of this equation is recognised as the definition of a minimum, and one might write

$$x\,[\![\Delta in]\!]\,S \quad\equiv\quad S \doteqdot \langle \mathrm{MIN}\,S' : x\in S' : S'\rangle$$

where the venturi tube "$\doteqdot$" indicates an equality assuming the well-definedness of the expression on its right side.

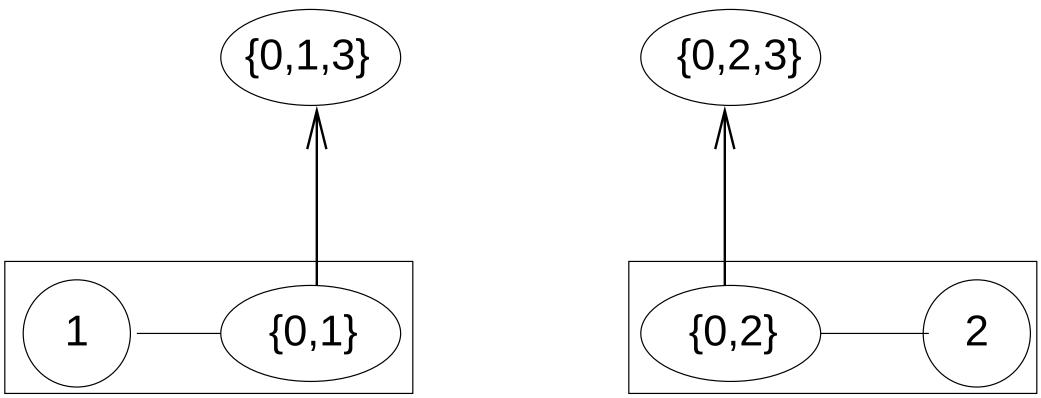

Figure 3: Diagonal of an Instance of the Membership Relation

Fig. 3 shows a particular instance. The set $\mathcal{X}$ is the set of numbers from 0 to 3. The set $\mathcal{S}$ is a subset of $2^{\{0,1,2,3\}}$; the chosen subset and the relation $in\backslash in$ for this choice are depicted by the directed graph forming the central portion of fig. 3. The relation Δin of type $\mathcal{X} \sim \mathcal{S}$ is depicted by the undirected graph whereby the atoms of the relation are depicted as rectangles. Note that the numbers 0 and 3 are not related by Δin to any of the elements of $\mathcal{S}$.

□

5.2 Basic Properties

Primarily for notational convenience, we note a simple property of the diagonal:

Lemma 72

$$(\Delta R)^{\cup} \;=\; \Delta(R^{\cup}) \;\;.$$

Proof

$$
\begin{array}{ll}
 & (\Delta R)^{\cup} \\
= & \quad \{ \quad \text{definition and distributivity} \quad \} \\
 & R^{\cup} \cap R\backslash R/R \\
= & \quad \{ \quad \text{factors} \quad \} \\
 & R^{\cup} \cap (R^{\cup} \backslash R^{\cup} / R^{\cup})^{\cup} \\
= & \quad \{ \quad \text{definition} \quad \} \\
 & \Delta(R^{\cup}) \quad .
\end{array}
$$

□

A consequence of lemma 72 is that we can write $\Delta R^{\cup}$ without ambiguity. This we do from now on.

Very straightforwardly, the relation $R \circ R^{\cup}$ is a per if R is difunctional. For a difunctional relation R, the relation $R \circ R^{\cup}$ is the left per domain of R. (Symmetrically, $R^{\cup} \circ R$ is the right per domain of R. See theorem 41, parts (iv) and (vi).) Thus $\Delta R \circ (\Delta R)^{\cup}$ is the left per domain of the diagonal of R. The following lemma is the basis of the construction, in certain cases, of an economic representation of the diagonal of R and, hence, of R itself.

Lemma 73 For all relations R,

$$(\Delta R)_{\prec} \;=\; (\Delta R)_{<} \circ R_{\prec} \;=\; R_{\prec} \circ (\Delta R)_{<} \quad .$$

Dually,

$$(\Delta R)_{\succ} \;=\; R_{\succ} \circ (\Delta R)_{>} \;=\; (\Delta R)_{>} \circ R_{\succ} \quad .$$

Proof We prove the first equation by mutual inclusion. First,

$$
\begin{array}{ll}
 & (\Delta R)_{\prec} \subseteq (\Delta R)_{<} \circ R_{\prec} \\
= & \quad \{ \quad \Delta R \text{ is difunctional, theorem 41; definition: (23)} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \subseteq (\Delta R)_{<} \circ R /\!\!/ R \\
\Leftarrow & \quad \{ \quad \text{domains and monotonicity} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \subseteq R /\!\!/ R \\
= & \quad \{ \quad \text{definition of } R /\!\!/ R \text{, converse and factors} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R \subseteq R \\
= & \quad \{ \quad \Delta R \subseteq R\,;\ \Delta R^{\cup} \subseteq R\backslash R/R \text{ and cancellation} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & (\Delta R)_{<} \circ R_{\prec} \subseteq (\Delta R)_{\prec} \\
= & \quad \{ \quad \Delta R \text{ is difunctional, theorem 41} \quad \} \\
 & (\Delta R)_{<} \circ R_{\prec} \subseteq \Delta R \circ \Delta R^{\cup} \\
\Leftarrow & \quad \{ \quad \text{domains and definition: (23)} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R /\!\!/ R \subseteq \Delta R \circ \Delta R^{\cup} \\
\Leftarrow & \quad \{ \quad \text{monotonicity and converse} \quad \} \\
 & R /\!\!/ R \circ \Delta R \subseteq \Delta R \\
= & \quad \{ \quad \text{definition of diagonal} \quad \} \\
 & R /\!\!/ R \circ \Delta R \subseteq R \;\;\wedge\;\; R /\!\!/ R \circ \Delta R \subseteq (R\backslash R/R)^{\cup}
\end{array}
$$

$$\begin{array}{ll}
\Leftarrow & \{ \quad \Delta R \subseteq R \text{ ; converse} \quad \} \\
& R /\!\!/ R \circ R \subseteq R \ \wedge \ \Delta R^{\cup} \circ R /\!\!/ R \subseteq R \backslash R / R \\
= & \{ \quad \text{cancellation; factors} \quad \} \\
& \text{true} \ \wedge \ R \circ \Delta R^{\cup} \circ R /\!\!/ R \circ R \subseteq R \\
\Leftarrow & \{ \quad \text{cancellation and } \Delta R^{\cup} \subseteq R \backslash R / R \quad \} \\
& R \circ R \backslash R / R \circ R \subseteq R \\
= & \{ \quad \text{cancellation} \quad \} \\
& \text{true} \ .
\end{array}$$

The remaining three equalities are simple consequences of the properties of converse, pers and coreflexives.

□

The following corollary of lemma 73 proves to be crucial later:

Lemma 74 For all relations R,

$$(\Delta R)\prec \ = \ R\prec \quad \equiv \quad (\Delta R)< \ = R< \ .$$

Dually,

$$(\Delta R)\succ \ = \ R\succ \quad \equiv \quad (\Delta R)> \ = R> \ .$$

Proof The proof is by mutual implication:

$$\begin{array}{ll}
& (\Delta R)< \ = R< \\
\Rightarrow & \{ \quad \text{lemma 73 and Leibniz} \quad \} \\
& (\Delta R)\prec \ = \ R< \circ R\prec \\
= & \{ \quad \text{dual of (30)} \quad \} \\
& (\Delta R)\prec \ = \ R\prec \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
& ((\Delta R)\prec)< \ = \ (R\prec)< \\
= & \{ \quad \text{dual of (30) with } R{:=}\Delta R \text{ and } R{:=}R \quad \} \\
& (\Delta R)< \ = \ R< \ .
\end{array}$$

□

5.3 Reduction to the Core

In this section our goal is to prove that if J is an index of relation R then ΔJ is an index of ΔR. Instantiating definition 63 with $J,R{:=}\Delta J,\Delta R$ the properties we have to prove are as follows.

(a) $\Delta J \subseteq \Delta R$,

(b) $\Delta R \circ \Delta J^{\cup} \circ \Delta R \ = \ \Delta R$.

(c) $(\Delta J)< \circ \Delta R \circ \Delta R^{\cup} \circ (\Delta J)< \ = \ (\Delta J)<$,

(d) $(\Delta J)> \circ \Delta R^{\cup} \circ \Delta R \circ (\Delta J)> \ = \ (\Delta J)>$.

Of these, the hardest to prove is (b). For properties (a), (c) and (d), all we need is that J is an arbitrary index of R. For property (b), we use the fact that an index of an arbitrary relation R is defined to be $J{\circ}R{\circ}K$ where J is an index of $R_\prec$ and K is an index of $R_\succ$.

We begin with the easier properties.

Lemma 75 Suppose J is an index of R. Then

$$\Delta J \subseteq \Delta R \quad .$$

Proof

$$
\begin{array}{ll}
 & \Delta J \subseteq \Delta R \\
= & \qquad \{ \quad \text{definition 67} \quad \} \\
 & J \cap (J\backslash J/J)^{\cup} \;\subseteq\; R \cap (R\backslash R/R)^{\cup} \\
= & \qquad \{ \quad \text{domains} \quad \} \\
 & J \cap J_< \circ (J\backslash J/J)^{\cup} \circ J_> \;\subseteq\; R \cap (R\backslash R/R)^{\cup} \\
\Leftarrow & \qquad \{ \quad J \text{ is an index of } R\text{, so } J \subseteq R\text{; monotonicity} \quad \} \\
 & J_< \circ (J\backslash J/J)^{\cup} \circ J_> \;\subseteq\; (R\backslash R/R)^{\cup} \\
= & \qquad \{ \quad \text{converse} \quad \} \\
 & J_> \circ J\backslash J/J \circ J_< \;\subseteq\; R\backslash R/R \\
= & \qquad \{ \quad \text{factors} \quad \} \\
 & R \circ J_> \circ J\backslash J/J \circ J_< \circ R \;\subseteq\; R \\
= & \qquad \{ \quad J \text{ is an index of } R\text{, definition 53(b); per domains} \quad \} \\
 & R_\prec \circ J \circ R_\succ \circ J_> \circ J\backslash J/J \circ J_< \circ R_\prec \circ J \circ R_\succ \;\subseteq\; R_\prec \circ R \circ R_\succ \\
\Leftarrow & \qquad \{ \quad \text{monotonicity} \quad \} \\
 & J \circ R_\succ \circ J_> \circ J\backslash J/J \circ J_< \circ R_\prec \circ J \;\subseteq\; R \quad .
\end{array}
$$

Continuing with the left side of the inclusion:

$$
\begin{array}{ll}
 & J \circ R_\succ \circ J_> \circ J\backslash J/J \circ J_< \circ R_\prec \circ J \\
= & \qquad \{ \quad \text{domains} \quad \} \\
 & J \circ J_> \circ R_\succ \circ J_> \circ J\backslash J/J \circ J_< \circ R_\prec \circ J_< \circ J \\
= & \qquad \{ \quad J \text{ is an index of } R\text{; definition 53(c) and (d)} \quad \} \\
 & J \circ J_> \circ J\backslash J/J \circ J_< \circ J \\
\subseteq & \qquad \{ \quad \text{domains and cancellation} \quad \} \\
 & J \\
\subseteq & \qquad \{ \quad J \text{ is an index of } R\text{; definition 53(a)} \quad \} \\
 & R \quad .
\end{array}
$$

