Difunctional and Block-Ordered Relations

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December 5, 2022

Abstract

Seventy years ago, in a series of publications, Jacques Riguet introduced the notions of a "relation difonctionelle", the "différence" of a relation and "relations de Ferrers". He also presented a number of properties of these notions, including an "analogie frappante" between "relations de Ferrers" and the "différence" of a relation. Riguet's definitions, particularly of the central concept of the "différence" of a relation, use formulae involving nested complements. Riguet's proofs make extensive use of natural language making them difficult to understand. The primary purpose of this paper is to bring Riguet's work up to date using modern calculational methods. Other goals are to document and extend Riguet's work as fully as possible, and to correct extant errors in the literature.

We call a "relation difonctionelle" a "difunctional relation", the "différence" of a relation we call the "diagonal" of a relation and a "relation de Ferrers" we call a "staircase relation" — a special case of a "block-ordered relation". We avoid as much as possible the use of nested complements by exploiting the left and right factor operators (aka division or residual operators) on relations.

We present complete, calculational proofs of two fundamental properties of difunctional relations: a relation is difunctional if and only if it can be represented by a pair of functional relations and that a relation is difunctional if and only if it is the union of a set of completely disjoint rectangles. The diagonal of a relation (Riguet's "différence") is a difunction that plays a very significant rôle in the study of block-ordered relations; accordingly, we study its properties in depth. For completeness, we also present a second method for constructing a difunction from an arbitrary relation: Riguet's "fermeture difonctionelle".

Riguet used an informal, mental picture of a staircase-like structure to introduce "relations de Ferrers" in the case of finite relations. Riguet also stated a necessary and sufficient condition for a "relation de Ferrers" to be the union of a totally ordered class of rectangles, where the ordering has a property that we call "polar". By omitting the totality requirement, we abstract the more general notion of a block-ordered relation. We explore conditions under which a given relation has a non-redundant, polar covering and when it is block-ordered. In doing so, we formulate and prove a

theorem establishing an equivalence between the property of a relation being block-ordered and properties of the diagonal of a relation. Our theorem generalises Riguet's "analogie frappante".

The primary novelty of our work is the introduction of the notion of the "core" of a relation. This is a notion that is of general applicability and not just in the context of block-ordered relations. For example, the core of a difunctional relation is a bijection, the core of a preorder is an ordering (a special case of the core of a block-ordered relation, which is also an ordering), and the core of a finite graph is an acyclic graph connecting its strongly connected components. Our generalisation of Riguet's "analogie frappante" shows how the core of a relation in combination with its diagonal is used —under certain conditions— to construct a non-redundant, injective polar covering of a given relation. The theorem may have practical application in the concise representation of very large databases.

Finally, we consider the special case of staircase relations. We consider different definitions that formalise Riguet's mental picture. Contrary to claims made in the published literature, we show that the definitions are not equivalent in general. We do prove their equivalence in the case of (block-)finite relations, a fact that is often taken for granted in the extant literature but of which we have never seen a proof.

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1 Introduction

The interface between requirements and specifications poses a major challenge for practising programmers because it is intrinsically a social process that is largely unsupported by mathematical method: requirements are informal and customer-led whereas specifications are formal (even if, as is often the case, the “specification” is the actual implementation of the requirements). There is no mathematically verifiable “correctness” relation between requirements and specifications.

The challenge of assuring the customer that their requirements have indeed been met can be overcome in different ways. We would argue that one of the most important ways is by deriving —by mathematical calculation— properties of the specification which are then interpreted in a way that can be understood by the customer. This process is vital to the integrity of the science of computing.

Seventy years ago, in a series of publications [Rig48, Rig50, Rig51], Jacques Riguet introduced the notions of a “relation difonctionelle”, the “différence” of a relation and “relations de Ferrers”. In the case of finite relations, he provided an informal mental picture of a “relation de Ferrers” in the form of a staircase-like structure. But his formal definition of a “relation de Ferrers” bears little or no resemblance to the mental picture and it is difficult to see how the formal corresponds to the informal. The name “relation de Ferrers” also gives little clue as to the practical relevance of the notion. Riguet’s definitions, particularly of the “différence” of a relation, use (in our view) over-complicated and outdated formulae involving nested complements that are better formulated using the factor operators (aka division or residual operators). Riguet also relies heavily on natural language justifications of important properties as well as asserting several properties without proof. More recent publications, some of which do not cite Riguet but which copy his definitions, introduce errors by failing to recognise the restrictions that Riguet made clear in his account of the properties of the notions.

The writing of this paper initially began as an exercise in applying modern calculational reasoning to bring Riguet’s work up to date and more accessible to a wider audience. In view of the extant errors in relatively recent publications and to try to avoid introducing yet more errors, we decided to include full details of all proofs. In the process, we decided that some changes in terminology were desirable: for reasons that we explain later, we call the “différence” of a relation the “diagonal” of the relation and we call “relations de Ferrers” staircase relations. We also realised that certain generalisations of Riguet’s work were desirable, the primary one being from “staircase” relations to “block-ordered relations”: the property of being a “staircase” relation demands a certain total ordering on “blocks” (“rectangles totalement ordonnées par inclusion” [Rig51]), being “block-ordered” does not require the ordering to be total. In summary, the goals of this paper are as follows:

1. To demonstrate the efficacy of modern calculational reasoning in developing a theory of block-ordered relations.

2. To document as fully as possible the precise relation between difunctional relations and block-ordered relations (Riguet's "analogie frappante").

3. To set the record straight with respect to the origin of the concepts and theorems relating difunctional relations to block-ordered relations.

4. To correct extant errors in the literature.

1.1 Mental Pictures

Partly as a consequence of our decision to include all proofs, this document has become quite long and it is inappropriate to introduce all parts in one go. In order to set the scene, this section gives a very informal account of the principle notions introduced. In doing so, we use notation that will be introduced in later sections. Readers unfamiliar with the notation are invited to read the section nevertheless,postponing full understanding until later.

For many, it is useful to have a "mental picture" of formal mathematical statements. Fig. 1 is such a mental picture of what we shall call a "staircase relation". (Riguet [Rig51] presents a similar picture of a "relation de Ferrers".) The shaded area depicts a binary relation on sets A and B, the vertical axis depicting the set A, the horizontal axis depicting the set B, and the shaded area depicting the set of pairs (a, b) for which the relation holds. Informally a staircase relation is any relation that can be depicted in such a way.

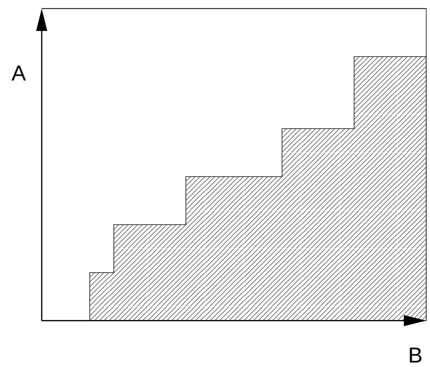


Figure 1: Mental Picture of a Staircase Relation

One of the problems we address in this paper is how to formulate the notion of a "staircase" relation in a way that is both amenable to mathematical calculation and captures the very informal definition just given. In the process of so doing, it is necessary to resolve ambiguities and/or misconceptions that inevitably arise from informal definitions.

Fig. 2 is a "mental picture" of a difunctional relation of type $A{\sim}B$. Informally, a difunctional relation is a (heterogeneous) relation that is the union of a collection of "completely disjoint rectangles[1]". The relation shown in fig. 2 is what we call the "diagonal" of the staircase relation shown in fig. 1.

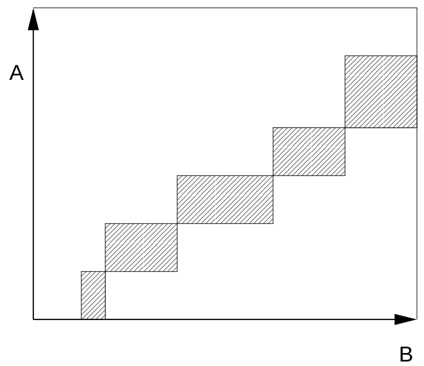


Figure 2: Mental Picture of a Difunctional Relation

The mental picture of a difunctional relation suggests a second property that appears to be folklore: each point a in the left domain and each point b in the right domain of a difunctional relation defines a rectangle whereby related pairs define the same rectangle. In this way, a difunctional relation is characterised by a pair of functional relations.

As mentioned earlier, Riguet [Rig51] uses the name "différence" for what we call the "diagonal". Fig. 3 explains in picture-form the reasoning behind the naming as well as how our formulation differs from Riguet's.

The four parts of fig. 3 depict in turn

(a) a relation R (coloured green),

(b) the factor $R^{\cup}\backslash R^{\cup}/R^{\cup}$ (in red, where $R^{\cup}$ denotes the converse of R),

(c) the diagonal of R (in blue — more precisely, the relation $R \cap R^{\cup}\backslash R^{\cup}/R^{\cup}$),

(d) the relation $R\circ\neg R^{\cup}\circ R$.

Informally, the diagonal of R (shown in fig. 3(c)) is that part of the relation R (shown in fig. 3(a)) that is common to the factor $R^{\cup}\backslash R^{\cup}/R^{\cup}$ (shown in fig. 3(b)).

Riguet formulated the diagonal as the "différence" between R and the relation $R\circ\neg R^{\cup}\circ R$, i.e. as $R\cap\neg(R\circ\neg R^{\cup}\circ R)$. (Note the nested complements, denoted by the

[1]See definition 123 for a formal definition of "completely disjoint rectangles".

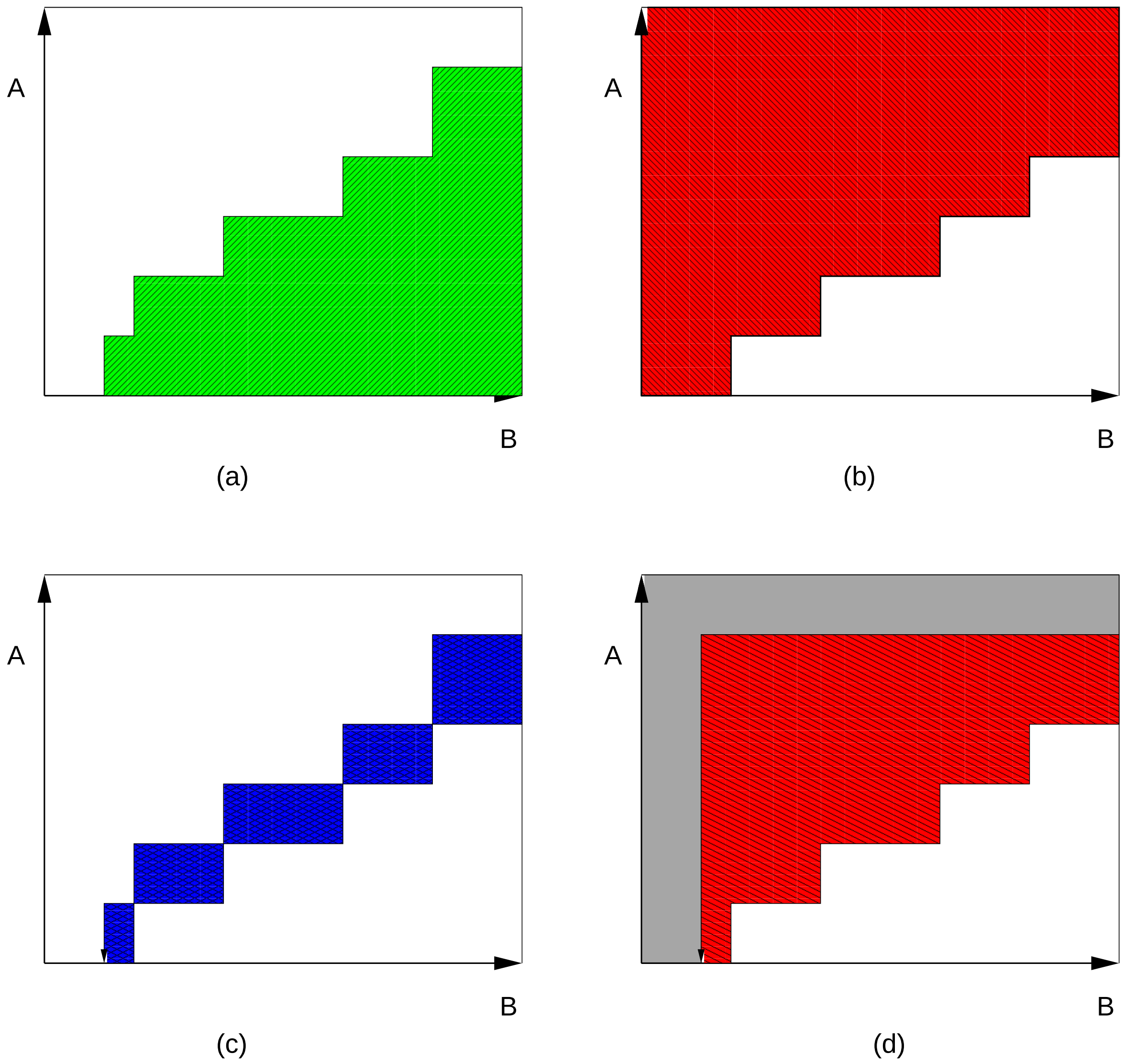


Figure 3: Riguet's "Différence"

symbol "$\neg$".) Fig. 3(d) shows the relation $R \circ \neg R^{\cup} \circ R$. It has two parts: the parts *not* coloured red, i.e. the shaded part and the white part. The part coloured red is the "useful" part $R_{<} \circ R^{\cup} \backslash R^{\cup} / R^{\cup} \circ R_{>}$ of the relation depicted in (b). Here $R_{<}$ and $R_{>}$ denote the left and right "domains" of R (not to be confused with the target and source of R). The shaded part of fig. 3(d) depicts the relation $R_{\bullet<} \circ \top\!\top \cup \top\!\top \circ R_{>\bullet}$: the set of pairs (a, b) such that either a is not related by R to any element of B or no element of A is related by R to b. Riguet's "différence" is the difference between the green part of fig. 3(a) and the non-red part of fig. 3(d).

Hopefully, by way of these informal pictures, we can now give an overview of the remainder of the paper.

1.2 Overview

To begin, we present the axiomatic basis for our formal reasoning in section 2. The basis for the axiom system originated in the work of De Morgan, Pierce, Schröder, Tarksi and, no doubt, many others. This section is an abbreviated version of the presentation in [BDGv21] to which the reader is referred for full details (including proofs of the stated theorems).

Section 3 goes into more detail on basic elements of relation algebra. At this point, we adhere to our maxim of providing proofs of all stated properties. Whilst the topics in this section —in particular factors (section 3.2), the domain operators (definition 42) and "provisional orderings" (definition 114)— all play a significant rôle later, we recommend that the reader skim the section briefly in the first instance, returning to it later as and when necessary. (The notion of a "provisional ordering" is new but the motivation for its introduction only becomes apparent later.)

Section 4 is the beginning of topics specific to block-orderings. "Blocks" or "rectangles" are particular sorts of relations that are pictured as rectangles. As pictured in fig. 2, a difunctional relation can be characterised as a collection of "completely disjoint rectangles". Section 4.1 presents a number of elementary properties of squares and rectangles whilst section 4.2 introduces some important definitions and properties: the notion of an "indexed set" of rectangles (definition 129), the notion of "completely disjoint rectangles" (definition 130) and the characterisation of an indexed set of completely disjoint rectangles by a pair of functional relations (theorem 141).

In section 5 we formulate properties of partial equivalence relations that will be familiar to most readers. The main topic is a theorem characterising a partial equivalence relation as a collection of disjoint squares. In more familiar terminology, a partial equivalence relation partitions its domain into disjoint equivalence classes. Note that we focus on *partial* equivalence relations (of which equivalence relations form a special case). In general, we are obliged to reason about the left and right domains of relations, particularly when reasoning about the diagonal of a relation (definition 183) — a topic that is central to this investigation. Recall our discussion of the shaded area of fig. 3(d).

We formulate several proofs of the per characterisation theorem, theorem 143, in section 5. Later we do the same for the characterisation of difunctional relations, theorem 161, one of the proofs being based on theorem 143. We do so in order to evaluate different calculational methods. In this case, contrary to the view we ourselves have propagated, the calculations exploiting points and the saturation axiom are preferable to the point-free calculations. Our formalism allows us to mitigate the negative aspects of pointwise reasoning so that points appear in formulae only where this is desirable. This is discussed further in section 12.

The main results of this investigation are presented in section 6 on difunctional relations, section 7 on the "diagonal" of a relation and sections 9 and 11 on block-ordered and staircase relations, respectively.

Section 6 is about the basis for the name "difunction": a difunctional relation is characterised by a pair of functional relations (theorem 161); moreover, such a characterisation is (essentially) unique (theorem 166). This is a well-known generalisation of the properties of partial equivalence relations and, as mentioned above, is included in order to evaluate different calculational methods.

For completeness, section 6.4 documents the properties of the "difunctional closure" of a relation: the "fermeture difonctionelle" introduced by Riguet [Rig50].

Section 7 is a detailed examination of the properties of the diagonal of a relation. Riguet's account of "relations de Ferrers" includes a theorem characterising such relations as the "réunion" of "rectangles" that have a very special property. Referring to fig. 1, each individual "tread" of a staircase relation defines a unique rectangle (exact details of which are given later) and the relation is the "réunion" of them all. With this as motivation, we abstract the notion of a "polar covering" and we prove a theorem that *every* relation has a polar covering. See definition 209 and theorem 211 in section 8. As a step towards Riguet's characterisation of "relations de Ferrers", we define the notion of a "non-redundant" polar covering. For finite relations, it is straightforward to show that a non-redundant polar covering can always be constructed from a given polar covering of the relation. The algorithm may, however, not be practical; moreover, there are infinite relations that do not have a non-redundant polar covering. (The less-than relation on real numbers is an example.) A focus of section 7 is to investigate when the diagonal of a relation defines a non-redundant polar covering of the relation. The main result in this section is thus theorem 222 (which we believe to be original to this paper).

Block-ordered relations are defined in section 9. Although we don't discuss it in any detail, the practical application of block-ordering a relation is efficient storage and recovery of information. Dividing the left and right domains of a relation into "blocks" is an obvious first step. We take the opportunity in section 9.1 to point out the pioneering contribution to information science made by Hartmanis and Stearns [HS64, HS66] in their study of so-called "pair algebras". The relevance to block-ordered relations is that so-called "perfect" Galois connections provide a rich source of examples. Section 9.2 relates block-orderings to diagonals. The section is entitled "analogie frappante" because the concluding theorem of the section (theorem 262) is a necessary and sufficient condition for a relation to be block-ordered expressed as a property of the diagonal of a relation, thus generalising Riguet's "analogie frappante" between the properties of a "relation de Ferrers" and difunctional relations. Theorem 234 proves that every block-ordered relation has a non-redundant polar covering, the non-redundancy of which is witnessed by the relation's diagonal.

Section 10 introduces a less-restrictive notion of "(possibly) imperfect" block-orderings. Every relation has an imperfect block-ordering as witnessed by the "grips" of the relation. The "grips" of a relation are "blocks" that are essentially the same as the so-called "Begriffen" ("concepts") of the relation [DP90].

Section 11 was the starting point of this investigation: principally, how should the informal mental picture of a "staircase" relation be made precise and what then are its properties? Unsurprisingly (at least to us) it turns out that pictures can be deceiving. We have been able to verify that all the claims made by Riguet are valid and much of the section is devoted to that task; in particular, theorem 334 establishes the (unproven) theorem in [Rig51] that every staircase relation has a linear polar covering. On the other hand, we provide examples showing that other claims in the extant literature are not valid. In particular, theorem 319 proves, by way of concrete examples, that not every staircase relation is block-ordered. It is the case, however, as correctly stated by Riguet [Rig51], that every finite staircase relation is block-ordered but we have been unable to find a proof anywhere in the literature. Theorem 335 and its proof rectify this lacuna.

Section 12 concludes the paper with a summary and discussion of publications in the last thirty years. (We have been unable to fill the forty-year gap —in respect of non-finite relations— from 1950 to 1990 and would welcome receiving information about relevant publications in that period.)

2 The Axiom System

We assume familiarity with a number of basic concepts of relation algebra: composition, converse, left and right domains, and left and right factors (aka "residuals"). Our presentation is based on the system of axioms formulated by Voermans [Voe99]; full details can be found in [BDGv21]. In addition to the axioms we give a pointwise *interpretation* of each of the operators. That is, we say, for each operator that we introduce, how the operator defines a set of pairs. In giving the interpretation we use the notation $[\![E]\!]$ to mean "the interpretation of E". Thus we write $x[\![R]\!]y$ instead of xRy; this enhances readability and also emphasises the difference between the objects of an abstract relation algebra and the interpretation of such objects as binary relations.

2.1 Point-Free Relation Algebra

We begin with a point-free axiomatisation of homogeneous relations. Later we extend the axiomatisation to heterogeneous relations (section 2.4) and to points (section 2.5).

The first unit is a lattice structure. Specifically, let $(\mathcal{A}, \subseteq)$ be a partially-ordered set. We postulate that $\mathcal{A}$ forms a complete, universally distributive lattice. The infimum and supremum operators will be denoted by $\cap$ and $\cup$, respectively. The top and bottom elements of the lattice will be denoted by $\top\!\!\top$ and $\perp\!\!\perp$, respectively. We call elements of $\mathcal{A}$ *relations* and denote them by variables R, S and T. The interpretation of $\mathcal{A}$ is the set of relations of some fixed type. The interpretation of a relation is a set; so $\mathcal{A}$ is a powerset.

As suggested by the choice of notation, the interpretation of $\subseteq$ is the subset ordering, the interpretation of $\cap$ is set intersection, and the interpretation of $\cup$ is set union. Formally,

$$[\![R\subseteq S]\!] \;\equiv\; \langle\forall x,y : x[\![R]\!]y : x[\![S]\!]y\rangle \quad ,$$

$$x\,[\![R\cap S]\!]\,y \;\equiv\; x[\![R]\!]y \wedge x[\![S]\!]y \quad \text{, and}$$

$$x\,[\![R\cup S]\!]\,y \;\equiv\; x[\![R]\!]y \vee x[\![S]\!]y \quad .$$

The interpretation of $\top\!\!\top$ is the universal relation and the interpretation of $\perp\!\!\perp$ is the empty relation. That is,

$$\langle\forall x,y \;::\; x[\![\top\!\!\top]\!]y \equiv \mathsf{true}\rangle \;\;\wedge\;\; \langle\forall x,y :: x[\![\perp\!\!\perp]\!]y \equiv \mathsf{false}\rangle \quad ,$$

This is the most complicated unit in the framework but one which should be familiar to the reader.

Every binary relation has a converse; the converse operator, denoted by a postfix "$^{\cup}$" symbol (pronounced "wok"), is interpreted by

$$x\,[\![R^{\cup}]\!]\,y \;\equiv\; y[\![R]\!]x$$

for all x and y. Axiomatically, we postulate the existence of a (total) unary function from relations to relations such that, for all relations R and S

(1) $\quad R^{\cup}\subseteq S \;\equiv\; R\subseteq S^{\cup}$.

The Galois connection (1) is all that is necessary to define the converse operator and its interface with the lattice structure. Its being a Galois connection makes it so attractive.

The set of homogeneous binary relations over some universe includes the identity relation, I, with the interpretation

$$x[\![I]\!]y \;\equiv\; x=y$$

for all x and y. Relations may also be composed via the binary composition operator, $\circ$, defined at the point level by

$$x\,[\![R\circ S]\!]\,z \;\equiv\; \langle\exists y :: x[\![R]\!]y \wedge y[\![S]\!]z\rangle \quad .$$

We capture these two notions axiomatically by demanding the existence of a relation I and a binary operator, $\circ$, mapping a pair of relations to a relation, such that $(\mathcal{A},\circ,I)$ is a monoid.

There are two interfaces to be specified. The interface with the converse operator is soon dealt with. Bearing in mind the intended relational interpretations of converse and composition we postulate

(2) $\quad (R\circ S)^{\cup} \;=\; S^{\cup}\circ R^{\cup}$,

for all relations R and S. For the interface with the lattice structure we postulate that a relation algebra is a regular algebra. In particular, we postulate that for all relations R the functions $(R\circ)$ and $(\circ R)$ are universally distributive. This is equivalent to postulating the existence of two factor operators; these are discussed in detail in section 3.2.

In the theory developed in this paper, the converse operator plays a very significant rôle. Because converse has such strong distributivity properties, it is frequently possible to "dualise" a property by simply applying the converse operator to obtain a property that is the mirror image of the original. (See, for example, (3) and (4).) Also, operators we define frequently have left and right variants with mirror properties. (See, for example, the domain operators introduced in definition 42.)

2.2 Operator Precedence

We have now introduced quite a large number of operators. In order to reduce the number of parentheses in formulae we should agree on a precedence between the different operators.

A general rule we use throughout is that all prefix and postfix operators as well as subscripting and superscripting take precedence over infix operators and infix operators in turn take precedence over multifix operators. When both prefix and postfix operators are applied to an expression, we use parentheses to clarify the order of evaluation. An exception is when a prefix and postfix operator obey an "associative" law, in which case we omit the parentheses. For example —as observed by De Morgan— complement and converse "associate". So we can safely write $\neg R^{\cup}$, parsing it as $\neg(R^{\cup})$ or as $(\neg R)^{\cup}$ depending on the calculational needs. Thus it remains to discuss the relative precedence of the infix operators.

For infix operators, the general rule is that metaoperators (operators like $\equiv$ and $\wedge$) have the lowest precedence. Next come relations like $\leq$ and $\subseteq$. The operators of relation algebra have the next highest precedence, and function application (which we denote by an infix dot) has the highest precedence of all. Among the infix operators of relation algebra the precedence is: intersection and union have the same, lowest precedence, and the highest precedence is given to composition.