□

Lemma 76 Suppose J is an index of R. Then

$$(\Delta J)_< \circ \Delta R \circ \Delta R^{\cup} \circ (\Delta J)_< \;=\; (\Delta J)_< \quad .$$

Dually,

$$(\Delta J)_> \circ \Delta R^{\cup} \circ \Delta R \circ (\Delta J)_> \;=\; (\Delta J)_> \quad .$$

Proof

$$
\begin{array}{ll}
& (\Delta J)_< \circ \Delta R \circ \Delta R^\cup \circ (\Delta J)_< \\
= & \{ \quad \Delta R \text{ is a difunction, theorem 41} \quad \} \\
& (\Delta J)_< \circ (\Delta R)_\prec \circ (\Delta J)_< \\
= & \{ \quad \text{lemma 73 (and symmetry)} \quad \} \\
& (\Delta J)_< \circ (\Delta R)_< \circ R_\prec \circ (\Delta R)_< \circ (\Delta J)_< \\
= & \{ \quad \text{by lemma 75 and monotonicity, } (\Delta J)_< \subseteq (\Delta R)_< \quad \} \\
& (\Delta J)_< \circ R_\prec \circ (\Delta J)_< \\
= & \{ \quad (\Delta J)_< \subseteq J_< \text{ (since } \Delta J \subseteq J\text{)} \quad \} \\
& (\Delta J)_< \circ J_< \circ R_\prec \circ J_< \circ (\Delta J)_< \\
= & \{ \quad J \text{ is an index of } R\text{, definition 53(c)} \quad \} \\
& (\Delta J)_< \circ J_< \circ (\Delta J)_< \\
= & \{ \quad (\Delta J)_< \subseteq J_< \text{ (since } \Delta J \subseteq J\text{)} \quad \} \\
& (\Delta J)_< \quad .
\end{array}
$$

□

In order to prove (b), we prove a more general theorem on cores. First, a lemma:

Lemma 77 Suppose R, C, λ and ρ are as in definition 54. Then

$$R_> \circ R\backslash R/R \circ R_< \;=\; \rho^\cup \circ C\backslash C/C \circ \lambda \quad .$$

Proof

$$
\begin{array}{ll}
& R_> \circ R\backslash R/R \circ R_< \\
= & \{ \quad (30) \quad \} \\
& (R_\succ)_> \circ R\backslash R/R \circ (R_\prec)_< \\
= & \{ \quad R_\prec = \lambda^\cup \circ \lambda\text{, } R_\succ = \rho^\cup \circ \rho\text{, and domains} \quad \} \\
& \rho_> \circ R\backslash R/R \circ \lambda_> \\
= & \{ \quad (56) \quad \} \\
& \rho_> \circ (\lambda^\cup \circ C \circ \rho)\backslash(\lambda^\cup \circ C \circ \rho)/(\lambda^\cup \circ C \circ \rho) \circ \lambda_> \\
= & \{ \quad \text{lemma 36 with } f,g,U,V,W := \rho,\lambda,C,C,C \quad \} \\
& \rho^\cup \circ (\lambda_< \circ C)\backslash C/(C \circ \rho_<) \circ \lambda \\
= & \{ \quad C = \lambda \circ R \circ \rho^\cup\text{; so } \lambda_< \circ C = C = C \circ \rho_< \quad \} \\
& \rho^\cup \circ C\backslash C/C \circ \lambda \quad .
\end{array}
$$

□

Theorem 78 Suppose R, C, λ and ρ are as in definition 54. Then

$$\Delta R \;=\; \lambda^\cup \circ \Delta C \circ \rho \quad \wedge \quad \Delta C \;=\; \lambda \circ \Delta R \circ \rho^\cup \quad .$$

In words, if λ and ρ witness that C is a core of R, then λ and ρ witness that ΔC is a core of ΔR.

Proof

$$
\begin{array}{ll}
 & \Delta R \\
= & \{ \quad \text{definition} \quad \} \\
 & R \cap (R\backslash R/R)^{\cup} \\
= & \{ \quad \text{domains and converse} \quad \} \\
 & R \;\cap\; (R_{>} \circ R\backslash R/R \circ R_{<})^{\cup} \\
= & \{ \quad \text{lemma 77} \quad \} \\
 & R \;\cap\; (\rho^{\cup} \circ C\backslash C/C \circ \lambda)^{\cup} \\
= & \{ \quad \text{straightforwardly) } R \;=\; \lambda^{\cup} \circ C \circ \rho \quad \} \\
 & \lambda^{\cup} \circ C \circ \rho \;\cap\; (\rho^{\cup} \circ C\backslash C/C \circ \lambda)^{\cup} \\
= & \{ \quad \text{distributivity of converse and functional relations} \quad \} \\
 & \lambda^{\cup} \circ (C \cap (C\backslash C/C)^{\cup}) \circ \rho \\
= & \{ \quad \text{definition 67} \quad \} \\
 & \lambda^{\cup} \circ \Delta C \circ \rho \quad .
\end{array}
$$

Hence

$$
\begin{array}{ll}
 & \lambda \circ \Delta R \circ \rho^{\cup} \\
= & \{ \quad \text{above} \quad \} \\
 & \lambda \circ \lambda^{\cup} \circ \Delta C \circ \rho \circ \rho^{\cup} \\
= & \{ \quad \lambda \text{ and } \rho \text{ are functional} \quad \} \\
 & \lambda_{<} \circ \Delta C \circ \rho_{<} \\
= & \{ \quad \Delta C \subseteq C\,;\text{ so } (\Delta C)_{<} \subseteq C_{<} \text{ and } (\Delta C)_{>} \subseteq C_{>} \\
 & \quad\;\; \text{(57) and domains} \quad \} \\
 & \Delta C \quad .
\end{array}
$$

□

We are now in a position to prove the final property (b) above.

Lemma 79 Suppose J is an index of R. Then

$$\Delta R \circ \Delta J^{\cup} \circ \Delta R \;=\; \Delta R \quad .$$

Proof We begin by noting that theorem 78 applies with C instantiated to J and λ and ρ defined by $\lambda = J_{<} \circ R_{\prec}$ and $\rho = J_{>} \circ R_{\succ}$. This is because J is a core of R. So

$$
\begin{array}{ll}
 & \Delta R \circ \Delta J^{\cup} \circ \Delta R \\
= & \{ \quad \text{theorem 78 with } C,\lambda,\rho := J\,,\, J_{<} \circ R_{\prec}\,,\, J_{>} \circ R_{\succ} \quad \} \\
 & \Delta R \circ (\lambda \circ \Delta R \circ \rho^{\cup})^{\cup} \circ \Delta R \\
= & \{ \quad \text{converse} \quad \} \\
 & \Delta R \circ \rho \circ \Delta R^{\cup} \circ \lambda^{\cup} \circ \Delta R \\
= & \{ \quad \text{definition of } \rho \text{ and } \lambda\,,\; (J_{<} \circ R_{\prec})^{\cup} = R_{\prec} \circ J_{<} \quad \} \\
 & \Delta R \circ J_{>} \circ R_{\succ} \circ \Delta R^{\cup} \circ R_{\prec} \circ J_{<} \circ \Delta R \\
= & \{ \quad \text{per domains} \quad \} \\
 & \Delta R \circ (\Delta R)_{\succ} \circ J_{>} \circ R_{\succ} \circ \Delta R^{\cup} \circ R_{\prec} \circ J_{<} \circ (\Delta R)_{\prec} \circ \Delta R \\
= & \{ \quad \text{lemma 73} \quad \} \\
 & \Delta R \circ (\Delta R)_{>} \circ R_{\succ} \circ J_{>} \circ R_{\succ} \circ \Delta R^{\cup} \circ R_{\prec} \circ J_{<} \circ R_{\prec} \circ (\Delta R)_{<} \circ \Delta R \\
= & \{ \quad \text{lemma 58} \quad \} \\
 & \Delta R \circ (\Delta R)_{>} \circ R_{\succ} \circ \Delta R^{\cup} \circ R_{\prec} \circ (\Delta R)_{<} \circ \Delta R
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{lemma 73} \quad \} \\
& \Delta R \circ (\Delta R)_{\succ} \circ \Delta R^{\cup} \circ (\Delta R)_{\prec} \circ \Delta R \\
= & \{ \quad \text{per domains} \quad \} \\
& \Delta R \circ \Delta R^{\cup} \circ \Delta R \\
= & \{ \quad \Delta R \ \text{is difunctional, theorem 41} \quad \} \\
& \Delta R \quad .
\end{array}
$$

□

Putting all the lemmas together, we have:

Theorem 80 Suppose J is an index of R. Then ΔJ is an index of ΔR.

Proof Lemmas 75, 76 and 79 combined with definition 63 (instantiated with $J,R := \Delta J, \Delta R$).

□

We conclude with a beautiful theorem.

Theorem 81 Suppose J is an index of R. Then

$$\Delta J \;=\; J_< \circ \Delta R \circ J_> \quad \wedge \quad \Delta R \;=\; R_\prec \circ \Delta J \circ R_\succ \quad .$$

Proof We first prove, by mutual implication, that the two equations are equivalent. Assume that

$$\Delta R \;=\; R_\prec \circ \Delta J \circ R_\succ \quad .$$

Then,

$$
\begin{array}{ll}
& J_< \circ \Delta R \circ J_> \\
= & \{ \quad \text{assumption} \quad \} \\
& J_< \circ R_\prec \circ \Delta J \circ R_\succ \circ J_> \\
= & \{ \quad \Delta J \subseteq J \text{, so } (\Delta J)_< \subseteq J_< \text{ and } (\Delta J)_> \subseteq J_> \text{; domains} \quad \} \\
& J_< \circ R_\prec \circ J_< \circ \Delta J \circ J_> \circ R_\succ \circ J_> \\
= & \{ \quad J \text{ is an index of } R\text{, definition 53(c) and (d)} \quad \} \\
& J_< \circ \Delta J \circ J_> \\
= & \{ \quad \text{reverse of middle step} \quad \} \\
& \Delta J \quad .
\end{array}
$$

Conversely, assume

$$\Delta J = J_< \circ \Delta R \circ J_> \quad .$$

Then,

$$
\begin{array}{ll}
& R_\prec \circ \Delta J \circ R_\succ \\
= & \{ \quad \text{assumption} \quad \} \\
& R_\prec \circ J_< \circ \Delta R \circ J_> \circ R_\succ \\
= & \{ \quad \text{lemma 73} \quad \} \\
& R_\prec \circ J_< \circ (\Delta R)_< \circ R_\prec \circ \Delta R \circ R_\succ \circ (\Delta R)_> \circ J_> \circ R_\succ
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{lemma 75 and domains} \quad \} \\
 & R{\prec} \circ J{<} \circ R{\prec} \circ \Delta R \circ R{\succ} \circ J{>} \circ R{\succ} \\
= & \{ \quad \text{definition 53(c) and 53(d)} \quad \} \\
 & R{\prec} \circ \Delta R \circ R{\succ} \\
= & \{ \quad \text{lemma 73 and domains} \quad \} \\
 & \Delta R \ .
\end{array}
$$