2.3 Modularity Rule and Cone Rule

Although composition distributes through suprema, it does *not* distribute through infima. This creates difficulties in calculations that combine infima with composition. The rule we now introduce to overcome this difficulty acts as an interface between all three units of the framework. Riguet [Rig48] named the rule after the famous mathematician J.W.R. Dedekind (he called it "la relation de Dedekind") because of its resemblance to the modular identity, a property of normal subgroups attributed to Dedekind. Schmidt and Ströhlein [SS93] have adopted Riguet's terminology (they refer to "the Dedekind formula") whereas Freyd and Ščedrov [Fv90] call it the *law of modularity* (possibly for the same reason as Riguet). We call it the *modularity rule*.

The modularity rule is that, for all relations R, S and T,

$$(3) \qquad R{\circ}S \cap T \subseteq R\circ(S \cap R^{\cup}\circ T) \quad .$$

The dual property, obtained by exploiting properties of the converse operator, is, for all relations R, S and T,

$$(4) \qquad S{\circ}R \cap T \subseteq (S \cap T\circ R^{\cup})\circ R \quad .$$

(This the first of many examples of mirror-image duality that we forewarned of in section 2.1.)

An additional rule, sometimes called "Tarski's rule", is called the *cone rule* below: for all relations R,

(5) $\langle\forall R :: \top\!\!\top\circ R\circ\top\!\!\top = \top\!\!\top \;\equiv\; R\neq\bot\!\!\bot\rangle$.

Axiom systems for relation algebra often include a complementation (negation) operator and, instead of the modularity rule, the so-called Schröder rule is postulated. Our formulation of Schröder's rule is slightly different from standard accounts, as we now explain.

Suppose we consider an algebra that obeys all the axioms of relation algebra except for the modularity rule. Suppose that the algebra is complemented (i.e. every relation has a complement); we denote the complement of relation R by $\neg R$. Then the *middle-exchange rule*: for all R, S, X and Y,

(6) $R\circ\neg X\circ S \subseteq \neg Y \;\equiv\; R^{\cup}\circ Y\circ S^{\cup} \subseteq X$

is equivalent to the modularity rule. Occasionally, its equivalent, the *rotation* rule:

(7) $R\circ S\subseteq\neg T^{\cup} \;\equiv\; T\circ R\subseteq\neg S^{\cup}$

is used.

The middle-exchange rule gets its name from the fact that the middle term in a composition is exchanged with the right side of an inclusion. It has an attractive, symmetric form, making it easy to remember in spite of having four free variables. The standard rule, due to Schröder, is the conjunction of the two equivalences obtained by instantiating R and S to the identity relation. The rotation rule (so called because of the way the variables are rotated) also has an attractive form.

This concludes our discussion of the point-free algebraic framework. In a few sentences, a relation algebra is a complete, universally distributive lattice on which is defined a monoid structure and a unary converse operator. Composition on the left and on the right are both universally distributive (with the implication that they both have upper adjoints: the factor operators to be introduced in section 3.2). Converse is a lattice isomorphism that preserves the unit of composition and distributes contravariantly through composition. Finally, the lattice structure, converse and the monoid structure are all interrelated via the modularity rule.

2.4 Heterogeneous Relations

A *heterogeneous* relation R has a *type* given by two sets A and B, which we call the *target* and *source* of R. We use the notation $A\sim B$ to denote the type of a relation.

Formally, a relation of type $A{\sim}B$ is a subset of $A{\times}B$. (Equivalently, it is a function with domain $A{\times}B$ and range Bool.) A *homogeneous* relation is a relation of type $A{\sim}A$ for some A.

The operators in the algebra of heterogeneous relations are typed. For example, the composition of two relations R and S, denoted as always by $R{\circ}S$, is only defined when the source of R equals the target of S. Moreover, the target of $R{\circ}S$ is the target of R and the source of $R{\circ}S$ is the source of S. That is, if R has type $A{\sim}B$ and S has type $B{\sim}C$ then $R{\circ}S$ has type $A{\sim}C$. We assume the reader is familiar with such rules.

The rules of the untyped calculus are applicable in the typed calculus, with some restrictions on types. Restrictions are necessary on types for, for example, the middle-exchange rule: (6).

Care must be taken with the overloading of notation. It is tempting, for example, to state the rule:

$$\top\!\!\top^{\cup} = \top\!\!\top$$

without qualification. But, if R has type $A{\sim}B$, its converse $R^{\cup}$ has type $B{\sim}A$. Thus the notation " $\top\!\!\top$ " on the left side of the equation denotes the universal relation of type $A{\sim}B$, for some types A and B; on the other hand, the notation " $\top\!\!\top$ " on the right side of the equation denotes the universal relation of type $B{\sim}A$. Rather than overload the notation in this way, we could decorate every occurrence of $\top\!\!\top$ with its type. For example, we could rephrase the rule as

$$({}_{A}\top\!\!\top_{B})^{\cup} \;=\; {}_{B}\top\!\!\top_{A} \quad .$$

The same applies to $\bot\!\!\bot$. We prefer not to do so because the type information is usually easy to infer. An exception is that we occasionally decorate the identity relation I with its type: I_A denotes the identity relation of type $A{\sim}A$.

Typed relation algebra, as briefly summarised above, extends category theory to what has been called *allegory theory*. See Freyd and Ščedrov [Fv90] for more details.

2.5 Points

The relations of a given type form a powerset. A powerset forms a complete, universally distributive, complemented lattice under the subset ordering. However, these properties do not characterise the properties of the *elements* of the sets in the powerset. For this, we need the notion of a "saturated", "atomic" lattice: elements of a set are modelled by so-called "atoms".

Let us recall the appropriate definitions , first in an arbitrary lattice and later specialising to relations.

Definition 8 (Atom and Atomicity) Consider an arbitrary poset ordered by the relation $\subseteq$ and with least element $\perp\!\!\perp$. Then the element p is an *atom* iff

$$\langle\forall q :: q\subseteq p \;\equiv\; q=p \vee q=\perp\!\!\perp\rangle \quad .$$

Note that $\perp\!\!\perp$ is an atom according to this definition. If p is an atom that is different from $\perp\!\!\perp$ we say that it is a *proper* atom. A lattice is said to be *atomic* if

$$\langle\forall q :: q\neq\perp\!\!\perp \;\equiv\; \langle\exists a : \mathrm{atom}.a \wedge a\neq\perp\!\!\perp : a\subseteq q\rangle\rangle \quad .$$

In words, a lattice is atomic if every proper element includes a proper atom.
□

Definition 9 (Saturated) A complete lattice is *saturated* iff

$$\langle\forall p :: p \;=\; \langle\cup a : \mathrm{atom}.a \wedge a\subseteq p : a\rangle\rangle \quad .$$

□

The following theorem is central to the use of saturated lattices as a model of powersets.

Theorem 10 Suppose $\mathcal{A}$ is a complete, universally distributive lattice. Then the following statements are equivalent.

(a) $\mathcal{A}$ is saturated,

(b) $\mathcal{A}$ is atomic and complemented,

(c) $\mathcal{A}$ is isomorphic to the powerset of its atoms.
□

Given a type A, the homogeneous relations of a given type $A{\sim}A$ form a powerset. A *coreflexive* relation is a relation of type $A{\sim}A$, for some A, that is a subset of the identity relation. (Coreflexives are also called partial identities, monotypes and tests.) To our axiom system, we add the following postulates.

1. For each type A, the poset of coreflexives is a complete, universally distributive, saturated lattice.

2. The *all-or-nothing* rule [Glü17]:

   $$\langle\forall a,b,R : \mathsf{AC}.a \wedge \mathsf{AC}.b : a\circ R\circ b=\perp\!\!\perp \vee a\circ R\circ b=a\circ\top\!\!\top\circ b\rangle$$

   where AC abbreviates "atomic and coreflexive".

The combination of these two properties is equivalent to the postulate that the lattice of relations of a given type is atomic and saturated. The proper atoms are events of the form $a\circ\top\!\!\top\circ b$ where a and b are proper atomic coreflexives; such an event models the pair (a, b) in conventional pointwise formulations of relation algebra.

Theorem 11 Suppose that, for all types A, the lattice of coreflexives of type $A{\sim}A$ is a complete, universally distributive, saturated lattice. Then, if the all-or-nothing rule is universally valid, the lattice of relations of type $A{\sim}B$ (for arbitrary types A and B) is also a saturated, atomic lattice; the atoms are elements of the form $a\circ\top\!\!\top\circ b$ where a and b are atoms of the lattice of coreflexives of types A and B, respectively. It follows that the lattice of relations is isomorphic to the powerset of the set of elements of the form $a\circ\top\!\!\top\circ b$ where a and b are atoms of the lattice of coreflexives.
□

(See Voermans [Voe99, section 4.5] for further discussion of so-called "extensionality" properties of relations. Note that Voermans gives the name "singleton" to proper atoms. Thus —perhaps confusingly— what we have just referred to as "pairs" are, in his terminology, also "singletons".)

In common with all coreflexives, a point is a homogeneous relation of type $A{\sim}A$. However, in keeping with the idea that points represent elements of type A, we often abbreviate the type $A{\sim}A$ to just A.

Definition 12 (Point) A *point* is a proper, atomic, coreflexive relation.
□

For the purposes of this paper, we don't need all the details of what is meant by "atomic". If A is a type, we use a, a' etc. to denote points of type A. Similarly for points of type B. Properties we use of a point a of type A are:

(13) $a\circ a \;=\; a \;=\; a^{\cup}$,

(14) $\top\!\!\top\circ a\circ\top\!\!\top \;=\; \top\!\!\top$,

(15) $a\circ\top\!\!\top\circ a \;=\; a$,

(16) $\langle\forall p :: p\subseteq I_A \;\equiv\; p = \langle\cup a : a\subseteq p : a\rangle\rangle$.

Also, for points a and a' of the same type,

(17) $a = a' \;\vee\; a\circ a' = \perp\!\!\perp$.

Property (14) is equivalent to the property that a point is non-empty ("proper"). The property is an instance of the rule we call the "cone rule" introduced earlier. In general, if a is a point of type A and b is a point of type B, the relation $a{\circ}\top\!\!\top{\circ}b$ represents the pair (a,b); given a relation R of type $A{\sim}B$ and points a and b of type A and B, respectively, the statement

$$a{\circ}\top\!\!\top{\circ}b \subseteq R$$

has the interpretation that the pair a and b are related by R. Specifically, for all relations R and points a and b of appropriate type,

$$(18)\qquad (a{\circ}R{\circ}b \neq \perp\!\!\perp) \;=\; (a{\circ}\top\!\!\top{\circ}b \subseteq R) \;=\; (a{\circ}\top\!\!\top{\circ}b = a{\circ}R{\circ}b) \quad .$$

(In conformance with long-standing mathematical practice, property (18) should be read conjunctionally: that is as the equality of three terms. In this case, each term is boolean.) The *saturation* property is that

$$(19)\qquad \langle\forall R :: R = \langle\cup\, a,b : a{\circ}\top\!\!\top{\circ}b \subseteq R : a{\circ}\top\!\!\top{\circ}b\rangle\rangle \quad .$$

The *irreducibility* property is that, if $\mathcal{R}$ is a function with range relations of type $A{\sim}B$ and source K, then, for all points a and b of appropriate type,

$$(20)\qquad a{\circ}\top\!\!\top{\circ}b \subseteq \cup\mathcal{R} \;\equiv\; \langle\exists k : k{\in}K : a{\circ}\top\!\!\top{\circ}b \subseteq \mathcal{R}.k\rangle \quad .$$

The *identity relation* I_A of type A has the property that, for all points a and a' of type A,

$$(21)\qquad a{\circ}\top\!\!\top{\circ}a' \subseteq I_A \;\equiv\; a = a' \quad .$$

Relations of the form $R{\circ}b{\circ}S$, where b is a point, play a central rôle in what follows. The interpretation of $R{\circ}b{\circ}S$ is a relation that holds between points a and c iff the relation R holds between a and b, and the relation S holds between b and c. This is expressed precisely by the property:

$$(22)\qquad a{\circ}\top\!\!\top{\circ}c \subseteq R{\circ}b{\circ}S \;\equiv\; a{\circ}\top\!\!\top{\circ}b \subseteq R \;\wedge\; b{\circ}\top\!\!\top{\circ}c \subseteq S \quad .$$

3 Basic Structures

This section contains a miscellany of topics that are referred to repeatedly in subsequent sections. We recommend that the reader scans it briefly in the first instance, postponing a more detailed reading until later.

3.1 Specifications

Sometimes we want to define functions indirectly via a property relating input and output values. The property is formalised and then it is shown that the formal specification relates each input value to exactly one output value. That is, the formal specification relates each input value to at most one and at least one output value. In order to reason within our axiom system, we then want to conclude that output values are points. See, for example, section 3.5, where we define the meaning of functionality and exhibit an expression that formulates, in very general terms, the result of applying a function to an argument.