Combining the two calculations, the two equations are equivalent and, therefore, it suffices to prove just one of them[3]. We prove the second by mutual inclusion:

$$
\begin{array}{ll}
 & \Delta R \\
= & \{ \quad \Delta R \text{ is difunctional} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ \Delta R \\
= & \{ \quad \text{lemma 79, converse} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ \Delta J \circ \Delta R^{\cup} \circ \Delta R \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 41(iv) and (vi)} \quad \} \\
 & (\Delta R){\prec} \circ \Delta J \circ (\Delta R){\succ} \\
= & \{ \quad \text{lemma 73} \quad \} \\
 & (\Delta R){<} \circ R{\prec} \circ \Delta J \circ R{\succ} \circ (\Delta R){>} \\
\subseteq & \{ \quad \text{domains are coreflexive} \quad \} \\
 & R{\prec} \circ \Delta J \circ R{\succ} \\
\subseteq & \{ \quad \text{lemma 75 and monotonicity} \quad \} \\
 & R{\prec} \circ \Delta R \circ R{\succ} \\
= & \{ \quad \text{lemma 73, domains} \quad \} \\
 & \Delta R \ .
\end{array}
$$

□

6 Block-Ordered Relations

In general, dividing a subset of a set A into blocks is formulated by specifying a functional relation f, say, with source[4] the set A; elements $a0$ and $a1$ are in the same block equivales $f.a0$ and $f.a1$ are both defined and $f.a0 = f.a1$. In mathematical terminology, a functional relation f defines the *partial equivalence relation* $f^{\cup} \circ f$ and the "blocks" are the equivalence classes of $f^{\cup} \circ f$. (Partiality means that some elements may not be in an equivalence class.)

Given functional relations f and g with sources A and B, respectively, and equal left domains, relation R of type $A{\sim}B$ is said to be *block-structured* by f and g if there is a relation S such that $R = f^{\cup} \circ S \circ g$. Informally, whether or not a and b are related by R depends entirely on the "block" $(f.a\, ,\, g.b)$ to which they belong. Note that it is not required that f and g be total functions: it suffices that $f{>} = R{<}$ and $g{>} = R{>}$. The type of S is $C{\sim}C$ where C includes $\{a\colon\ a \circ f{>} = a\colon\ f.a\}$ (equally $\{b\colon\ b \circ f{>} = b\colon\ g.b\}$).

[3] It is not necessary to prove the equivalence of the two statements in order to prove the theorem; we could have omitted the second calculation. But some redundancy in proofs enhances their reliability.

[4] In the terminology we use, a relation of type $A{\sim}B$ has *target* A and *source* B.

Definition 82 (Block-Ordered Relation) Suppose T is a relation of type $C{\sim}C$, f is a relation of type $C{\sim}A$ and g is a relation of type $C{\sim}B$. Suppose further that T is a provisional ordering and that f and g are functional and onto the domain of T. That is, suppose

(83) $$f \circ f^{\cup} \;=\; f{<} \;=\; T \cap T^{\cup} \;=\; g{<} \;=\; g \circ g^{\cup} \;.$$

Then we say that the relation $f^{\cup} \circ T \circ g$ is a *block-ordered relation*. A relation R of type $A{\sim}B$ is said to be *block-ordered* by f, g and T if $R = f^{\cup} \circ T \circ g$ and $f^{\cup} \circ T \circ g$ is a block-ordered relation.

□

The archetypical example of a block-ordered relation is a preorder. Informally, if R is a preorder, its symmetric closure $R \cap R^{\cup}$ is an equivalence relation, and the relation R defines a partial ordering on the equivalence classes. Equivalently, if a representative element is chosen for each equivalence class, the relation R is a partial ordering on the representatives. Theorem 84 makes this precise.

Theorem 84 Suppose T is a provisional preorder and suppose J is a (coreflexive) index of $T{\prec}$. Then $J{\circ}T{\circ}J$ is an index of T and is a provisional ordering. Hence, T is a block-ordered relation.

Proof That $J{\circ}T{\circ}J$ is an index of T is the combination of lemma 48 and theorem 61. It is a provisional preorder because T is a preorder and J is coreflexive. So, it remains to show that $J{\circ}T{\circ}J$ is provisionally anti-symmetric. That is, we must show that $J{\circ}T{\circ}J \cap (J{\circ}T{\circ}J)^{\cup} \subseteq I$.

$$\begin{array}{ll}
 & J{\circ}T{\circ}J \cap (J{\circ}T{\circ}J)^{\cup} \\
= & \{ \quad J \text{ is coreflexive, distributivity} \quad \} \\
 & J{\circ}(T \cap T^{\cup}){\circ}J \\
\subseteq & \{ \quad 48 \quad \} \\
 & J \circ T{\prec} \circ J \\
= & \{ \quad J \text{ is an index of } T{\prec}\text{, definition 59(b) with } P := T{\prec} \quad \} \\
 & J \\
\subseteq & \{ \quad J \text{ is coreflexive} \quad \} \\
 & I \;.
\end{array}$$

□

Identifying a block-ordering of a relation —if it exists— is important for efficiency. Although a relation is defined to be a set of pairs, relations —even relations on finite sets— are rarely stored as such; instead some base set of pairs is stored and an algorithm used to generate, on demand, additional information about the relation. This is particularly so of ordering relations. For example, a test $m < n$ on integers m and n in a computer program is never implemented as a table lookup; instead an algorithm is used to infer from the basic relations $0 < 1$ together with the internal representation of m and n what the value of the test is. In the case of block-structured relations, functional relations f and g define partial equivalence relations $f^{\cup} \circ f$ and $g^{\cup} \circ g$ on their respective sources. (The relations $f^{\cup} \circ f$ and $g^{\cup} \circ g$ are partial because f and g are not required to be total.) Combining the functional relations with an ordering relation on their (common) target is an effective way of implementing a relation (assuming the ordering relation is also implemented effectively).

Example 85 Suppose G is the edge relation of a finite graph. The relation G^* is, of course, a preorder and so is block-ordered. The block-ordering of G^* given by theorem 84 is, however, not very useful. For practical purposes a block-ordering constructed from G (rather than G^*) is preferable. Here we outline how this is done.

Recall from example 69, that the diagonal $\Delta(G^*)$ is the relation $G^* \cap (G^{\cup})^*$ and that this is an equivalence relation on the nodes of G, whereby the equivalence classes are the *strongly connected components* of G. Let N denote the nodes of G and C denote the set of strongly connected components of G. By theorem 62, there is a function sc of type $C \leftarrow N$ such that

(86) $$G^* \cap (G^{\cup})^* \;=\; \mathsf{sc}^{\cup} \circ \mathsf{sc} \;\;.$$

The relation $\mathcal{A}$ defined by

$$\mathsf{sc} \circ G \circ \mathsf{sc}^{\cup} \;\cap\; \neg I_C$$

is a homogeneous relation on the strongly connected components of G, i.e. a relation of type $C \sim C$. Informally, it is a graph obtained from the graph G by coalescing the nodes in a strongly connected component of G into a single node whilst retaining the edges of G that connect nodes in distinct strongly connected components. A fundamental theorem is that

(87) $$G^* \;=\; \mathsf{sc}^{\cup} \circ \mathcal{A}^* \circ \mathsf{sc} \;\;.$$

Moreover, $\mathcal{A}$ is acyclic. That is,

(88) $$I_C \cap \mathcal{A}^+ \;=\; \perp\!\!\!\perp \;\;.$$

(See [BDGv22, Bac22] for the details of the proof of (87) and (88). In fact the theorem is valid for all relations G; finiteness is not required.)

The relation $\mathcal{A}^*$ is, of course, transitive. It is also reflexive; combined with its acyclicity, it follows that

(89) $$\mathcal{A}^* \cap (\mathcal{A}^*)^{\cup} \;=\; I_C \;\;.$$

That is, $\mathcal{A}^*$ is a (total) provisional ordering on C. The conclusion is that G^* is block-ordered by sc, sc and $\mathcal{A}^*$.

Informally, a finite graph can always be decomposed into its strongly connected components together with an acyclic graph connecting the components.

Although the informal interpretation of this theorem is well-known, the formal proof is non-trivial. Although not formulated in the same way, it is essentially the "transitive reduction" of an arbitrary (not necessarily acyclic) graph formulated by Aho, Garey and Ullman [AGU72, Theorem 2].

The decomposition (87) is (implicitly) exploited when computing the inverse A^{-1} of a real matrix A in order to minimise storage requirements: using an elimination technique, a so-called "product form" is computed for each strongly connected component, whilst the process of "forward substitution" is applied to the acyclic-graph structure.
□

Theorem 90 makes precise the statement that block orderings —where they exist— are unique "up to isomorphism".

Theorem 90 Suppose T is a provisional ordering. Suppose also that f and g are functional and onto the domain of T. That is, suppose

$$f\circ f^{\cup} \;=\; f< \;=\; T\cap T^{\cup} \;=\; g< \;=\; g\circ g^{\cup} \quad .$$

Suppose further[5] that S, h and k satisfy the same properties as T, f and g (respectively) and that

(91) $\quad f^{\cup}\circ T\circ g \;=\; h^{\cup}\circ S\circ k \quad .$

Then

(92) $\quad f{>}=h{>} \;\wedge\; g{>}=k{>} \quad ,$

(93) $\quad f^{\cup}\circ g = h^{\cup}\circ k \quad ,$

(94) $\quad f^{\cup}\circ T^{\cup}\circ g \;=\; h^{\cup}\circ S^{\cup}\circ k$, and

(95) $\quad f\circ h^{\cup} \;=\; g\circ k^{\cup} \quad .$

Also, letting ϕ denote $f\circ h^{\cup}$ (equally, by (95), $g\circ k^{\cup}$),

(96) $\quad \phi\circ\phi^{\cup} = T\cap T^{\cup} \;\wedge\; \phi^{\cup}\circ\phi = S\cap S^{\cup} \;\wedge\; \phi\circ T = S\circ\phi \quad .$

In words, ϕ is an order isomorphism of the domains of T and S.

Proof In combination with the assumption (91), properties (92), (94) and (93) are immediate from (105), (106) and (107), respectively.

Proof of (95) is a step on the way to proving (96). From symmetry considerations, it is an obvious first step.

$$
\begin{array}{ll}
 & f\circ h^{\cup} \\
= & \{ \quad \text{assumption:} \quad k\circ k^{\cup} = h< \quad \} \\
 & f\circ h^{\cup}\circ k\circ k^{\cup} \\
= & \{ \quad (93) \quad \} \\
 & f\circ f^{\cup}\circ g\circ k^{\cup} \\
= & \{ \quad \text{assumption:} \quad f\circ f^{\cup} = g< \quad \} \\
 & g\circ k^{\cup} \quad .
\end{array}
$$

Now,

[5] The types of T and S may be different. The types of f and h, and of g and k will then also be different. As in lemma 104, the requirement is that the types are compatible with the type restrictions on the operators in all assumed properties. The symbol " I " in (96) is overloaded: if the type of T is $A{\sim}A$ and the type of S is $B{\sim}B$, $\phi\circ\phi^{\cup}$ has type $A{\sim}A$ and $\phi^{\cup}\circ\phi$ has type $B{\sim}B$.

$$\begin{array}{ll}
& \phi\circ\phi^{\cup} \\
= & \{ \quad \text{definition of } \phi\text{, converse} \quad \} \\
& f\circ h^{\cup}\circ h\circ f^{\cup} \\
= & \{ \quad (95) \quad \} \\
& g\circ k^{\cup}\circ h\circ f^{\cup} \\
= & \{ \quad (93) \text{ and converse} \quad \} \\
& g\circ g^{\cup}\circ f\circ f^{\cup} \\
= & \{ \quad \text{assumption:} \quad f\circ f^{\cup} = T\cap T^{\cup} = g\circ g^{\cup} \quad \} \\
& T\cap T^{\cup} \quad .
\end{array}$$

Symmetrically, $\phi^{\cup}\circ\phi = T\cap T^{\cup}$. Finally,

$$\begin{array}{ll}
& T\circ\phi \\
= & \{ \quad \text{definition of } \phi \quad \} \\
& T\circ f\circ h^{\cup} \\
= & \{ \quad \text{assumptions:} \quad f\circ f^{\cup} = T\cap T^{\cup} = g\circ g^{\cup} \\
& \qquad T \;=\; (T\cap T^{\cup})\circ T\circ(T\cap T^{\cup}) \quad \} \\
& f\circ f^{\cup}\circ T\circ g\circ g^{\cup}\circ f\circ h^{\cup} \\
= & \{ \quad \text{assumption:} \quad f^{\cup}\circ T\circ g \;=\; h^{\cup}\circ S\circ k \quad \text{, (93) and converse} \quad \} \\
& f\circ h^{\cup}\circ S\circ k\circ k^{\cup}\circ h\circ h^{\cup} \\
= & \{ \quad \text{assumption:} \quad h\circ h^{\cup} = S\cap S^{\cup} = k\circ k^{\cup} \quad \} \\
& f\circ h^{\cup}\circ S \\
= & \{ \quad \text{definition of } \phi \quad \} \\
& \phi\circ S \quad .
\end{array}$$

□

6.1 Pair Algebras and Galois Connections

In order to find lots of examples of block-ordered relations one need look no further than the theory of Galois connections (which are, of course, ubiquitous). In this section, we briefly review the notion of a "pair algebra" —due to Hartmanis and Stearns [HS64, HS66]— and its relation to Galois connections.

Hartmanis and Stearns limited their analysis to finite, complete posets, and their analysis was less general than is possible. This work was extended in [Bac98] to non-finite posets and the current section is a short extract.

A Galois connection involves two posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$ and two functions, $F\in\mathcal{A}\leftarrow\mathcal{B}$ and $G\in\mathcal{B}\leftarrow\mathcal{A}$. These four components together form a *Galois connection* iff for all $b\in\mathcal{B}$ and $a\in\mathcal{A}$

(97) $\quad F.b\sqsubseteq a \equiv b\preceq G.a \quad .$

We refer to F as the *lower adjoint* and to G as the *upper adjoint*.

A Galois connection is thus a connection between two functions between posets. Typical accounts of the properties of Galois connections (e.g. [GHK+80]) focus on the properties of these *functions*. For example, given a function F, one may ask whether F is a lower adjoint in a Galois connection. The question posed by Hartmanis and Stearns was, however, rather different.

To motivate their question, note that the statement $F.b \sqsubseteq a$ defines a *relation* between $\mathcal{B}$ and $\mathcal{A}$. So too does $b \preceq G.a$. The existence of a Galois connection states that these two relations are equal. A natural question is therefore: under which conditions does an arbitrary (binary) relation between two posets define a Galois connection between the sets?

Exploring the question in more detail leads to two separate questions. The first is: suppose R is a relation between posets $(\mathcal{A}, \sqsubseteq)$ and $(\mathcal{B}, \preceq)$. What is a necessary and sufficient condition that there exist a function F such that

$$(a,b) \in R \;\equiv\; F.b \sqsubseteq a \quad ?$$

The second is the dual of the first: given relation R, what is a necessary and sufficient condition that there exist a function G such that

$$(a,b) \in R \;\equiv\; b \preceq G.a \quad ?$$

The conjunction of these two conditions is a necessary and sufficient condition for a relation R to define a Galois connection. Such a relation is called a *pair algebra*.

Example 98 It is easy to demonstrate that the two questions are separate. To this end, fig. 4 depicts two posets and a relation between them. The posets are $\{\alpha,\beta\}$ and $\{A,B\}$; both are ordered lexicographically: the reflexive-transitive reduction of the lexicographic ordering is depicted by the directed edges. The relation of type $\{\alpha,\beta\} \sim \{A,B\}$ is depicted by the undirected edges.

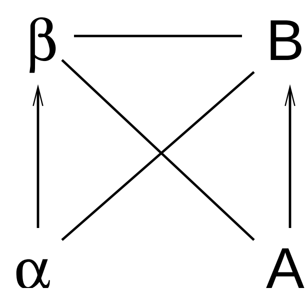

Figure 4: A Relation on Two Posets

Let the relation be denoted by R. Define the function F of type $\{\alpha,\beta\} \leftarrow \{A,B\}$ by $F.B = \alpha$ and $F.A = \beta$. Then it is easy to check that. for $a \in \{\alpha,\beta\}$ and $b \in \{A,B\}$,

$$(a,b) \in R \;\equiv\; F.b \sqsubseteq a \quad .$$

(There are just four cases to be considered.) On the other hand, there is no function G of type $\{A,B\} \leftarrow \{\alpha,\beta\}$ such that

$$(a,b) \in R \;\equiv\; b \preceq G.a \quad .$$

To check that this is indeed the case, it suffices to check that the assignment $G.A = \alpha$ is invalid (because $\alpha \sqsubseteq \alpha$ but $(\alpha, A) \notin R$) and the assignment $G.A = \beta$ is also invalid (because $\alpha \sqsubseteq \beta$ but $(\alpha, A) \notin R$).

□

Example 99 A less artificial, general way to demonstrate that the two questions are separate is to consider the membership relation. Specifically, suppose $\mathcal{S}$ is a set. Then the membership relation, denoted as usual by the —overloaded— symbol "$\in$", is a heterogeneous relation of type $\mathcal{S}\sim 2^{\mathcal{S}}$ (where $2^{\mathcal{S}}$ denotes the type of subsets of $\mathcal{S}$). Now, for all x of type $\mathcal{S}$ and X of type $2^{\mathcal{S}}$,

$$x\in X \;\equiv\; \{x\}\subseteq X \quad .$$

The right side of this equation has the form $F.b\sqsubseteq a$ where F is the function that maps an element into a singleton set and the ordering is the subset ordering. Also, its left side has the form $(a,b)\in R$, where the relation R is the membership relation and a and b are x and X, respectively. (This is where the overloading of notation can become confusing, for which our apologies!) It is, however, not possible to express $x\in X$ in the form $x\preceq G.X$ (except in the trivial cases where $\mathcal{S}$ has cardinality at most one). We leave the proof to the reader.
□

Example 100 An example of a Galois connection is the definition of the ceiling function on real numbers: for all real numbers x, $\lceil x\rceil$ is an integer such that, for all integers m,

$$x\leq m \;\equiv\; \lceil x\rceil\leq m \quad .$$

To properly fit the definition of a Galois connection, it is necessary to make explicit the implicit coercion from integers to real numbers in the left side of this equation. Specifically, we have, for all real numbers x and integers m,

$$x \leq_{\mathbb{R}} \mathsf{real}.m \quad\equiv\quad \lceil x\rceil \leq_{\mathbb{Z}} m$$

where real denotes the function that "coerces" an integer to a real, and $\leq_{\mathbb{R}}$ and $\leq_{\mathbb{Z}}$ denote the (homogeneous) at-most relations on, respectively, real numbers and integers. If, however, we consider the symbol "$\leq$" on the left side of the equation to denote the heterogeneous at-most relation of type $\mathbb{R}\sim\mathbb{Z}$, the fact that

$$x\leq m \quad\equiv\quad \lceil x\rceil \leq_{\mathbb{Z}} m$$

gives a representation of the (heterogeneous) "$\leq$" relation of type $\mathbb{R}\sim\mathbb{Z}$ as a block-ordered relation: referring to definition 82, the provisional ordering is $\leq_{\mathbb{Z}}$, f is the ceiling function and g is the identity function.

More interesting is if we take the contrapositive. We have, for all real numbers x and integers m,

$$m<x \;\equiv\; m\leq\lceil x\rceil-1 \quad .$$

On the right of this equation is the (homogeneous) at-most relation on integers. On the left is the (heterogeneous) less-than relation of type $\mathbb{Z}\sim\mathbb{R}$. The equation demonstrates that this relation is block-ordered; the "blocks" of real numbers being all the numbers that have the same ceiling. (The functional f is the identity function, the functional g maps real number x to $\lceil x\rceil-1$ and the provisional ordering is the ordering $\leq_{\mathbb{Z}}$.) The example is interesting because the (homogeneous) less-than relation on real numbers is *not* block-ordered. This is because its diagonal is empty. See examples 70 and 127.
□

Returning to the discussion immediately preceding example 98, the two separate questions are each of interest in their own right: a positive answer to either question may predict that a given relation has a block-ordering of a specific form: in the case of the first question, where the functional g in definition 82 is the identity function, and, in the case of the second question, where the functional f in definition 82 is the identity function. In both cases, a further step is to check the requirement on f and g: in the first case, one has to check that the function F is surjective and in the second case that the function G is surjective. (A Galois connection is said to be "perfect" if both F and G are surjective.) For example, the fact that

$$x \leq m \quad \equiv \quad x \leq_{\mathbb{R}} \mathsf{real}.m$$

does *not* define a block-ordering because the function real is not surjective.

The relevant theory predicting exactly when the first of the two questions has a positive answer is as follows. Suppose $(\mathcal{B},\sqsubseteq)$ is a complete poset. Let $\sqcap$ denote the infimum operator for $\mathcal{B}$ and suppose p is a predicate on $\mathcal{B}$. Then we define *inf-preserving* by

(101) $\quad p \text{ is inf-preserving} \quad \equiv \quad \langle \forall g :: p.(\sqcap g) \equiv \langle \forall x :: p.(g.x) \rangle \rangle$.

So, for a given a, the predicate $\langle b :: (a,b) \in R \rangle$ is inf-preserving equivales

$$\langle \forall g \ :: \ (a, \sqcap g) \in R \ \equiv \ \langle \forall x :: (a,\ g.x) \in R \rangle \rangle \ .$$

Then we have:

Theorem 102 Suppose $\mathcal{A}$ is a set and $(\mathcal{B},\sqsubseteq)$ is a complete poset. Suppose $R \subseteq \mathcal{A} \times \mathcal{B}$ is a relation between the two sets. Define F by

(103) $\quad F.a \quad = \quad \langle \sqcap b \ : \ (a,b) \in R \ : \ b \rangle$.