Although the process seems to be obvious, we want to stick to our goal of validating every step within our axiom system. For this reason, we now present the technical justification. As just mentioned, we refer the reader to section 3.5 for a concrete example.

In the following lemmas, p is a coreflexive relation and dummies a and a' are points of the same type as p.

We begin with the consequence of showing that specification p has at least one solution.

Lemma 23

$$p \neq \perp\!\!\perp \;\equiv\; \langle \exists a :: a \subseteq p \rangle \enspace .$$

Proof

$$
\begin{array}{ll}
 & p \neq \perp\!\!\perp \\
= & \{ \quad \text{cone rule: (5)} \quad \} \\
 & \top\!\!\top \circ p \circ \top\!\!\top = \top\!\!\top \\
= & \{ \quad \text{saturation property: (19)} \quad \} \\
 & \top\!\!\top \circ \langle \cup a : a \subseteq p : a \rangle \circ \top\!\!\top \;=\; \top\!\!\top \\
= & \{ \quad \text{distributivity} \quad \} \\
 & \langle \cup a : a \subseteq p : \top\!\!\top \circ a \circ \top\!\!\top \rangle \;=\; \top\!\!\top \\
= & \{ \quad a \text{ ranges over points: (15)} \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & \langle \cup a : a \subseteq p : \top\!\!\top \rangle \;=\; \top\!\!\top \\
\Rightarrow & \{ \quad \langle \cup a : \mathsf{false} : \top\!\!\top \rangle = \perp\!\!\perp \text{ and } \perp\!\!\perp \neq \top\!\!\top \quad \} \\
 & \langle \exists a :: a \subseteq p \rangle \\
\Rightarrow & \{ \quad a \text{ ranges over points: so } \perp\!\!\perp \neq a \\
 & \qquad \text{predicate calculus, (details left to the reader)} \quad \} \\
 & p \neq \perp\!\!\perp \;\; .
\end{array}
$$

□

Next we formulate the consequence of showing that specification p has at most one solution.

Lemma 24

$$
\langle \forall a : a \subseteq p : a = p \rangle \;\;\equiv\;\; \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \;\; .
$$

Proof

$$
\begin{array}{ll}
 & \langle \forall a : a \subseteq p : a = p \rangle \\
= & \{ \quad \text{anti-symmetry} \quad \} \\
 & \langle \forall a : a \subseteq p : a \supseteq p \rangle \\
= & \{ \quad \text{saturation: (16)} \quad \} \\
 & \langle \forall a : a \subseteq p : a \supseteq \langle \cup a' : a' \subseteq p : a' \rangle \rangle \\
= & \{ \quad \text{suprema} \quad \} \\
 & \langle \forall a : a \subseteq p : \langle \forall a' : a' \subseteq p : a \supseteq a' \rangle \rangle \\
\Leftarrow & \{ \quad \text{reflexivity of the subset relation} \quad \} \\
 & \langle \forall a : a \subseteq p : \langle \forall a' : a' \subseteq p : a = a' \rangle \rangle \\
= & \{ \quad \text{nesting of quantifications} \quad \} \\
 & \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \\
\Leftarrow & \{ \quad \text{Leibniz and predicate calculus} \quad \} \\
 & \langle \forall a : a \subseteq p : a = p \rangle \;\; .
\end{array}
$$

□

Theorem 25 Suppose p is a coreflexive relation. Then p is a point equivales

$$
\langle \exists a :: a \subseteq p \rangle \;\;\wedge\;\; \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \;\; .
$$

(As above, dummies a and a' range over points of the same type as p.)

In words, a specification p defines a point iff it has at least one solution and at most one solution.

Proof In the following dummy q ranges over coreflexives of the same type as p and a ranges over points of the same type as p.

$$\begin{array}{ll}
& p \text{ is atomic} \\
= & \{ \quad \text{definition 8} \quad \} \\
& \langle \forall q : q \subseteq p : q = p \vee q = \perp\!\!\perp \rangle \\
= & \{ \quad \text{trading} \quad \} \\
& \langle \forall q : q \subseteq p \wedge q \neq \perp\!\!\perp : q = p \rangle \\
= & \{ \quad \text{lemma 23} \quad \} \\
& \langle \forall q : q \subseteq p \wedge \langle \exists a :: a \subseteq q \rangle : q = p \rangle \\
= & \{ \quad \text{distributivity (of conjunction over disjunction),} \\
& \qquad \text{range disjunction} \quad \} \\
& \langle \forall q,a : a \subseteq q \subseteq p : q = p \rangle \\
\Leftarrow & \{ \quad \text{anti-symmetry} \quad \} \\
& \langle \forall a : a \subseteq p : a = p \rangle \\
= & \{ \quad \text{lemma 24} \quad \} \\
& \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \quad .
\end{array}$$

Also,

$$\begin{array}{ll}
& p \text{ is atomic} \\
= & \{ \quad \text{definition 8} \quad \} \\
& \langle \forall q : q \subseteq p : q = p \vee q = \perp\!\!\perp \rangle \\
\Rightarrow & \{ \quad \text{points } a \text{ and } a' \text{ are coreflexives, weakening} \quad \} \\
& \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : (a = p \vee a = \perp\!\!\perp) \wedge (a' = p \vee a' = \perp\!\!\perp) \rangle \\
= & \{ \quad \text{points are proper (i.e. } a \neq \perp\!\!\perp \text{ and } a' \neq \perp\!\!\perp \text{)} \quad \} \\
& \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = p \wedge a' = p \rangle \\
\Rightarrow & \{ \quad \text{transitivity of equality} \quad \} \\
& \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \quad .
\end{array}$$

Combining the two calculations, we have established by mutual implication that

(26) $\quad p \text{ is atomic} \;\equiv\; \langle\forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a'\rangle$.

It follows that, for all coreflexives p,

$$
\begin{array}{ll}
 & p \text{ is a point} \\
= & \{ \quad \text{definitions 8 and 12, assumption: } p \text{ is coreflexive} \quad \} \\
 & p \neq \perp\!\!\perp \;\wedge\; p \text{ is atomic} \\
= & \{ \quad \text{lemma 23 and (26)} \quad \} \\
 & \langle\exists a :: a \subseteq p\rangle \;\wedge\; \langle\forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a'\rangle \;.
\end{array}
$$

□

3.2 Factors

If R is a relation of type $A{\sim}B$ and S is a relation of type $A{\sim}C$, the relation $R\backslash S$ of type $B{\sim}C$ is defined by the Galois connection, for all T (of type $B{\sim}C$),

$$R\backslash S \supseteq T \;\equiv\; S \supseteq R{\circ}T \;.$$

Similarly, if R is a relation of type $A{\sim}B$ and S is a relation of type $C{\sim}B$, the relation R/S of type $A{\sim}C$ is defined by the Galois connection, for all T,

$$R/S \supseteq T \;\equiv\; R \supseteq T{\circ}S \;.$$

(The existence of these operators is equivalent to the universal distributivity of composition over union.)

In relation algebra, factors are also known as "residuals". We prefer the term "factor" because it emphasises calculational properties whereas "residual" emphasises an operational understanding (what is left after taking something away). In particular, factors have the *cancellation* properties:

$$T\circ T\backslash U \;\subseteq\; U \quad\wedge\quad R/S\circ S \;\subseteq\; R \;.$$

The factor operators (which we pronounce "under" and "over" respectively) are mutually associative. That is

(27) $\quad R\backslash(S/T) \;=\; (R\backslash S)/T$.

This means that it is unambiguous to write $R\backslash S/T$ — which we shall do in order to promote the associativity property by making its use invisible (in the same way that the use of the associativity of composition is made invisible).

The relations $R\backslash R$ (of type $B{\sim}B$ if R has type $A{\sim}B$) and R/R (of type $A{\sim}A$ if R has type $A{\sim}B$) play a central rôle in what follows. As is easily verified, both are *preorders*. That is, both are *transitive*:

$$R\backslash R \circ R\backslash R \;\subseteq\; R\backslash R \quad \wedge \quad R/R \circ R/R \;\subseteq\; R/R$$

and both are *reflexive*:

$$I \subseteq R\backslash R \quad \wedge \quad I \subseteq R/R \ \ .$$

(The notation " I " is overloaded in the above equation. In the left conjunct, it denotes the identity relation of type $B{\sim}B$ and, in the right conjunct, it denotes the identity relation of type $A{\sim}A$, assuming R has type $A{\sim}B$. We often overload constants in this way. Note, however, that we do not attempt to combine the two inclusions into one.) In addition, for all R,

(28) $$R \circ R\backslash R \;=\; R \;=\; R/R \circ R \ \ ,$$

(29) $$R/(R\backslash R) \;=\; R \;=\; (R/R)\backslash R \ \ ,$$

(30) $$(R\backslash R)/(R\backslash R) \;=\; R\backslash R \;=\; (R\backslash R)\backslash(R\backslash R) \ \ \text{and}$$

(31) $$(R/R)\backslash(R/R) \;=\; R/R \;=\; (R/R)/(R/R) \ \ .$$

In fact, we don't use (29) directly; its relevance is as the initial step in proving the leftmost equations of (30) and (31). We choose not to exploit the associativity of the over and under operators in (30) and (31) —by writing, for example, $(R\backslash R)/(R\backslash R)$ as $R\backslash R/(R\backslash R)$— in order to emphasise their rôle as properties of the preorders $R\backslash R$ and R/R.

In relation algebra (as opposed to regular algebra) it is possible to eliminate the factor operators altogether because they can be expressed in terms of complements and converse. The rule for doing so is given in lemma 32. Although the elimination of factors is highly undesirable, we are obliged to introduce complements and it is useful to exploit the lemma occasionally.

Lemma 32 For all R, S and T,

$$R\backslash S/T \;=\; \neg(R^{\cup} \circ \neg S \circ T^{\cup}) \ \ .$$

Proof We have, for all X,

$$
\begin{array}{ll}
& X \subseteq R\backslash S/T \\
= & \{ \quad \text{definition of factors} \quad \} \\
& R{\circ}X{\circ}T \subseteq S \\
= & \{ \quad \text{middle-exchange: (6)} \quad \} \\
& R^{\cup} \circ \neg S \circ T^{\cup} \subseteq \neg X \\
= & \{ \quad \text{complements} \quad \} \\
& X \subseteq \neg(R^{\cup} \circ \neg S \circ T^{\cup}) \quad .
\end{array}
$$

The lemma follows by indirect equality (i.e. by instantiating X to the left and right sides of the claimed equality and then using reflexivity and anti-symmetry of the subset ordering).
□

For the purpose of providing examples, extreme cases are often the most illuminating. Instantiating lemma 32 with $R,S,T := \neg I\,,\neg I\,,I$, and $R,S,T := I\,,\neg I\,,\neg I$ (where I denotes an identity relation of some unspecified type), we get

(33) $\quad \neg I \backslash \neg I \;=\; I \;=\; \neg I / \neg I \quad .$

Thus the equality relation on a type is the preorder of the form $R\backslash R$ (or R/R) obtained by the instantiation $R{:=}\neg I$.

Let $\mathbb{1}$ denote the type with exactly one element. Then the universal relation ${}_{\mathbb{1}}\top\top_{\mathbb{1}}$ equals the identity relation $I_{\mathbb{1}}$. Thus the type $\mathbb{1}$ is an example of a finite, non-empty type such that $\neg I_{\mathbb{1}}$ is the empty relation ${}_{\mathbb{1}}\bot\bot_{\mathbb{1}}$.

Property (28) exemplifies how much easier calculations with factors can be compared to calculations that combine complements with converses. The property is very easy to spot and apply. Expressed using lemma 32, it is equivalent to

$$R \circ \neg(R^{\cup} \circ \neg R) \;=\; R \;=\; \neg(\neg R \circ R^{\cup}) \circ R \quad .$$

In this form, the property is difficult to spot and its correct application is difficult to check.