Then the following two statements are equivalent.

- $\langle \forall\, a,b \ : \ a \in \mathcal{A} \wedge b \in \mathcal{B} \ : \ (a,b) \in R \ \equiv \ F.a \sqsubseteq b \rangle$.
- For all a, the predicate $\langle b :: (a,b) \in R \rangle$ is inf-preserving.

□

The answer to the second question is, of course, obtained by formulating the dual of theorem 102.

In general, for most relations occurring in practical information systems the answer to the pair-algebra questions will be negative: the required inf- and sup-preserving properties just do not hold. However, a common way to define a pair algebra is to extend a given relation to a relation between sets in such a way that the infimum and supremum preserving properties are automatically satisfied. Hartmanis and Stearns' [HS64, HS66] solution to the state assignment problem was to consider the lattice of partitions of a given set; in so-called "concept analysis", the technique is to extend a given relation to a relation between rectangles.

An important property of Galois connections is the theorem we call the "unity of opposites": if F and G are the adjoint functions in a Galois connection of the posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$, then there is an isomorphism between the posets $(F.\mathcal{B}\,,\ \sqsubseteq)$ and $(G.\mathcal{A}\,,\preceq)$. ($F.\mathcal{B}$ denotes the "image" of the function F, and similarly for $G.\mathcal{A}$.) Knowledge of the unity-of-opposites theorem suggests theorem 90, which expresses an isomorphism between different representations of block-ordered relations.

6.2 Analogie Frappante

In this section, we relate block-orderings to diagonals. The main result is theorem 114; we call theorem 114 the "analogie frappante" because it generalises Riguet's "analogie frappante" connecting "relation de Ferrers" to diagonals.

Some elements of the following lemma have been recorded earlier by Winter [Win04]. We think the overlap is justified because Winter's calculations make very heavy use of complementation whereas our calculations avoid its use altogether.

Lemma 104 Suppose T is a provisional ordering of type $C{\sim}C$. Suppose also that f and g are functional and onto the domain of T. That is, suppose[6] that

$$f\circ f^{\cup} \;=\; f{<} \;=\; T\cap T^{\cup} \;=\; g{<} \;=\; g\circ g^{\cup} \quad .$$

Let R denote $f^{\cup}\circ T\circ g$. Then

(105) $\quad R{<} = f{>} \;\wedge\; R{>} = g{>} \quad ,$

(106) $\quad f^{\cup}\circ T^{\cup}\circ g \;=\; R{<}\circ(R\backslash R/R)^{\cup}\circ R{>}$, and

(107) $\quad f^{\cup}\circ g \;=\; \Delta R \quad ,$

(108) $\quad R{<} = (\Delta R){<} \;\wedge\; R{>} = (\Delta R){>} \quad ,$

(109) $\quad R{\prec} \;=\; \Delta R\circ\Delta R^{\cup} \;=\; f^{\cup}\circ f \;\wedge\; R{\succ} \;=\; \Delta R^{\cup}\circ\Delta R \;=\; g^{\cup}\circ g \quad .$

Proof Property (105) is a straightforward application of domain calculus:

$$
\begin{array}{ll}
 & R{>} \\
= & \quad\{ \quad \text{definition: } R = f^{\cup}\circ T\circ g \quad \} \\
 & (f^{\cup}\circ T\circ g){>} \\
= & \quad\{ \quad \text{domains (specifically, } [\,(U\circ V){>} = (U{>}\circ V){>}\,] \text{ and } [\,(U^{\cup}){>} = U{<}\,]) \quad \} \\
 & (f{<}\circ T\circ g){>} \\
= & \quad\{ \quad \text{assumption: } T = f{<}\circ T\circ g{<} \text{ (so } T = f{<}\circ T) \quad \} \\
 & (T\circ g){>} \\
= & \quad\{ \quad \text{domains (specifically, } [\,(U\circ V){>} = (U{>}\circ V){>}\,]) \quad \} \\
 & (T{>}\circ g){>} \\
= & \quad\{ \quad \text{lemma 50 and assumption: } T\cap T^{\cup} = g{<} \quad \} \\
 & g{>} \quad .
\end{array}
$$

By a symmetric argument, $(f^{\cup}\circ T\circ g){<} = f{>}$.

Now we consider (106). The raison d'être of (106) is that it expresses the left side as a function of $f^{\cup}\circ T\circ g$. In a pointwise calculation a natural step is to use indirect ordering. In a point-free calculation, this corresponds to using factors. That is, we exploit lemma 47:

[6] The ordering T must be homogeneous but f and g may be heterogeneous and of different type, so long as both have target C.

$$
\begin{array}{ll}
& f^{\cup}\circ T^{\cup}\circ g \\
= & \{ \quad \text{assumption: } T \text{ is a provisional ordering} \\
& \quad\ \ \text{lemmas 44, 48 and 47} \quad \} \\
& f^{\cup}\circ(T\cap T^{\cup})\circ T^{\cup}\backslash T^{\cup}/T^{\cup}\circ(T\cap T^{\cup})\circ g \\
= & \{ \quad \text{assumption: } f< \,=\, T\cap T^{\cup} \,=\, g< \quad \} \\
& f^{\cup}\circ T^{\cup}\backslash T^{\cup}/T^{\cup}\circ g \\
= & \{ \quad \text{lemma 36 and assumption: } T = f<\circ T\circ g< \quad \} \\
& f>\circ(g^{\cup}\circ T^{\cup}\circ f)\backslash(g^{\cup}\circ T^{\cup}\circ f)/(g^{\cup}\circ T^{\cup}\circ f)\circ g> \\
= & \{ \quad \text{(105) and definition of } R \quad \} \\
& R<\circ R^{\cup}\backslash R^{\cup}/R^{\cup}\circ R> \\
= & \{ \quad \text{factors} \quad \} \\
& R<\circ(R\backslash R/R)^{\cup}\circ R> \quad .
\end{array}
$$

Note the use of lemma 36. The discovery of this lemma is driven by the goal of the calculation.

The pointwise interpretation of $f^{\cup}\circ g$ is a relation expressing equality between values of f and g. This suggests that, in order to prove (107), we begin by exploiting the anti-symmetry of T:

$$
\begin{array}{ll}
& f^{\cup}\circ g \\
= & \{ \quad f< \,=\, T\cap T^{\cup} \,=\, g< \text{ and domains} \quad \} \\
& f^{\cup}\circ(T\cap T^{\cup})\circ g \\
= & \{ \quad \text{distributivity (valid because } f \text{ and } g \text{ are functional)} \quad \} \\
& f^{\cup}\circ T\circ g \ \cap\ f^{\cup}\circ T^{\cup}\circ g \\
= & \{ \quad \text{definition of } R \text{ and (106)} \quad \} \\
& f^{\cup}\circ T\circ g \ \cap\ f>\circ((f^{\cup}\circ T\circ g)\backslash(f^{\cup}\circ T\circ g)/(f^{\cup}\circ T\circ g))^{\cup}\circ g> \\
= & \{ \quad \text{(110) (see below)} \quad \} \\
& f>\circ f^{\cup}\circ T\circ g\circ g> \ \cap\ ((f^{\cup}\circ T\circ g)\backslash(f^{\cup}\circ T\circ g)/(f^{\cup}\circ T\circ g))^{\cup} \\
= & \{ \quad \text{domains (specifically, } f>\circ f^{\cup} = f^{\cup} \text{ and } g\circ g> = g\text{)} \quad \} \\
& f^{\cup}\circ T\circ g \ \cap\ ((f^{\cup}\circ T\circ g)\backslash(f^{\cup}\circ T\circ g)/(f^{\cup}\circ T\circ g))^{\cup} \\
= & \{ \quad \text{definition of } R \text{ and } \Delta R \quad \} \\
& \Delta R \quad .
\end{array}
$$

A crucial step in the above calculation is the use of the property

(110) $\quad U\cap p\circ V\circ q \ = \ p\circ(U\cap V)\circ q \ = \ p\circ U\circ q\cap V$

for all relations U and V and coreflexive relations p and q. This is a frequently used property of domain restriction.

The remaining equations (108) and (109) are straightforward. First

$$
\begin{array}{ll}
& (\Delta R)< \\
= & \{ \quad (107) \quad \} \\
& (f^{\cup}\circ g)< \\
= & \{ \quad \text{domains and assumption: } f< = g< \quad \} \\
& f> \\
= & \{ \quad \text{assumption: } f< \,=\, T\cap T^{\cup} \quad \}
\end{array}
$$

$$((T \cap T^{\cup}) \circ f)_{>}$$
$= \quad \{ \quad$ domains and converse $\quad \}$
$$(f^{\cup} \circ (T \cap T^{\cup}))_{<}$$
$= \quad \{ \quad$ lemma 50 and domains $\quad \}$
$$(f^{\cup} \circ T)_{<}$$
$= \quad \{ \quad$ domains and assumption: $g_{<} = T \cap T^{\cup}$ and lemma 50 $\quad \}$
$$(f^{\cup} \circ T \circ g)_{<} \quad .$$

That is $(\Delta R)_{<} = R_{<}$. The dual equation $(\Delta R)_{>} = R_{>}$ is immediate from the fact that $(\Delta R)^{\cup} = \Delta(R^{\cup})$ and properties of the domain operators. For the per domains, we have:

$$R_{\prec}$$
$= \quad \{ \quad R_{<} = (\Delta R)_{<}$ and $R_{>} = (\Delta R)_{>}$ (above); lemma 74 $\quad \}$
$$(\Delta R)_{\prec}$$
$= \quad \{ \quad \Delta R$ is difunctional, theorem 41 (with $R := \Delta R$) $\quad \}$
$$\Delta R \circ \Delta R^{\cup}$$
$= \quad \{ \quad$ lemma 104 and definition of ΔR $\quad \}$
$$f^{\cup} \circ g \circ (f^{\cup} \circ g)^{\cup}$$
$= \quad \{ \quad$ converse and $f_{<} = g_{<} = g \circ g^{\cup}$ $\quad \}$
$$f^{\cup} \circ f \quad .$$

Again, the dual equation is immediate.
□

We now prove the converse of lemma 104.

Lemma 111 A relation R is block-ordered if $R_{<} = (\Delta R)_{<}$ and $R_{>} = (\Delta R)_{>}$.

Proof Suppose $R_{<} = (\Delta R)_{<}$ and $R_{>} = (\Delta R)_{>}$. Our task is to construct relations f, g and T such that

$$R = f^{\cup} \circ T \circ g \quad ,$$

$$T \cap T^{\cup} \subseteq I \quad \wedge \quad T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \wedge \quad T \circ T \subseteq T \quad \text{and}$$

$$f \circ f^{\cup} = f_{<} = T \cap T^{\cup} = g_{<} = g \circ g^{\cup} \quad .$$

Since ΔR is difunctional, theorem 64 is the obvious place to start. Applying the theorem, we can construct f and g such that

$$\Delta R = f^{\cup} \circ g \quad \wedge \quad f \circ f^{\cup} = f_{<} = g \circ g^{\cup} = g_{<} \quad .$$

Using standard properties of the domain operators together with the assumption that $R_{<} = (\Delta R)_{<}$ and $R_{>} = (\Delta R)_{>}$, it follows that

$$R_{<} = f_{>} \quad \wedge \quad R_{>} = g_{>} \quad .$$

It remains to construct the provisional ordering T. The appropriate construction is suggested by lemma 104, in particular (106). Specifically, we define T by the equation

(112) $\quad T = g \circ R \backslash R / R \circ f^{\cup} \quad .$

The proof that $R = f^{\cup} \circ T \circ g$ is by mutual inclusion. First note that

(113) $\quad f^{\cup} \circ T \circ g \;=\; \Delta R \circ R\backslash R/R \circ \Delta R$

since

$$
\begin{array}{ll}
 & f^{\cup} \circ T \circ g \\
= & \quad \{ \qquad (112) \quad \} \\
 & f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \circ g \\
= & \quad \{ \qquad \Delta R = f^{\cup} \circ g \quad \} \\
 & \Delta R \circ R\backslash R/R \circ \Delta R \quad .
\end{array}
$$

So

$$
\begin{array}{ll}
 & f^{\cup} \circ T \circ g \\
\subseteq & \quad \{ \qquad (113) \text{ and } \Delta R \subseteq R \quad \} \\
 & R \circ R\backslash R/R \circ R \\
\subseteq & \quad \{ \qquad \text{cancellation} \quad \} \\
 & R \quad .
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & R \subseteq f^{\cup} \circ T \circ g \\
= & \quad \{ \qquad (113) \quad \} \\
 & R \subseteq \Delta R \circ R\backslash R/R \circ \Delta R \\
= & \quad \{ \qquad \text{per domains} \quad \} \\
 & R{\prec} \circ R \circ R{\succ} \subseteq \Delta R \circ R\backslash R/R \circ \Delta R \\
= & \quad \{ \qquad \text{assumption: } R{<} = (\Delta R){<} \text{ and } R{>} = (\Delta R){>}\text{, lemma 74} \quad \} \\
 & (\Delta R){\prec} \circ R \circ (\Delta R){\succ} \subseteq \Delta R \circ R\backslash R/R \circ \Delta R \\
= & \quad \{ \qquad \Delta R \text{ is difunctional, theorem 41 (with } R := \Delta R\text{)} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R \circ \Delta R^{\cup} \circ \Delta R \subseteq \Delta R \circ R\backslash R/R \circ \Delta R \\
\Leftarrow & \quad \{ \qquad \text{monotonicity} \quad \} \\
 & \Delta R^{\cup} \circ R \circ \Delta R^{\cup} \subseteq R\backslash R/R \\
\Leftarrow & \quad \{ \qquad \Delta R^{\cup} \subseteq R\backslash R/R\text{, monotonicity} \quad \} \\
 & R\backslash R/R \circ R \circ R\backslash R/R \subseteq R\backslash R/R \\
= & \quad \{ \qquad \text{factors} \quad \} \\
 & R \circ R\backslash R/R \circ R \circ R\backslash R/R \circ R \subseteq R \\
= & \quad \{ \qquad \text{cancellation} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

Combining the two inclusions we conclude that indeed $R = f^{\cup} \circ T \circ g$.

We now establish the requirements on T. First,

$$
\begin{array}{ll}
 & T \cap T^{\cup} \\
= & \quad \{ \qquad \text{definition and converse} \quad \} \\
 & g \circ R\backslash R/R \circ f^{\cup} \;\cap\; f \circ (R\backslash R/R)^{\cup} \circ g^{\cup} \\
\subseteq & \quad \{ \qquad \text{modular law} \quad \} \\
 & f \circ (f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \circ g \;\cap\; (R\backslash R/R)^{\cup}) \circ g^{\cup}
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & f \circ (\Delta R \circ R\backslash R/R \circ \Delta R \ \cap \ (R\backslash R/R)^{\cup}) \circ g^{\cup} \\
\subseteq & \{ \quad \Delta R \subseteq R \text{, monotonicity and cancellation} \quad \} \\
 & f \circ (R \ \cap \ (R\backslash R/R)^{\cup}) \circ g^{\cup} \\
= & \{ \quad \Delta R = R \ \cap \ (R\backslash R/R)^{\cup} \quad \} \\
 & f \circ \Delta R \circ g^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & f \circ f^{\cup} \circ g \circ g^{\cup} \\
= & \{ \quad f \circ f^{\cup} \ = \ f< \ = \ g \circ g^{\cup} \ = \ g< \quad \} \\
 & f< \ \ .
\end{array}
$$

Thus $T \cap T^{\cup} \subseteq f<$. So $T \cap T^{\cup} \subseteq I$. Now

$$
\begin{array}{ll}
 & f< \ \subseteq T \cap T^{\cup} \\
= & \{ \quad \text{infima and } f< \text{ is coreflexive} \quad \} \\
 & f< \ \subseteq T \\
\Leftarrow & \{ \quad \text{domains} \quad \} \\
 & f \circ f^{\cup} \ \subseteq \ T \\
\Leftarrow & \{ \quad \text{definition of } T \text{ and monotonicity} \quad \} \\
 & f \ \subseteq \ g \circ R\backslash R/R \\
\Leftarrow & \{ \quad f< \ = \ g \circ g^{\cup} \text{, domains and monotonicity} \quad \} \\
 & g^{\cup} \circ f \ \subseteq \ R\backslash R/R \\
= & \{ \quad f^{\cup} \circ g = \Delta R \quad \} \\
 & \Delta R^{\cup} \subseteq R\backslash R/R \\
= & \{ \quad \Delta R \ = \ R \cap (R\backslash R/R)^{\cup} \text{, converse} \quad \} \\
 & \text{true} \ \ .
\end{array}
$$

So, by anti-symmetry we have established that $T \cap T^{\cup} = f<$. Since also $f< = g<$, we conclude from the definition of T and properties of domains that

$$T \ = \ (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \ \ .$$

The final task is to show that T is transitive:

$$
\begin{array}{ll}
 & T \circ T \\
= & \{ \quad \text{definition} \quad \} \\
 & g \circ R\backslash R/R \circ f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & g \circ R\backslash R/R \circ \Delta R \circ R\backslash R/R \circ f^{\cup} \\
\subseteq & \{ \quad \Delta R \subseteq R \quad \} \\
 & g \circ R\backslash R/R \circ R \circ R\backslash R/R \circ f^{\cup} \\
\subseteq & \{ \quad \text{factors} \quad \} \\
 & g \circ R\backslash R/R \circ f^{\cup} \\
= & \{ \quad \text{definition} \quad \} \\
 & T \ \ .
\end{array}
$$

□

It is interesting to reflect on the proof of lemma 111. The hardest part is to find appropriate definitions of f, g and T such that $R = f^{\cup} \circ T \circ g$. The key to constructing f and g is Riguet's "analogie frappante" [Rig51] whereby he introduced the "différence" —the diagonal ΔR — of the relation R. Expressing the diagonal in terms of factors as we have done makes many parts of the calculations very straightforward. One much less straightforward step is the use of lemma 74 in the proof that $R \subseteq f^{\cup} \circ T \circ g$. Note how calculational needs drive the search for the lemma: in order to simplify the inclusion it is necessary to expose the term $R \backslash R / R$ on the right side, and that is precisely what the lemma enables.

We conclude with the theorem that we call the "analogie frappante". It is not the theorem that Riguet suggested but we have chosen to give it this name in order to recognise Riguet's contribution.

Theorem 114 (Analogie Frappante) A relation R is block-ordered if and only if $R{<} = (\Delta R){<}$ and $R{>} = (\Delta R){>}$.

Proof Lemma 104 establishes "only-if" and lemma 111 establishes "if".
□

Example 115 A generic way to construct examples of relations that are not block-ordered is to exploit example 71. In order to avoid unnecessary repetition, we refer the reader to that example for the definition of the relation in given a finite set $\mathcal{X}$ and a set $\mathcal{S}$ of subsets of $\mathcal{X}$.

Recall that the diagonal $\Delta\mathsf{in}$ of type $\mathcal{X} \sim \mathcal{S}$ is injective. It follows that the size of $(\Delta\mathsf{in}){<}$ is at most the size of $\mathcal{S}$. If, however, the set $\mathcal{S}$ has $\mathcal{X}$ as one of its elements, the size of $\mathsf{in}{<}$ equals the size of $\mathcal{X}$. Theorem 114 thus predicts that, if $\mathcal{X}$ is an element of $\mathcal{S}$, a necessary condition for in to be block-ordered is that the sizes of $\mathcal{X}$ and $\mathcal{S}$ must be equal; conversely, if $\mathcal{X}$ is an element of $\mathcal{S}$, in is not block-ordered if the sizes of $\mathcal{X}$ and $\mathcal{S}$ are different.

Fig. 3 (example 71) shows that, even if the sizes of $\mathcal{X}$ and $\mathcal{S}$ are equal, the relation in may not be block-ordered: as remarked then, for the choice of $\mathcal{S}$ shown in fig. 3, $\mathsf{in}{<}$ and $(\Delta\mathsf{in}){<}$ are different since 0 and 3 are elements of the former but not the latter.

It is straightforward to construct instances of $\mathcal{X}$ and $\mathcal{S}$ such that the relation in is block-ordered. It suffices to ensure that three conditions are satisfied: $\mathcal{X}$ is an element of $\mathcal{S}$, the sizes of $\mathcal{X}$ and $\mathcal{S}$ are equal, and, for each x in $\mathcal{X}$, the set of sets represented by $(x \circ \mathsf{in}){>}$ is totally ordered. Fig. 5 is one such. Referring to definition 82, the functional f is $\Delta\mathsf{in}^{\cup}$ (depicted by rectangles) and the functional g is $I_{\mathcal{S}}$; the ordering relation is the subset relation $\mathsf{in} \backslash \mathsf{in}$ (depicted by the directed graph).
□

The following theorem is a corollary of theorem 78. In combination with theorem 114 it states that a relation is block-ordered iff its core is block-ordered. Testing whether or not a given relation is block-ordered can thus be decomposed into computing a core of the relation and then testing whether or not that is block-ordered. (For practical purposes computing an index of the relation is to be preferred.)