It is useful to record the distributivity properties of converse over the factor operators:

Lemma 34 For all R and S,

(35) $\quad R^{\cup} \backslash S^{\cup} \;=\; (S/R)^{\cup} \;=\; \neg R / \neg S \quad .$

Symmetrically,

(36) $\quad R^{\cup} / S^{\cup} \;=\; (S\backslash R)^{\cup} \;=\; \neg R \backslash \neg S \quad .$

Also,

(37) $\quad (R\backslash S/T)^{\cup} \;=\; T^{\cup} \backslash S^{\cup} / R^{\cup} \quad .$

Proof We prove the first equation of (35) using indirect equality. For any R, S and T, we have:

$$
\begin{array}{ll}
 & T \subseteq (S/R)^{\cup} \\
= & \{ \quad \text{converse: (1)} \quad \} \\
 & T^{\cup} \subseteq S/R \\
= & \{ \quad \text{Galois connection defining factors} \quad \} \\
 & T^{\cup} \circ R \subseteq S \\
= & \{ \quad \text{converse: (1) and (2)} \quad \} \\
 & R^{\cup} \circ T \subseteq S^{\cup} \\
= & \{ \quad \text{Galois connection defining factors} \quad \} \\
 & T \subseteq R^{\cup} \backslash S^{\cup} \quad .
\end{array}
$$

The second equation of (35) is proved using the property

(38) $\quad R \backslash S \;=\; \neg(R^{\cup} \circ \neg S) \quad \wedge \quad S/T \;=\; \neg(\neg S \circ T^{\cup}) \quad .$

We have:

$$
\begin{array}{ll}
 & \neg R / \neg S \\
= & \{ \quad \text{(38) with } S,T := \neg R, \neg S, \\
 & \quad\;\; \text{properties of negation and converse} \quad \} \\
 & \neg(R \circ \neg S^{\cup}) \\
= & \{ \quad \text{(38) with } R,S := R^{\cup}, S^{\cup} \\
 & \quad\;\; \text{properties of negation and converse} \quad \} \\
 & R^{\cup} \backslash S^{\cup} \\
= & \{ \quad \text{first equality} \quad \} \\
 & (S/R)^{\cup} \quad .
\end{array}
$$

Property (36) proved using symmetrical calculations and (37) is a combination of (35) and (36).

(Note how the associativity property $\neg(R^{\cup}) = (\neg R)^{\cup}$ is used silently in the above calculation.)

□

The following corollary is relevant to section 11 on staircase relations.

Corollary 39 For all R,

$$R\backslash R \cup (R\backslash R)^{\cup} \;=\; (R^{\cup}/R^{\cup})^{\cup} \cup R^{\cup}/R^{\cup} \;=\; \neg R\backslash\neg R \cup (\neg R\backslash\neg R)^{\cup} \;\; .$$

Proof

$$
\begin{array}{ll}
 & R\backslash R \cup (R\backslash R)^{\cup} \\
= & \{ \quad \text{converse and lemma 34} \\
 & \quad\;\; \text{(in particular (35) with } R,S := R^{\cup}, R^{\cup}\text{)} \quad \} \\
 & (R^{\cup}/R^{\cup})^{\cup} \cup R^{\cup}/R^{\cup} \\
= & \{ \quad \text{lemma 34} \\
 & \quad\;\; \text{(in particular (36) with } R,S := \neg R, \neg R\text{)} \quad \} \\
 & (\neg R\backslash\neg R)^{\cup} \cup \neg R\backslash\neg R \;\; .
\end{array}
$$

□

When considering concrete examples, it is sometimes necessary to know the pointwise definition of the factor operators. The following lemma is needed in theorem 319 where we exhibit a concrete counterexample to an error in the extant literature.

Lemma 40 For all relations R and points a and b (of appropriate type),

$$a{\circ}\top\!\top{\circ}b \subseteq (R\backslash R/R)^{\cup} \;\;\equiv\;\; \langle\forall a',b' : a'{\circ}\top\!\top{\circ}b \subseteq R \wedge a{\circ}\top\!\top{\circ}b' \subseteq R : a'{\circ}\top\!\top{\circ}b' \subseteq R\rangle \;\; .$$

Proof

$$
\begin{array}{ll}
 & a{\circ}\top\!\top{\circ}b \subseteq (R\backslash R/R)^{\cup} \\
= & \{ \quad \text{definition of converse and factors} \quad \} \\
 & R{\circ}b{\circ}\top\!\top{\circ}a{\circ}R \subseteq R \\
= & \{ \quad \text{saturation property: (19)} \quad \} \\
 & \langle\forall a',b' :: a'{\circ}R{\circ}b{\circ}\top\!\top{\circ}a{\circ}R{\circ}b' \subseteq a'{\circ}R{\circ}b'\rangle \\
= & \{ \quad \text{all-or-nothing: theorem 11} \quad \} \\
 & \langle\forall a',b' : a'{\circ}\top\!\top{\circ}b \subseteq R \wedge a{\circ}\top\!\top{\circ}b' \subseteq R : a'{\circ}\top\!\top{\circ}b{\circ}\top\!\top{\circ}a{\circ}\top\!\top{\circ}b' \subseteq R\rangle \\
= & \{ \quad \text{cone rule, } a \text{ and } b \text{ are points} \quad \} \\
 & \langle\forall a',b' : a'{\circ}\top\!\top{\circ}b \subseteq R \wedge a{\circ}\top\!\top{\circ}b' \subseteq R : a'{\circ}\top\!\top{\circ}b' \subseteq R\rangle \;\; .
\end{array}
$$

□

3.3 The Domain Operators

Within relation algebra, there are various ways that sets can be represented as relations. Schmidt and Ströhlein [SS93] use "conditions" (relations of the form $R{\circ}\top\!\!\top$ or $\top\!\!\top{\circ}R$ — called "vectors" by Schmidt and Ströhlein), Freyd and Ščedrov [Fv90] use coreflexives. A third possibility is to use "squares" (as suggested by Voermans [Voe99]).

Definition 41 A (homogeneous) relation R is a *square* iff $R = R \circ \top\!\!\top \circ R^{\cup}$.
□

Points are squares. Also if a and b are points (of appropriate type), the relations $R^{\cup} \circ a \circ R$ and $R \circ b \circ R^{\cup}$ are squares. (This is an easy consequence of the properties (13) and (15).) We see later (lemma 57) that $R^{\cup} \circ a \circ R$ represents the set of all points b such that a and b are related by R, and similarly for $R \circ b \circ R^{\cup}$.

Formally, coreflexives, conditionals and squares are isomorphic representations of sets. Nevertheless, choosing which to use can make a considerable difference to concise calculation. Squares have the disadvantage that they are not closed under union (although squares are closed under intersection); coreflexives and conditionals are both closed under union and intersection. The only advantage of using conditionals over coreflexives and squares is that they are closed under negation but the advantage is not significant. (Schmidt and Ströhlein [SS93] make extensive use of negation but most can be eliminated by the use of factors.) The overwhelming advantage of using coreflexives is their convenience in expressing restrictions on the left and right domain of relations, in combination with the associativity of composition. So, if p is a coreflexive, $R{\circ}p{\circ}S$ simultaneously restricts the right domain of R and the left domain of S to elements in the set represented by p . If conditions are used, one must choose between using a right condition to restrict the right domain of R and a left condition to restrict the left domain of S . Squares can also be used to restrict the left or right domain of a relation —there are several instances in section 6.3.1— but cannot be used to simultaneously restrict the right and left domains of two relations. For this reason, we generally prefer to use coreflexives to represent sets, except in very special circumstances.

Definition 42 (Domain Operators) Given relation R of type $A{\sim}B$, the *coreflexive representation* $R{<}$ of the *left domain* of R is defined by the equation

$$R{<} \;\; = \;\; I \cap R \circ R^{\cup}$$

and the *coreflexive representation* $R{>}$ of the *right domain* of R is defined by the equation

$$R{>} \;\; = \;\; I \cap R^{\cup} \circ R \;\; .$$

□

The name "domain operator" is chosen because of the fundamental properties: for all R and all coreflexives p,

(43) $\quad R = R{\circ}p \;\equiv\; R{>} \subseteq p$

and

(44) $\quad R = p{\circ}R \;\equiv\; R{<} \subseteq p$.

A simple, often used consequence of (43) and (44) is the property:

(45) $\quad R{<} \circ R \;=\; R \;=\; R \circ R{>}$.

In words, $R{>}$ is the least coreflexive p such that restricting the "domain" of R on the right has no effect on R. It is in this sense that $R{<}$ and $R{>}$ represent the set of points on the left and on the right on which the relation R is "defined", i.e. its left and right "domains".

Aside Freyd and Ščedrov [Fv90] call $R{<}$ the "domain" of R; they do not appear to give a name to $R{>}$. Like us, they also use the names "source" and "target". In their account a relation of type $A{\sim}B$ has source A and target B; we reverse the names. (See the warning above.) Bird and De Moor [BdM97] call $R{>}$ the "domain" of R and $R{<}$ the "range" of R. **End of Aside**

In our earlier work on relation algebra, the domain operators play a very significant rôle, and the same is true here. We regard knowledge of their properties as so fundamental that we often explain steps making use of domain calculus with the simple hint "domains". The most fundamental property of the domain operators —monotonicity— we use silently. Sometimes (for example in the proof of lemma 55) we state the properties within everywhere brackets.

For readers unfamiliar with the domain operators, we summarise their properties below. We restrict our attention here to the right-domain operator. The reader is requested to dualise the results to the left-domain operator.

The intended interpretation of $R{>}$ (read R "right") for relation R is $\{x \mid \langle \exists y :: y[\![R]\!]x \rangle\}$. Two ways we can reformulate this requirement without recourse to points are formulated in the following theorem.

Theorem 46 (Right Domain) For all relations R and coreflexives p,

(47) $\quad R{>} \subseteq p \;\equiv\; R \subseteq \top\!\!\top{\circ}p$

and

(48) $\quad R{>} \subseteq p \;\equiv\; R = R{\circ}p$.

□

The characterisations (47) and (48) predict a number of useful calculational properties of the right domain operator. Some are immediate, some involve a little bit of work for their verification. Immediate from (47) —a Galois connection— is that the right domain operator is universally $\cup$-junctive, and $(\top\!\top\circ)$ is universally distributive over infima of coreflexives. In particular,

$$\top\!\top\circ(p\cap q) = (\top\!\top\circ p)\cap(\top\!\top\circ q) \quad ,$$

$$(R\cup S)_{>} = R_{>}\cup S_{>} \quad ,$$

and

$$\perp\!\!\perp_{>} = \perp\!\!\perp \quad .$$

The last of these can in fact be strengthened to

(49) $\quad R_{>} = \perp\!\!\perp \;\equiv\; R = \perp\!\!\perp \quad .$

The proof is straightforward: use (47) in combination with $\top\!\top\circ\perp\!\!\perp = \perp\!\!\perp$.

From (47) we may also deduce a number of cancellation properties. But, in combination with the modularity rule, the cancellation properties can be strengthened. We leave their proofs together with a couple of other interesting applications of Galois connections as exercises.

Theorem 50 For all relations R, S and T

(a) $\top\!\top\circ R_{>} = \top\!\top\circ R \quad ,$

(b) $R\cap S\circ\top\!\top\circ T \;=\; S_{<}\circ R\circ T_{>} \quad ,$

(c) $(R^{\cup})_{>} = R_{<} \quad ,$

(d) $(R\cap S\circ T)_{>} = (S^{\cup}\circ R\cap T)_{>} \quad ,$

(e) $(R\circ\top\!\top\circ S)_{>} = S_{>} \;\Leftarrow\; R\neq\perp\!\!\perp \quad .$

□

We complete this section by documenting the isomorphism between coreflexives and conditions. Recall that the right conditions are, by definition, the fixed points of the function $(\top\!\top\circ)$.

Theorem 51 The coreflexives are the fixed points of the right domain operator. That is, for all R,

(a) $\quad R = R_{>} \equiv R\subseteq I \quad .$

Also, for all coreflexives p and all right conditions C,

(b) $(\top\!\!\top\circ p)> = p$, and
(c) $\top\!\!\top \circ C> = C$.

Moreover, for all relations R and S,

(d) $R> \subseteq S> \;\equiv\; \top\!\!\top\circ R \subseteq \top\!\!\top\circ S$.

Hence,

(e) $R> = S> \;\equiv\; \top\!\!\top\circ R = \top\!\!\top\circ S$.

The right-domain operator is thus a poset isomorphism mapping the set of right conditions to the set of coreflexives and its inverse is the function $(\top\!\!\top\circ)$.
□

Some powerful and far from obvious theorems about coreflexives are proved by mapping the theorems to statements about conditionals and then exploiting the characteristic properties of $\top\!\!\top$ — $\top\!\!\top \supseteq R$ for all R, and $\top\!\!\top = \top\!\!\top^{\cup}$ — to prove these statements. An illustration of the technique is afforded by the proof of the following lemma.

(52) $(R\circ S)> = (R> \circ S)>$.

We begin the proof by invoking theorem 51

$$
\begin{array}{ll}
 & (R\circ S)> = (R> \circ S)> \\
= & \{ \quad \text{theorem 51(e)} \quad \} \\
 & \top\!\!\top\circ R\circ S = \top\!\!\top \circ R> \circ S \\
= & \{ \quad \top\!\!\top \circ R> = \top\!\!\top\circ R \quad \} \\
 & \top\!\!\top\circ R\circ S = \top\!\!\top\circ R\circ S \\
= & \{ \quad \text{reflexivity} \quad \} \\
 & \mathsf{true} \;\; .
\end{array}
$$

Another useful property is:

(53) $X = \perp\!\!\perp \equiv X> = \perp\!\!\perp$.