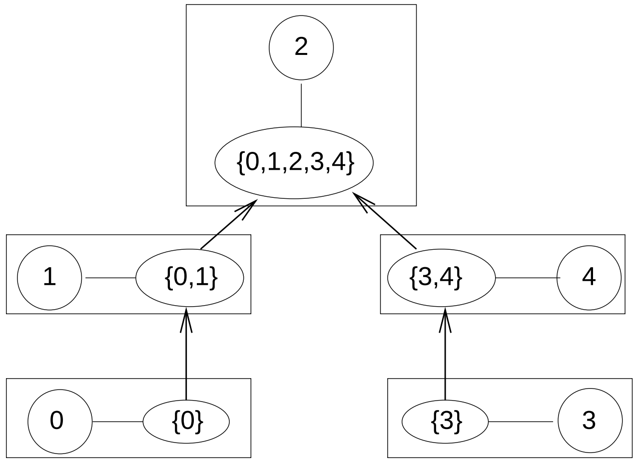

Figure 5: A Block-Ordered Membership Relation

Theorem 116 Suppose R is an arbitrary relation and suppose C is a core of R as witnessed by λ and ρ. Then

$$R_< = (\Delta R)_< \quad \equiv \quad C_< = (\Delta C)_< \;\;.$$

Dually,

$$R_> = (\Delta R)_> \quad \equiv \quad C_> = (\Delta C)_> \;\;.$$

Proof Suppose R, C, λ and ρ are as in definition 54. Then

$$
\begin{array}{ll}
 & C_< = (\Delta C)_< \\
= & \quad \{ \quad \text{definition 54 and theorem 78} \quad \} \\
 & (\lambda \circ R \circ \rho^{\cup})_< = (\lambda \circ \Delta R \circ \rho^{\cup})_< \\
\Rightarrow & \quad \{ \quad \text{Leibniz} \quad \} \\
 & (\lambda^{\cup} \circ (\lambda \circ R \circ \rho^{\cup})_<)_< = (\lambda^{\cup} \circ (\lambda \circ \Delta R \circ \rho^{\cup})_<)_< \\
= & \quad \{ \quad \text{domains} \quad \} \\
 & (\lambda^{\cup} \circ \lambda \circ R \circ \rho^{\cup})_< = (\lambda^{\cup} \circ \lambda \circ \Delta R \circ \rho^{\cup})_< \\
= & \quad \{ \quad \lambda^{\cup} \circ \lambda \circ R = R_{\prec} \circ R = R \,, \\
 & \quad\quad\;\; (\rho^{\cup})_< = (\rho^{\cup} \circ \rho)_< = (R_{\succ})_< = R_> \text{, and domains} \quad \} \\
 & R_< = (\lambda^{\cup} \circ \lambda \circ \Delta R \circ \rho^{\cup})_< \\
= & \quad \{ \quad (\rho^{\cup})_< = (\rho^{\cup} \circ \rho)_< \text{ and domains} \quad \} \\
 & R_< = (\lambda^{\cup} \circ \lambda \circ \Delta R \circ \rho^{\cup} \circ \rho)_< \\
= & \quad \{ \quad \text{theorem 78} \quad \} \\
 & R_< = (\lambda^{\cup} \circ \Delta C \circ \rho)_< \\
= & \quad \{ \quad \text{theorem 78} \quad \} \\
 & R_< = (\Delta R)_< \;\;.
\end{array}
$$

Similarly,

$$
\begin{array}{ll}
 & R_< = (\Delta R)_< \\
= & \quad \{ \quad \text{definition 54, theorem 78 and Leibniz} \quad \}
\end{array}
$$

$$(\lambda^{\cup}\circ C\circ\rho)< \;=\; (\lambda^{\cup}\circ\Delta C\circ\rho)<$$

$\Rightarrow$ { Leibniz and domains }

$$(\lambda\circ\lambda^{\cup}\circ C\circ\rho)< \;=\; (\lambda\circ\lambda^{\cup}\circ\Delta C\circ\rho)<$$

$=$ { $\rho< \;=\; (\rho\circ\rho^{\cup})<$ and domains }

$$(\lambda\circ\lambda^{\cup}\circ C\circ\rho\circ\rho^{\cup})< \;=\; (\lambda\circ\lambda^{\cup}\circ\Delta C\circ\rho\circ\rho^{\cup})<$$

$=$ { theorem 78 (applied twice) }

$$C< \;=\; (\Delta C)< \;\;.$$

The property

$$R< \;=\; (\Delta R)< \quad\equiv\quad C< \;=\; (\Delta C)<$$

follows by mutual implication. The dual follows by instantiating R to $R^{\cup}$ and applying the properties of converse.

□

By combining the definition of block-ordering with theorem 78, it is immediately clear that R is block-ordered if its core C is a provisional ordering. In general, a core of a block-ordered relation will not be a provisional ordering. This is because the types of the targets of the components λ and ρ in the definition of a core are not required to be the same; on the other hand, orderings are required to be homogeneous relations. However by carefully restricting the choice of core, it is possible to construct a core that is indeed a provisional ordering.

Theorem 117 Suppose R is an arbitrary relation. Then if R is block-ordered it has a core that is a provisional ordering.

Proof Suppose R is block-ordered. That is, suppose that f, g and T are relations such that T is a provisional ordering,

$$R = f^{\cup}\circ T\circ g$$

and

$$f\circ f^{\cup} \;=\; f< \;=\; T\cap T^{\cup} \;=\; g< \;=\; g\circ g^{\cup} \;\;.$$

Then, by lemma 104, $R\prec = f^{\cup}\circ f$ and , $R\succ = g^{\cup}\circ g$. Thus f and g satisfy the conditions for witnessing a core C of R. (Cf. definition 54 with $\lambda,\rho := f,g$.) Consequently,

$$C$$

$=$ { definition 54 }

$$f\circ R\circ g^{\cup}$$

$=$ { $R = f^{\cup}\circ T\circ g$ }

$$f\circ f^{\cup}\circ T\circ g\circ g^{\cup}$$

$=$ { $f\circ f^{\cup} \;=\; f< \;=\; T\cap T^{\cup} \;=\; g< \;=\; g\circ g^{\cup}$ }

$$(T\cap T^{\cup})\circ T\circ(T\cap T^{\cup})$$

$=$ { T is a provisional ordering, lemma 50 and domains }

$$T \;\;.$$

We conclude that C is the provisional ordering T.
□

Combining theorem 117 with the theorem that all cores of a given relation are isomorphic, we conclude that any core of a block-ordered relation is isomorphic to a provisional ordering. Loosely speaking, block-ordered relations are provisional orderings up to isomorphism and reduction to the core.

Example 118 From the Galois connection, for all reals x and integers m,

$$\lceil x \rceil \leq m \;\equiv\; x \leq m$$

defining the ceiling function, we deduce that the heterogeneous relation ${}_{\mathbb{R}}{\leq}_{\mathbb{Z}}$ has core the provisional ordering $\leq_{\mathbb{Z}}$. This is because the ceiling function is surjective. Its core in *not* the ordering $\leq_{\mathbb{R}}$ because the coercion real from integers to reals is not surjective. (See also example 100.)

On the other hand, if a Galois connection

$$F.b \sqsubseteq a \equiv b \preceq G.a$$

of posets $(\mathcal{A}, \sqsubseteq)$ and $(\mathcal{B}, \preceq)$ is "perfect" (i.e. both F and G are surjective), both the orderings $\sqsubseteq$ and $\preceq$ are cores of the defined heterogeneous relation. That the orderings are isomorphic is an instance of the unity-of-opposites theorem [Bac02].
□

7 Staircase Relations

For any binary relation R, the relations $R\backslash R$ and R/R are preorders. That is, both are transitive and reflexive. (If R has type $A{\sim}B$ then $R\backslash R$ has type $B{\sim}B$ and R/R has type $A{\sim}A$.) That relation R is a "staircase" relation means formally that the preorder $R\backslash R$ is linear[7]. For brevity, we denote the property of being a staircase relation by SC. That is:

Definition 119 The predicate SC on (binary) relations is defined by, for all R,

$$\mathsf{SC}.R \quad\equiv\quad R\backslash R \cup (R\backslash R)^{\cup} = \top\!\!\top \quad .$$

A relation that satisfies the predicate SC is called a *staircase* relation.
□

The pointwise formulation of the relation $R\backslash R$ is

$$b0[\![R\backslash R]\!]b1 \quad\equiv\quad \langle \forall a : a[\![R]\!]b0 : a[\![R]\!]b1 \rangle \quad .$$

In terms of the mental picture shown in fig. 1, a vertical line through a point b depicts the set of points a such that $a[\![R]\!]b$; a staircase relation is one such that the points of type B can be ordered in such a way that the preorder $R\backslash R$ is depicted by the left-to-right ordering

[7] An ordering S —of any sort— is said to be *linear* if $S \cup S^{\cup} = \top\!\!\top$. Sometimes the word "total" is used instead of linear. For example, Riguet [Rig51] uses the term "totalement ordonnées".

of points on the B-axis, and the length of the vertical lines increases monotonically (although not always strictly) as one proceeds from left to right in the diagram.

Various equivalent definitions of the predicate SC are given below. Riguet [Rig51] defined a "relation de Ferrers" to be a relation satisfying (124). The equivalence of (121) and (122) corresponds in the mental picture of a staircase relation to the property that the vertical lines being increasing in length is equivalent to the horizontal lines being decreasing in length. (Cf. the statement of Riguet's theorem quoted in the introduction.)

Lemma 120 The following are all equivalent formulations of $SC.R$:

(121) $R\backslash R \cup (R\backslash R)^{\cup} \;=\; \top$,

(122) $R/R \cup (R/R)^{\cup} \;=\; \top$,

(123) $R \cup (R\backslash R/R)^{\cup} \;=\; \top$,

(124) $R\circ\neg R^{\cup}\circ R \;\subseteq\; R$.

Proof We prove first that (122) and (124) are equivalent:

$$
\begin{array}{ll}
 & R\circ\neg R^{\cup}\circ R \;\subseteq\; R \\
= & \quad\{\quad \text{factors} \quad\} \\
 & R\circ\neg R^{\cup} \;\subseteq\; R/R \\
= & \quad\{\quad \text{complements} \quad\} \\
 & \top \;\subseteq\; R/R \cup \neg(R\circ\neg R^{\cup}) \\
= & \quad\{\quad (10) \text{ with } R,S := R^{\cup}, R^{\cup} \text{ (and } R = (R^{\cup})^{\cup}\text{)} \quad\} \\
 & \top \;\subseteq\; R/R \;\cup\; R^{\cup}\backslash R^{\cup} \\
= & \quad\{\quad (13) \text{ with } R,S := R,R \quad\} \\
 & \top \;\subseteq\; R/R \cup (R/R)^{\cup} \\
= & \quad\{\quad [S \subseteq \top] \text{ with } S := R/R \cup (R/R)^{\cup} \text{ and anti-symmetry} \quad\} \\
 & \top \;=\; R/R \cup (R/R)^{\cup} \quad .
\end{array}
$$

A symmetric argument establishes the equivalence of (121) and (124):

$$
\begin{array}{ll}
 & R\circ\neg R^{\cup}\circ R \;\subseteq\; R \\
= & \quad\{\quad \text{factors} \quad\} \\
 & \neg R^{\cup}\circ R \;\subseteq\; R\backslash R \\
= & \quad\{\quad \text{complements} \quad\} \\
 & \top \;\subseteq\; R\backslash R \cup \neg(\neg R^{\cup}\circ R) \\
= & \quad\{\quad (10) \text{ with } S,T := R^{\cup}, R^{\cup} \quad\} \\
 & \top \;\subseteq\; R\backslash R \;\cup\; R^{\cup}/R^{\cup} \\
= & \quad\{\quad (14) \text{ with } R,S := R,R \text{ (and } R = (R^{\cup})^{\cup}\text{)} \quad\} \\
 & \top \;\subseteq\; R\backslash R \cup (R\backslash R)^{\cup} \\
= & \quad\{\quad [S \subseteq \top] \text{ with } S := R\backslash R \cup (R\backslash R)^{\cup} \text{ and anti-symmetry} \quad\} \\
 & \top \;=\; R\backslash R \cup (R\backslash R)^{\cup} \quad .
\end{array}
$$

Finally,

$$
\begin{array}{ll}
 & R\circ\neg R^{\cup}\circ R \;\subseteq\; R \\
= & \quad\{\qquad \text{factors} \quad\} \\
 & \neg R^{\cup} \;\subseteq\; R\backslash R/R \\
= & \quad\{\qquad \text{converse and complements} \quad\} \\
 & \mathbb{T} \;\subseteq\; R\cup(R\backslash R/R)^{\cup} \\
= & \quad\{\qquad [\,S\subseteq\mathbb{T}\,] \text{ with } S := R\cup(R\backslash R/R)^{\cup} \text{ and anti-symmetry} \quad\} \\
 & \mathbb{T} \;=\; R\cup(R\backslash R/R)^{\cup} \quad .
\end{array}
$$

□

An example of a staircase relation predicted by lemma 125 is the at-most relation — on natural numbers, integers, rational numbers or reals.

Two general methods for identifying examples of staircase relations are given in lemmas 125 and 126.

Lemma 125 A linear preorder is a staircase relation. That is, for all (homogeneous) R,

$$SC.R \quad\Leftarrow\quad R\circ R\subseteq R \;\wedge\; I\subseteq R \;\wedge\; R\cup R^{\cup}=\mathbb{T} \quad .$$

Proof We have

$$R=R\backslash R/R \quad\Leftarrow\quad R\circ R\subseteq R \;\wedge\; I\subseteq R$$

since

$$
\begin{array}{ll}
 & R\subseteq R\backslash R/R \\
= & \quad\{\qquad \text{factors} \quad\} \\
 & R\circ R\circ R\subseteq R \\
\Leftarrow & \quad\{\qquad \text{monontonicity and transitivity} \quad\} \\
 & R\circ R\subseteq R
\end{array}
$$

and

$$
\begin{array}{ll}
 & R\backslash R/R\subseteq R \\
= & \quad\{\qquad [\,R=I\backslash R/I\,] \quad\} \\
 & R\backslash R/R\subseteq I\backslash R/I \\
\Leftarrow & \quad\{\qquad \text{(anti)monotonicity} \quad\} \\
 & I\subseteq R \quad .
\end{array}
$$

Also,

$$R^{\cup}\circ R^{\cup}\subseteq R^{\cup} \;\wedge\; I\subseteq R^{\cup} \quad\equiv\quad R\circ R\subseteq R \;\wedge\; I\subseteq R \quad .$$

(The converse of a preorder is a preorder.) So

$$
\begin{array}{ll}
 & SC.R \\
= & \quad\{\qquad \text{lemma 120, in particular (123)} \quad\} \\
 & R\cup(R\backslash R/R)^{\cup} \;=\; \mathbb{T} \\
= & \quad\{\qquad \text{assumption: } R \text{ is a preorder} \\
 & \qquad\qquad (\text{hence, } R^{\cup} \text{ is a preorder and } R^{\cup}=R^{\cup}\backslash R^{\cup}/R^{\cup}) \\
 & \qquad\qquad \text{lemma 12, in particular (15)} \quad\} \\
 & R\cup R^{\cup} \;=\; \mathbb{T} \\
= & \quad\{\qquad \text{assumption: } R \text{ is linear (i.e. } R\cup R^{\cup}=\mathbb{T}) \quad\} \\
 & \text{true} \quad .
\end{array}
$$

□

The second way of constructing a staircase relation is to reduce a linear preorder by eliminating its reflexive part (making it so-called “strict”). For example, the less-than relation (on natural numbers, integers, rational numbers or reals) is a staircase relation. Formally, we have:

Lemma 126 For all (homogeneous) R,

$$SC.R \;\Leftarrow\; R\circ R\subseteq R \;\wedge\; R\cup I\cup R^{\cup} = \mathbb{T} \;\;.$$

Proof

$$\begin{array}{ll}
 & SC.R \\
= & \{\quad (123) \quad\} \\
 & R\cup(R\backslash R/R)^{\cup} \;=\; \mathbb{T} \\
= & \{\quad [X\subseteq\mathbb{T}] \text{ and antisymmetry} \quad\} \\
 & \mathbb{T}\subseteq R\cup(R\backslash R/R)^{\cup} \\
\Leftarrow & \{\quad \text{assumption: } R\cup I\cup R^{\cup}=\mathbb{T}\text{, so } \mathbb{T}\subseteq R\cup I\cup R^{\cup} \\
 & \quad\;\; \text{monotonicity and transitivity} \quad\} \\
 & I\cup R^{\cup}\subseteq(R\backslash R/R)^{\cup} \\
= & \{\quad \text{converse, factors and distributivity} \quad\} \\
 & R\circ I\circ R\cup R\circ R\circ R \;\subseteq\; R \\
= & \{\quad \text{supremum and monotonicity} \quad\} \\
 & R\circ R\subseteq R \\
= & \{\quad \text{assumption} \quad\} \\
 & \text{true} \;\;.
\end{array}$$

□

Example 127 The less-than relations on the integers, $<_{\mathbb{Z}}$, on the rationals, $<_{\mathbb{Q}}$, and on the reals, $<_{\mathbb{R}}$, are all staircase relations since in each case $<\backslash<$ is the at-most relation, $\leq$. See example 70 for details of the preorder in each case. The less-than relation on the integers is a (linearly) block-ordered relation but the less-than relation on the rationals and the less-than relation on the reals are not block-ordered. This is because, as shown in example 70, the less-than relations on the rationals and on the reals both have empty diagonals.
□

That the less-than relation on the real numbers is not block-ordered is a consequence of the fact that if $x<y$ the interval between x and y can always be subdivided at will. (That is, it is always possible to find a real number z such that $x<z$ and $z<y$.) The same is also true of the rationals. Abstracting from the details of the less-than relation, we get the following theorem. (Winter [Win04] proves a similar theorem. See section 8 for further discussion.)

Theorem 128 Suppose R is a homogeneous relation such that

$$R\neq\perp\!\!\!\perp \;\wedge\; I\cap R=\perp\!\!\!\perp \;\wedge\; R=R\circ R \;\wedge\; R\cup I\cup R^{\cup}=\mathbb{T} \;\;.$$

Then R is a staircase relation and $\Delta R = \perp\!\!\!\perp$.

It follows that any such relation is *not* block-ordered.

Proof Lemma 126 proves that R is a staircase relation.

Comparing the above conditions on R with those in lemma 126, the additions are the non-emptiness property $R \neq \perp\!\!\!\perp$, the "strictness" property $I \cap R = \perp\!\!\!\perp$ and the "subdivision" property $R \subseteq R \circ R$. (The less-than relation on real numbers has the subdivision property whereas the less-than relation on the integers does not.) Applying lemma 129 (below), the subdivision and strictness properties imply that $\Delta R = \perp\!\!\!\perp$. That R is not block-ordered follows from theorem 114 and the assumption that $R \neq \perp\!\!\!\perp$.

□

The lemma used to prove theorem 128 is the following:

Lemma 129

$$R \subseteq R \circ R \Rightarrow (\Delta R = \perp\!\!\!\perp \equiv I \cap R \subseteq \perp\!\!\!\perp) \quad .$$

Proof

$$
\begin{array}{ll}
 & R \;\subseteq\; R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \qquad \{ \quad \text{monotonicity} \quad \} \\
 & I \cap R \;\subseteq\; I \,\cap\, R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \qquad \{ \quad \text{modular law} \quad \} \\
 & I \cap R \;\subseteq\; R \circ (R^{\cup} \circ R^{\cup} \cap \neg R^{\cup}) \circ R \\
= & \qquad \{ \quad \text{assumption: } R \subseteq R \circ R \quad \} \\
 & I \cap R \;\subseteq\; R \circ (R^{\cup} \cap \neg R^{\cup}) \circ R \\
= & \qquad \{ \quad \text{complements} \quad \} \\
 & I \cap R \subseteq \perp\!\!\!\perp \\
= & \qquad \{ \quad I = I^{\cup} \text{, converse and shunting} \quad \} \\
 & I \subseteq \neg R^{\cup} \\
\Rightarrow & \qquad \{ \quad \text{monotonicity} \quad \} \\
 & R \circ R \;\subseteq\; R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \qquad \{ \quad \text{assumption: } R \subseteq R \circ R \text{ and transitivity} \quad \} \\
 & R \;\subseteq\; R \circ \neg R^{\cup} \circ R \quad .
\end{array}
$$

That is,

(130) $\quad R \subseteq R \circ R \;\Rightarrow\; (R \;\subseteq\; R \circ \neg R^{\cup} \circ R \;\equiv\; I \cap R \subseteq \perp\!\!\!\perp) \quad .$

So

$$
\begin{array}{ll}
 & \Delta R = \perp\!\!\!\perp \\
= & \qquad \{ \quad [\perp\!\!\!\perp \subseteq X] \text{ and antisymmetry, definition of } \Delta R \quad \} \\
 & R \cap (R \backslash R / R)^{\cup} \subseteq \perp\!\!\!\perp \\
= & \qquad \{ \quad \text{shunting} \quad \} \\
 & R \subseteq \neg (R \backslash R / R)^{\cup} \\
= & \qquad \{ \quad (11) \quad \} \\
 & R \;\subseteq\; R \circ \neg R^{\cup} \circ R \\
= & \qquad \{ \quad \text{assumption: } R \subseteq R \circ R \text{, (130)} \quad \} \\
 & I \cap R \subseteq \perp\!\!\!\perp \quad .
\end{array}
$$

□

8 Conclusion

The primary novel contribution of this paper is the combination of theorems 78 and 117: essentially, a block-ordered relation is a relation whose core is a provisional ordering. The discovery of these theorems was inspired by Riguet's suggestion of an "analogie frappante" linking the notion of a "relation de Ferrers" and the (difunctional) "différence" of a relation. Theorem 114 is a precise statement of an "analogie" linking the notions of a block-ordered relation and the diagonal of a relation.

A secondary, but nevertheless important, contribution of this paper is our (almost) exclusive use of the properties of factors of a relation, particularly with respect to formulating and reasoning about the notion of the diagonal of a relation (Riguet's "différence"), as opposed to Riguet's use of nested complements. Indeed, our only use of complements is in section 7 where we formulated the notion of a staircase relation —in terms of factors— and showed its equivalence to Riguet's notion of a "relation de Ferrers" —which he formulated in terms of nested complements— .

Our motivation for including section 7 is partly to give proper credit to Riguet's contribution but also to rectify misleading/incorrect statements in the extant literature[8]. Specifically, the claim that a "relation de Ferrers ... can be rewritten in staircase block form" [SS93, Definition 4.4.11] is, at best, confused: as shown in example 127, the less-than relation on real numbers is a staircase relation but not block-ordered. The lesson to be learnt is, in our view, that mental pictures, such as the one of a staircase relation shown in fig. 1 and informal natural-language statements, can never be relied on. Ultimately, it is vital that informal notions are formalised and the properties of the formal notions are explored in detail in order to confirm that they do indeed conform to their intended meaning.

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