The proof is by mutual implication. First,

$$X = \perp\!\!\perp \;\Rightarrow \{\text{Leibniz}\}\;\; X> = \perp\!\!\perp> \;\Rightarrow \{\perp\!\!\perp> = \perp\!\!\perp\}\;\; X> = \perp\!\!\perp \;\; .$$

Second,

$X{>} = \perp\!\!\perp$

$=$ { $\perp\!\!\perp$ is least relation }

$X{>} \subseteq \perp\!\!\perp$

$=$ { theorem 46 }

$I \cap \top\!\!\top \circ X \subseteq \perp\!\!\perp$

$\Rightarrow$ { monotonicity of composition,

preparing for use of the modularity rule }

$(I \cap X \circ \top\!\!\top) \circ \top\!\!\top \subseteq \perp\!\!\perp$

$\Rightarrow$ { modularity rule: (3), $\top\!\!\top = \top\!\!\top^{\cup}$ }

$\top\!\!\top \cap X \subseteq \perp\!\!\perp$

$=$ { $\top\!\!\top$ is greatest relation, $\perp\!\!\perp$ is least relation }

$X = \perp\!\!\perp$.

We conclude this section with a basic property that becomes very obvious with a little knowledge of the domain operators. Specifically, we have, for all relations R,

(54) $R \subseteq R \circ R^{\cup} \circ R$

The proof is easy:

$R \subseteq R \circ R^{\cup} \circ R$

$\Leftarrow$ { $R{>} \subseteq R^{\cup} \circ R$ and monotonicity of composition }

$R = R \circ R{>}$

$=$ { domains }

true .

3.4 Properties of Points

This section documents properties of points with respect to domains and factors.

Lemma 55 For all relations R and points a and b (of appropriate type),

$$a \subseteq R{<} \ \equiv \ (a \circ R){>} \neq \perp\!\!\perp \text{ , and}$$

$$b \subseteq R{>} \ \equiv \ (R \circ b){<} \neq \perp\!\!\perp \text{ .}$$

Proof We prove the second equation.

$$
\begin{array}{ll}
 & (R{\circ}b){<} \neq \bot\!\!\bot \\
= & \{ \quad \text{cone rule: (5)} \quad \} \\
 & \top\!\!\top \circ (R{\circ}b){<} \circ \top\!\!\top \;=\; \top\!\!\top \\
= & \{ \quad [\, R{<} \circ \top\!\!\top \;=\; R{\circ}\top\!\!\top \,] \text{ with } R{:=}R{\circ}b \quad \} \\
 & \top\!\!\top{\circ}R{\circ}b{\circ}\top\!\!\top = \top\!\!\top \\
= & \{ \quad [\, \top\!\!\top \circ R{>} \;=\; \top\!\!\top{\circ}R \,] \quad \} \\
 & \top\!\!\top \circ R{>} \circ b \circ \top\!\!\top \;=\; \top\!\!\top \\
= & \{ \quad \text{cone rule: (5)} \quad \} \\
 & R{>} \circ b \neq \bot\!\!\bot \\
= & \{ \quad R{>} \circ b \subseteq b \;; \\
 & \qquad \text{so, by atomicity of } b\text{, } R{>} \circ b = b \vee R{>} \circ b = \bot\!\!\bot \;; \\
 & \qquad \text{also, } b \neq \bot\!\!\bot \quad \} \\
 & R{>} \circ b = b \\
= & \{ \quad R{>} \circ b = R{>} \cap b \quad \} \\
 & b \subseteq R{>} \;\;.
\end{array}
$$

□

For a point b the square $R \circ b \circ R^{\cup}$ represents the set of all points a such that a and b are related by R. This is made precise in lemma 56 and its corollary, lemma 57.

Lemma 56 For all relations R of type $A{\sim}B$, all coreflexives p of type $A{\sim}A$ and all points b of type B,

$$p \subseteq R \circ b \circ R^{\cup} \quad\equiv\quad p{\circ}\top\!\!\top{\circ}b \subseteq R \;\;.$$

Symmetrically, for all relations R of type $A{\sim}B$, all coreflexives q of type $B{\sim}B$ and all points a of type A,

$$q \subseteq R^{\cup} \circ a \circ R \quad\equiv\quad a{\circ}\top\!\!\top{\circ}q \subseteq R \;\;.$$

Proof By mutual implication:

$$
\begin{array}{ll}
 & p \subseteq R \circ b \circ R^{\cup} \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & p{\circ}\top\!\!\top{\circ}b \;\subseteq\; R \circ b \circ R^{\cup} \circ \top\!\!\top \circ b \\
\Rightarrow & \{ \quad R^{\cup} \circ \top\!\!\top \subseteq \top\!\!\top \;;\; b \text{ is a point: (15) and } b \subseteq I \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & p\circ\top\!\!\top\circ b \;\subseteq\; R \\
\Rightarrow & \{ \quad \text{converse and monotonicity} \quad \} \\
 & p\circ\top\!\!\top\circ b\circ b\circ\top\!\!\top\circ p^{\cup} \;\subseteq\; R\circ b\circ R^{\cup} \\
\Rightarrow & \{ \quad b \text{ is a point: (13) and (14)} \\
 & \quad\;\; p \text{ is a coreflexive, so } p^{\cup}=p\,;\text{ monotonicity} \quad \} \\
 & p\circ\top\!\!\top\circ p \;\subseteq\; R\circ b\circ R^{\cup} \\
\Rightarrow & \{ \quad I\subseteq\top\!\!\top \text{ and } p\circ p=p \quad \} \\
 & p \;\subseteq\; R\circ b\circ R^{\cup} \quad .
\end{array}
$$

□

Property (18) is the most basic formulation of membership of pairs in a relation. It can also be formulated in terms of squares and in terms of domains:

Lemma 57 For all relations R and points a and b (of appropriate type),

$$(a \subseteq R\circ b\circ R^{\cup}) \;=\; (a\circ\top\!\!\top\circ b \subseteq R) \;=\; (b \subseteq R^{\cup}\circ a\circ R) \quad .$$

Proof Straightforward instantiation of lemma 56:

$$
\begin{array}{ll}
 & a \subseteq R\circ b\circ R^{\cup} \\
= & \{ \quad \text{lemma 56 with } p:=a \quad \} \\
 & a\circ\top\!\!\top\circ b \subseteq R \\
= & \{ \quad \text{lemma 56 with } p:=b \quad \} \\
 & b \subseteq R^{\cup}\circ b\circ R \quad .
\end{array}
$$

□

Lemma 58 For all relations R and points a and b (of appropriate type),

$$(a \subseteq (R\circ b)_{<}) \;=\; (a\circ\top\!\!\top\circ b \subseteq R) \;=\; (b \subseteq (a\circ R)_{>}) \quad .$$

Proof

$$
\begin{array}{ll}
 & a\circ\top\!\!\top\circ b \subseteq R \\
\Rightarrow & \{ \quad \text{monotonicity and } a \text{ is a coreflexive, so } a\circ a=a \quad \} \\
 & a\circ\top\!\!\top\circ b \subseteq a\circ R \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & (a\circ\top\!\!\top\circ b)_{>} \subseteq (a\circ R)_{>}
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{domains: definition 42, } a \text{ and } b \text{ are points: (14) and (15)} \quad \} \\
 & b \subseteq (a{\circ}R)_{>} \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b \ \subseteq \ a \circ \top\!\!\top \circ (a{\circ}R)_{>} \\
= & \{ \quad \text{domains: } [\ \top\!\!\top \circ R_{>} \ = \ \top\!\!\top{\circ}R\ ] \text{ with } R{:=}\,a{\circ}R \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b \ \subseteq \ a{\circ}\top\!\!\top{\circ}a{\circ}R \\
= & \{ \quad a \text{ is a point, so } a{\circ}\top\!\!\top{\circ}a = a \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b \subseteq a{\circ}R \\
\Rightarrow & \{ \quad a \text{ is a coreflexive, monotonicity} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b \subseteq R \ .
\end{array}
$$

That is, we have shown by mutual implication that

$$a{\circ}\top\!\!\top{\circ}b \subseteq R \ \equiv \ b \subseteq (a{\circ}R)_{>} \ .$$

A symmetric calculation establishes that

$$a{\circ}\top\!\!\top{\circ}b \subseteq R \ \equiv \ a \subseteq (R{\circ}b)_{<} \ .$$

□

Combined with property (18), lemmas 57 and 58 give six alternative ways of formulating the membership relation $a{\circ}\top\!\!\top{\circ}b \subseteq R$. All are useful.

Lemma 59 For all relations R and points a (of appropriate type),

$$a \subseteq R_{<} \ \equiv \ \langle \exists b : b \subseteq R_{>} : a{\circ}\top\!\!\top{\circ}b \subseteq R \rangle \ .$$

Also, for all relations R and points b (of appropriate type),

$$b \subseteq R_{>} \ \equiv \ \langle \exists a : a \subseteq R_{<} : a{\circ}\top\!\!\top{\circ}b \subseteq R \rangle \ .$$

Proof We prove the first equation:

$$
\begin{array}{ll}
 & a \subseteq R_{<} \\
= & \{ \quad \text{lemma 55} \quad \} \\
 & (a{\circ}R)_{>} \neq \perp\!\!\perp \\
= & \{ \quad \text{lemma 23} \quad \} \\
 & \langle \exists b :: b \subseteq (a{\circ}R)_{>} \rangle
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{lemma 58} \quad \} \\
& \langle \exists b :: a \circ \top\!\!\top \circ b \subseteq R \rangle \\
= & \{ \quad \text{domains (specifically, } a \circ \top\!\!\top \circ b \subseteq R \Rightarrow b \subseteq R_{>} \text{)} \quad \} \\
& \langle \exists b : b \subseteq R_{>} : a \circ \top\!\!\top \circ b \subseteq R \rangle \quad .
\end{array}
$$

□

Lemma 60 gives a pointwise interpretations of the factor operators. Although we typically try to avoid pointwise reasoning, the lemma is sometimes indispensable.

Lemma 60 For all relations R of type $A \sim C$ and S of type $B \sim C$ (for some A, B and C) and all points a and b,

$$a \circ \top\!\!\top \circ b \subseteq R/S \quad \equiv \quad (b \circ S)_{>} \subseteq (a \circ R)_{>} \quad .$$

Dually, for all relations R of type $C \sim A$ and S of type $C \sim B$, and all points a and b,

$$a \circ \top\!\!\top \circ b \subseteq R \backslash S \quad \equiv \quad (R \circ a)_{<} \subseteq (S \circ b)_{<} \quad .$$

Proof By mutual implication:

$$
\begin{array}{ll}
& a \circ \top\!\!\top \circ b \subseteq R/S \\
= & \{ \quad \text{definition of factor} \quad \} \\
& a \circ \top\!\!\top \circ b \circ S \subseteq R \\
\Rightarrow & \{ \quad a \text{ and } b \text{ are points, monotonicity and domains} \\
& \quad \text{(see initial steps in proof of lemma 58)} \quad \} \\
& (b \circ S)_{>} \subseteq (a \circ R)_{>} \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
& a \circ \top\!\!\top \circ (b \circ S)_{>} \subseteq a \circ \top\!\!\top \circ (a \circ R)_{>} \\
= & \{ \quad \text{domains} \quad \} \\
& a \circ \top\!\!\top \circ b \circ S \subseteq a \circ \top\!\!\top \circ a \circ R \\
= & \{ \quad a \text{ is a point (so } a \circ \top\!\!\top \circ a = a \text{)} \quad \} \\
& a \circ \top\!\!\top \circ b \circ S \subseteq a \circ R \\
\Rightarrow & \{ \quad a \text{ is a coreflexive} \quad \} \\
& a \circ \top\!\!\top \circ b \circ S \subseteq R \\
= & \{ \quad \text{definition of factor} \quad \} \\
& a \circ \top\!\!\top \circ b \subseteq R/S \quad .
\end{array}
$$

The second equivalence is proved similarly.

$$
\begin{array}{ll}
 & a{\circ}\top\!\!\top{\circ}b \subseteq R\backslash S \\
= & \{ \quad \text{definition of factor} \quad \} \\
 & R{\circ}a{\circ}\top\!\!\top{\circ}b \subseteq S \\
\Rightarrow & \{ \quad \text{monotonicity and coreflexives} \\
 & \quad\;\; \text{(see initial steps in proof of lemma 58)} \quad \} \\
 & (R{\circ}a)_< \subseteq (S{\circ}b)_< \\
\Rightarrow & \{ \quad \text{(as in above calculation)} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b \subseteq R\backslash S \quad .
\end{array}
$$

□

For relations R and S with the same source, the relation $R/S \cap (S/R)^{\cup}$ is the "symmetric left division" of R and S. Dually, for relations R and S with the same target, the relation $R\backslash S \cap (S\backslash R)^{\cup}$ is their "symmetric right division". (See the discussion at the beginning of section 3.7.) The following corollary of lemma 60 gives a pointwise interpretation of these "division" operators.

Corollary 61 For all relations R and S with the same source, and all points a and b (of appropriate type),

$$a{\circ}\top\!\!\top{\circ}b \subseteq R/S \cap (S/R)^{\cup} \equiv (a{\circ}R)_> = (b{\circ}S)_> \quad .$$

Dually, for all relations R and S with the same target, and all points a and b (of appropriate type),

$$a{\circ}\top\!\!\top{\circ}b \subseteq R\backslash S \cap (S\backslash R)^{\cup} \equiv (R{\circ}a)_< = (S{\circ}b)_< \quad .$$

Proof Straightforward application of lemma 60 and anti-symmetry:

$$
\begin{array}{ll}
 & a{\circ}\top\!\!\top{\circ}b \subseteq R/S \cap (S/R)^{\cup} \\
= & \{ \quad \text{infima and converse} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b \subseteq R/S \;\wedge\; b{\circ}\top\!\!\top{\circ}a \subseteq S/R \\
= & \{ \quad \text{lemma 60} \quad \} \\
 & (b{\circ}S)_> \subseteq (a{\circ}R)_> \;\wedge\; (a{\circ}R)_> \subseteq (b{\circ}S)_> \\
= & \{ \quad \text{anti-symmetry} \quad \} \\
 & (a{\circ}R)_> = (b{\circ}S)_> \quad .
\end{array}
$$

□

3.5 Functionality

A relation R of type $A{\sim}B$ is said to be *functional* if $R \circ R^{\cup} \subseteq I_A$. A relation R of type $A{\sim}B$ is said to be *surjective* if $R \circ R^{\cup} \supseteq I_A$. Equivalently, a relation R of type $A{\sim}B$ is surjective if $R{<} = I_A$. A relation R of type $A{\sim}B$ that is both functional and surjective is thus defined by the property $R \circ R^{\cup} = I_A$.

(Other words used for functional are "quasi-fonctionelle" [Rig48], "simple" [Fv90, BdM97] and "univalent" [SS93].)

Dual to functionality and surjectivity are the notions of injectivity and totality, respectively. A relation R of type $A{\sim}B$ is said to be *injective* if $R^{\cup} \circ R \subseteq I_B$. A relation R of type $A{\sim}B$ is said to be *total* if $R^{\cup} \circ R \supseteq I_B$. Equivalently, a relation R of type $A{\sim}B$ is surjective if $R{>} = I_B$.

Typically, we use lowercase letters f , g , h to denote functional relations. As the terminology suggests, these point-free definitions correspond to notions that are more usually defined in terms of points. The pointwise interpretations are explained below, beginning with the interpretation of a functional relation as what others might call a "partial function".

The standard notion of a partial function is a relation that defines a unique output value for each input value in its domain. In our axiom system we formulate this as follows.

Suppose R of type $A{\sim}B$ is functional and suppose b is a point of type B such that $b \subseteq R{>}$. We assert that the equation

(62) $\quad a{:}\;\; a{\in}A{:}\;\; a{\circ}\top\!\top{\circ}b \subseteq R$

has exactly one solution. Conversely, we assert that if equation (62) has exactly one solution for all points b such that $b \subseteq R{>}$, the relation R is functional. (In (62) the expression " $a{\in}A$ " limits the range of the dummy a to points of type A ; this notation will be used later where the range of a dummy cannot be deduced from other considerations.)

Equation (62) is an example of the sort of indirect specification anticipated in section 2.5. (See in particular theorem 25.) More formally, for functional relation f and point b such that $b \subseteq f{>}$, equation (62) defines $f.b$ as the unique solution of the equation:

$$a{::}\;\; \mathrm{point}.a \;\wedge\; a{\circ}\top\!\top{\circ}b \subseteq f \quad .$$

Suppose we denote this unique solution by $f.b$. The defining property of $f.b$ is thus

(63) $\quad \langle \forall\, a,b \;:\; b \subseteq f{>} \;:\; a{\circ}\top\!\top{\circ}b \subseteq f \;\equiv\; a = f.b \rangle \quad .$

But it is not immediately obvious that $f.b$ is well-defined in our axiom system. Theorem 64 provides a formal justification.

Theorem 64 Suppose relation R has type $A{\sim}B$. Then

(65) $\quad R\circ R^{\cup}\subseteq I_A \;\equiv\; \langle\forall b : b\subseteq R{>} : \mathsf{point}.(R\circ b\circ R^{\cup})\rangle$.

Moreover, if f is a relation of type $A{\sim}B$ and $f\circ f^{\cup}\subseteq I_A$, the relation $f\circ b\circ f^{\cup}$ is a point of type A and

(66) $\quad \langle\forall a,b \;:\; b\subseteq f{>} \;:\; a\circ\top\!\!\top\circ b\subseteq f \;\equiv\; a = f\circ b\circ f^{\cup}\rangle$.

In words, f is functional iff, for all points b in the right domain of f, the relation $f\circ b\circ f^{\cup}$ defines a unique point of type A, which point we denote by $f.b$.

Proof We prove (65) by mutual implication. First,

$$
\begin{array}{ll}
 & R\circ R^{\cup}\subseteq I_A \\
= & \{ \quad \text{domains} \quad \} \\
 & R\circ R{>}\circ R^{\cup}\subseteq I_A \\
= & \{ \quad \text{saturation axiom: (16)} \quad \} \\
 & R\circ\langle\cup b \;:\; b\subseteq R{>} \;:\; b\rangle\circ R^{\cup} \;\subseteq\; I_A \\
= & \{ \quad \text{distributivity} \quad \} \\
 & \langle\forall b \;:\; b\subseteq R{>} \;:\; R\circ b\circ R^{\cup}\subseteq I_A\rangle \\
\Leftarrow & \{ \quad \text{definition 12 of a point} \quad \} \\
 & \langle\forall b : b\subseteq R{>} : \mathsf{point}.(R\circ b\circ R^{\cup})\rangle \quad .
\end{array}
$$

Thus we have established the “if” part of the equivalence. Now, for the “only-if”, assume $R\circ R^{\cup}\subseteq I_A$.

We first note that, for all b such that $b\subseteq R{>}$, equation (62) has *at most one* solution since, for all points a and a' of type A,

$$
\begin{array}{ll}
 & a\circ\top\!\!\top\circ b\subseteq R \;\;\wedge\;\; a'\circ\top\!\!\top\circ b\subseteq R \\
\Rightarrow & \{ \quad \text{converse and monotonicity} \quad \} \\
 & a\circ\top\!\!\top\circ b\circ b\circ\top\!\!\top\circ a' \;\subseteq\; R\circ R^{\cup} \\
= & \{ \quad b \text{ is a point, so } \top\!\!\top\circ b\circ b\circ\top\!\!\top = \top\!\!\top \quad \} \\
 & a\circ\top\!\!\top\circ a' \;\subseteq\; R\circ R^{\cup} \\
\Rightarrow & \{ \quad \text{assumption: } R\circ R^{\cup}\subseteq I_A\text{, transitivity of the subset relation} \quad \} \\
 & a\circ\top\!\!\top\circ a'\subseteq I_A \\
\Rightarrow & \{ \quad a \text{ and } a' \text{ are points: (21)} \quad \} \\
 & a = a' \quad .
\end{array}
$$

That is,

(67) $\quad \langle\forall b : b \subseteq R{>} : \langle\forall\, a,a' : a{\circ}\top\!\top{\circ}b \subseteq R \;\wedge\; a'{\circ}\top\!\top{\circ}b \subseteq R : a = a'\rangle\rangle$.

By lemma 55, equation (62) has *at least one* solution for all points b such that $b \subseteq R{>}$. That is,

(68) $\quad \langle\forall b : b \subseteq R{>} : \langle\exists a :: a{\circ}\top\!\top{\circ}b \subseteq R\rangle\rangle$.

Thus equation (62) has *exactly one* solution for all points b such that $b \subseteq f{>}$. So:

$$
\begin{array}{ll}
& \langle\forall b : b \subseteq R{>} : \mathsf{point}.(R \circ b \circ R^{\cup})\rangle \\
= & \{ \quad \begin{array}{ll} & R \circ b \circ R^{\cup} \\ \subseteq & \{ \quad \text{assumption: } b \subseteq R{>}\text{, monotonicity} \quad \} \\ & R \circ R{>} \circ R^{\cup} \\ = & \{ \quad \text{domains} \quad \} \\ & R \circ R^{\cup} \\ \subseteq & \{ \quad \text{assumption: } R \circ R^{\cup} \subseteq I_A \quad \} \\ & I_A \quad , \end{array} \\
& \quad \text{theorem 25 with } p := R \circ b \circ R^{\cup} \quad \} \\
& \quad \langle\forall b : b \subseteq R{>} : \langle\exists a :: a \subseteq R \circ b \circ R^{\cup}\rangle\rangle \\
\wedge & \quad \langle\forall b : b \subseteq R{>} : \langle\forall\, a,a' : a \subseteq R \circ b \circ R^{\cup} \wedge a' \subseteq R \circ b \circ R^{\cup} : a = a'\rangle\rangle \\
= & \{ \quad \text{lemma 57} \quad \} \\
& \quad \langle\forall b : b \subseteq R{>} : \langle\exists a :: a{\circ}\top\!\top{\circ}b \subseteq R\rangle\rangle \\
\wedge & \quad \langle\forall b : b \subseteq R{>} : \langle\forall\, a,a' : a{\circ}\top\!\top{\circ}b \subseteq R \;\wedge\; a'{\circ}\top\!\top{\circ}b \subseteq R : a = a'\rangle\rangle \\
= & \{ \quad \text{(67) and (68)} \quad \} \\
& \mathsf{true} \quad .
\end{array}
$$

This concludes the proof of (65).

Now, assuming that $f \circ f^{\cup} \subseteq I$, it follows from (65) (with $R := f$) that $f \circ b \circ f^{\cup}$ is a point. Also, for all points a and b (of types A and B, respectively),

$$
\begin{array}{ll}
& b \subseteq f{>} \;\wedge\; a{\circ}\top\!\top{\circ}b \subseteq f \\
= & \{ \quad \text{lemma 58 (aiming to eliminate first conjunct)} \quad \} \\
& b \subseteq f{>} \;\wedge\; b \subseteq (a{\circ}f){>} \;\wedge\; a{\circ}\top\!\top{\circ}b \subseteq f \\
= & \{ \quad \text{monotonicity and lemma 58} \quad \}
\end{array}
$$

$$a\circ\top\!\top\circ b \subseteq f$$
$$= \quad \{ \quad \text{lemma 57} \quad \}$$
$$a \subseteq f\circ b\circ f^{\cup}$$
$$= \quad \{ \quad f\circ b\circ f^{\cup} \text{ is a point, definitions 12 and 8} \quad \}$$
$$a = f\circ b\circ f^{\cup} \quad .$$

□

Occasionally we need to define a functional relation. Sometimes we specify the relation by means of an equation: "we define f of type ... by $f.b = \ldots$". More often, we use the notation $\langle b :: \ldots\rangle$ to denote a total function, or $\langle b : \ldots : \ldots\rangle$ to denote a (non-total) functional, the range part being used to specify a restriction on the domain. This is consistent with our notation for suprema and infima (such as in universal and existential quantifications).

A consequence of the unicity property expressed by (63) is the property that, for all functional relations f of type $C{\sim}A$ and g of type $C{\sim}B$, and all points a and b,

(69) $\quad a\circ\top\!\top\circ b \subseteq f^{\cup}\circ g \quad\equiv\quad a\subseteq f{>} \wedge f.a = g.b \wedge b\subseteq g{>} \quad .$

The proof exploits the irreducibility of points:

$$a\circ\top\!\top\circ b \subseteq f^{\cup}\circ g$$
$$= \quad \{ \quad \text{domains, saturation axiom: (16) and distributivity} \quad \}$$
$$a\circ\top\!\top\circ b \subseteq \langle\cup c : c{\in}C : f^{\cup}\circ c\circ g\rangle$$
$$= \quad \{ \quad \text{points are irreducible: (20)} \quad \}$$
$$\langle\exists c : c{\in}C : a\circ\top\!\top\circ b \subseteq f^{\cup}\circ c\circ g\rangle$$
$$= \quad \{ \quad (22) \quad \}$$
$$\langle\exists c : c{\in}C : a\circ\top\!\top\circ c \subseteq f^{\cup} \wedge c\circ\top\!\top\circ b \subseteq g\rangle$$
$$= \quad \{ \quad \text{converse, lemma 58 and (63)} \quad \}$$
$$\langle\exists c : c{\in}C : a\subseteq f{>} \wedge c = f.a \wedge b\subseteq g{>} \wedge c = g.b\rangle$$
$$= \quad \{ \quad \text{Leibniz and predicate calculus} \quad \}$$
$$a\subseteq f{>} \wedge f.a = g.b \wedge b\subseteq g{>} \quad .$$

Now suppose R is a surjective relation of type $A{\sim}B$. In this case, for all points a of type A, the equation

(70) $\quad b:\ \ b{\in}B:\ \ a\circ\top\!\top\circ b \subseteq R$

has *at least one* solution since:

$$
\begin{array}{ll}
 & I_B \subseteq R^\cup \circ R \\
= & \{ \quad \text{saturation axiom: (16) and supremum} \quad \} \\
 & \langle \forall b : b \in B : b \subseteq R^\cup \circ R \rangle \\
= & \{ \quad \text{saturation axiom: (16) and distributivity} \quad \} \\
 & \langle \forall b : b \in B : b \subseteq \langle \cup a : a \in A : R^\cup \circ a \circ R \rangle \rangle \\
= & \{ \quad \text{points are irreducible: (20)} \quad \} \\
 & \langle \forall b : b \in B : \langle \exists a : a \in A : b \subseteq R^\cup \circ a \circ R \rangle \rangle \\
= & \{ \quad \text{lemma 57} \quad \} \\
 & \langle \forall b : b \in B : \langle \exists a : a \in A : a \circ \top\!\!\top \circ b \subseteq R \rangle \rangle \quad .
\end{array}
$$

In the same way, pointwise formulations of the dual notions of injectivity and totality can be derived. Our terminology reflects a bias in the interpretation of relations as having output on the left and input on the right. A more neutral terminology such as "left-functional", "right-functional", "left-total" and "right-total" would be preferable.

Care must be taken when using the above pointwise definitions in our axiom system. The problem is the overloading of the symbol $\top\!\!\top$: sometimes the type information is essential. For example, the left-domain operator (which we denote by the postfix symbol $<$) defines a total function of type $Cor.A \leftarrow (A \sim B)$, for all types A and B, where $Cor.A$ denotes the set of coreflexives of type A. Denoting this function by $Ldom$, we must be careful to distinguish between $Ldom.R$ and $R<$. This is because, according to (62),

$$ (71) \qquad Ldom.R \circ \top\!\!\top \circ R \subseteq Ldom \quad ; $$

on the other hand,

$$ (72) \qquad R< \circ \top\!\!\top \circ R = R \circ \top\!\!\top \circ R $$

and it doesn't make sense to write

$$ R< \circ \top\!\!\top \circ R \;\subseteq\; < \;\; ! $$

In equation (71), both R and $Ldom.R$ are *points* of type $A \sim B$ and $Cor.A$, respectively, and the symbol "$\top\!\!\top$" has type $Cor.A \sim (A \sim B)$ whereas in equation (72) $R<$ is not a point, the leftmost occurrence of the symbol "$\top\!\!\top$" has type $A \sim A$ and its rightmost occurrence has type $A \sim B$.

We conclude this section with a number of properties of functional relations. The properties stem from the observation that functionality can be defined via a Galois connection. Specifically, the relation f is functional iff, for all relations R and S (of appropriate type),

$$ (73) \qquad f \circ R \subseteq S \quad \equiv \quad f> \circ R \subseteq f^\cup \circ S \quad . $$

It is a simple exercise to show that (73) is equivalent to the property $f \circ f^{\cup} \subseteq I$. (Although (73) doesn't immediately fit the standard definition of a Galois connection, it can be turned into standard form by restricting the range of the dummy R to relations that satisfy $f> \circ R = R$, i.e. relations R such that $R< \subseteq f>$.)

The converse-dual of (73) is also used frequently: g is functional iff, for all R and S,

(74) $\quad R \circ g^{\cup} \subseteq S \quad \equiv \quad R \circ g> \subseteq S \circ g \quad .$

Comparing the Galois connections defining the over and under operators (see section 3.2) with the Galois connection defining functionality (see (73)) suggests a formal relationship between "division" by a functional relation and composition with the relation's converse. The precise form of this relationship is given by the following lemma.

Lemma 75 For all R and all functional relations f,

$$f> \circ f \backslash R \;=\; f^{\cup} \circ R \quad .$$

Proof We use the anti-symmetry of the subset relation. First,

$$
\begin{array}{ll}
 & f^{\cup} \circ R \;\subseteq\; f> \circ f \backslash R \\
= & \{ \quad \text{domains} \quad \} \\
 & f> \circ f^{\cup} \circ R \;\subseteq\; f> \circ f \backslash R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & f^{\cup} \circ R \;\subseteq\; f \backslash R \\
= & \{ \quad \text{factors} \quad \} \\
 & f \circ f^{\cup} \circ R \;\subseteq\; R \\
\Leftarrow & \{ \quad \text{definition and monotonicity} \quad \} \\
 & f \text{ is functional} \quad .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & f> \circ f \backslash R \;\subseteq\; f^{\cup} \circ R \\
\Leftarrow & \{ \quad f> \subseteq f^{\cup} \circ f \,;\ \text{monotonicity and transitivity} \quad \} \\
 & f^{\cup} \circ f \circ f \backslash R \;\subseteq\; f^{\cup} \circ R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & f \circ f \backslash R \;\subseteq\; R \\
= & \{ \quad \text{cancellation} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

□

Two lemmas that will be needed later now follow. Lemma 76 allows the converse of a functional relation (i.e. an injective relation) to be cancelled, whilst lemma 77 expresses a distributivity property.

Lemma 76 For all R and all functional relations f,

$$f< \circ f^{\cup} \backslash (f^{\cup} \circ R) \;=\; f< \circ R \;.$$

Proof

$$
\begin{array}{ll}
 & f< \circ f^{\cup} \backslash (f^{\cup} \circ R) \\
= & \{ \quad \text{assumption: } f \text{ is functional} \quad \} \\
 & f \circ f^{\cup} \circ f^{\cup} \backslash (f^{\cup} \circ R) \\
\subseteq & \{ \quad \text{cancellation} \quad \} \\
 & f \circ f^{\cup} \circ R \\
= & \{ \quad \text{assumption: } f \text{ is functional} \quad \} \\
 & f< \circ R \;.
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & f< \circ R \subseteq f< \circ f^{\cup} \backslash (f^{\cup} \circ R) \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R \subseteq f^{\cup} \backslash (f^{\cup} \circ R) \\
= & \{ \quad \text{factors} \quad \} \\
 & \mathsf{true} \;.
\end{array}
$$

The lemma follows by anti-symmetry of the subset relation.
□

Lemma 77 For all R and S and all functional relations f,

$$R \backslash (S \circ f) \circ f> \;=\; R \backslash S \circ f \;.$$

Proof

$$R\backslash(S{\circ}f)\circ f{>}\ \subseteq\ R\backslash S\circ f$$

$\Leftarrow$ { $f{>} \subseteq f^{\cup}\circ f$, monotonicity }

$$R\backslash(S{\circ}f)\circ f^{\cup}\ \subseteq\ R\backslash S$$

$=$ { factors }

$$R\circ R\backslash(S{\circ}f)\circ f^{\cup}\ \subseteq\ S$$

$\Leftarrow$ { cancellation }

$$S\circ f\circ f^{\cup}\ \subseteq\ S$$

$=$ { assumption: f is functional }

$$\mathsf{true}\quad .$$

Also,

$$R\backslash S\circ f\ \subseteq\ R\backslash(S{\circ}f)\circ f{>}$$

$\Leftarrow$ { monotonicity, $f = f\circ f{>}$ }

$$R\backslash S\circ f\ \subseteq\ R\backslash(S{\circ}f)$$

$=$ { factors and cancellation }

$$\mathsf{true}\quad .$$

The lemma follows by anti-symmetry of the subset relation.

□

The following lemma is crucial to fully understanding Riguet's "analogie frappante"; see lemma 248 in section 9.2. (The lemma is complicated by the fact that it has five free variables. Simpler, possibly better known, instances can be obtained by instantiating one or more of f, g, U and W to the identity relation.)

Lemma 78 Suppose f and g are functional. Then for all U, V and W,

$$f^{\cup}\circ(g{<}\circ U)\backslash V/(W\circ f{<})\circ g$$
$$=\ f{>}\circ(g^{\cup}\circ U\circ f)\backslash(g^{\cup}\circ V\circ f)/(g^{\cup}\circ W\circ f)\circ g{>}\quad .$$

Proof Guided by the assumed functionality of f and g, we use the rule of indirect equality. Specifically, we have, for all R, U, V and W,

$$f{>}\circ R\circ g{>}\ \subseteq\ f^{\cup}\circ(g{<}\circ U)\backslash V/(W\circ f{<})\circ g$$

$=$ { assumption: f and g are functional, (73) and (74) }

$$f\circ R\circ g^{\cup}\ \subseteq\ (g{<}\circ U)\backslash V/(W\circ f{<})$$

$$= \quad \{ \quad \text{factors} \quad \}$$

$$g< \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f< \;\subseteq\; V$$

$$= \quad \{ \quad \text{assumption: } f \text{ and } g \text{ are functional}$$

$$\text{i.e.} \quad f \circ f^{\cup} = f< \wedge\ g \circ g^{\cup} = g< \quad \}$$

$$g \circ g^{\cup} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f \circ f^{\cup} \;\subseteq\; V$$

$$= \quad \{ \quad \text{assumption: } f \text{ and } g \text{ are functional, (73) and (74)} \quad \}$$

$$g> \circ g^{\cup} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f \circ f> \;\subseteq\; g^{\cup} \circ V \circ f$$

$$= \quad \{ \quad \text{domains (four times)} \quad \}$$

$$g^{\cup} \circ U \circ f \circ f> \circ R \circ g> \circ g^{\cup} \circ W \circ f \;\subseteq\; g^{\cup} \circ V \circ f$$

$$= \quad \{ \quad \text{factors} \quad \}$$

$$f> \circ R \circ g> \;\subseteq\; (g^{\cup} \circ U \circ f)\backslash(g^{\cup} \circ V \circ f)/(g^{\cup} \circ W \circ f)$$

$$= \quad \{ \quad f> \text{ and } g> \text{ are coreflexives} \quad \}$$

$$f> \circ R \circ g> \;\subseteq\; f> \circ (g^{\cup} \circ U \circ f)\backslash(g^{\cup} \circ V \circ f)/(g^{\cup} \circ W \circ f) \circ g>$$

The lemma follows by instantiating R to the left and right sides of the claimed equation, simplifying using domain calculus, and then applying the reflexivity and anti-symmetry of the subset relation.

□

The final lemma in this section anticipates the discussion of per domains in section 3.8.

Lemma 79 Suppose relations R, f and g are such that

$$f \circ f^{\cup} \;=\; f< \;=\; R< \quad \wedge \quad g< \;=\; g \circ g^{\cup} \quad .$$

Then, for all S,

(80) $$g> \circ (f^{\cup} \circ R \circ g)\backslash(f^{\cup} \circ S) \quad = \quad g^{\cup} \circ R\backslash S \quad .$$

It follows that

(81) $$g> \circ (f^{\cup} \circ R \circ g)\backslash(f^{\cup} \circ R \circ g) \circ g> \quad = \quad g^{\cup} \circ R\backslash R \circ g \quad .$$

Proof The proof of (80) is as follows.

$$g> \circ (f^{\cup} \circ R \circ g)\backslash(f^{\cup} \circ S)$$

$$= \quad \{ \quad \text{factors} \quad \}$$

$$g> \circ g\backslash((f^{\cup} \circ R)\backslash(f^{\cup} \circ S))$$

$$
\begin{array}{ll}
= & \{ \quad \text{lemma 75 with } f,R := g\,,(f^{\cup}\circ R)\backslash(f^{\cup}\circ S) \quad \} \\
& g^{\cup}\circ(f^{\cup}\circ R)\backslash(f^{\cup}\circ S) \\
= & \{ \quad \text{factors} \quad \} \\
& g^{\cup}\circ R\backslash(f^{\cup}\backslash(f^{\cup}\circ S)) \\
= & \{ \quad [\, R\backslash S = R\backslash(R_{<}\circ S)\,] \text{ with } R,S := R\,,\, f^{\cup}\backslash(f^{\cup}\circ S) \\
& \quad\;\; \text{assumption: } f_{<} = R_{<} \quad \} \\
& g^{\cup}\circ R\backslash(f_{<}\circ f^{\cup}\backslash(f^{\cup}\circ S)) \\
= & \{ \quad \text{lemma 76 with } f,R := f,S \quad \} \\
& g^{\cup}\circ R\backslash(f_{<}\circ S) \\
= & \{ \quad \text{assumption: } f_{<} = R_{<}\,,\ [\, R\backslash S = R\backslash(R_{<}\circ S)\,] \quad \} \\
& g^{\cup}\circ R\backslash S \quad .
\end{array}
$$

Now we prove (81).

$$
\begin{array}{ll}
& g_{>}\circ(f^{\cup}\circ R\circ g)\backslash(f^{\cup}\circ R\circ g)\circ g_{>} \\
= & \{ \quad \text{(80) with } S := R\circ g \quad \} \\
& g^{\cup}\circ R\backslash(R\circ g)\circ g_{>} \\
= & \{ \quad \text{lemma 77} \quad \} \\
& g^{\cup}\circ R\backslash R\circ g \quad .
\end{array}
$$

□

3.6 Isomorphic Relations

Several theorems we present "characterise" classes of relations in terms of functional relations. Typically these characterisations are not unique but unique "up to isomorphism". See, for example, section 5.2. The definition of "isomorphic" relations and some properties of the notion are given below.

Definition 82 Suppose R and S are two relations (not necessarily of the same type). Then we say that R and S are *isomorphic* and write $R \cong S$ iff

$$
\begin{array}{l}
\langle \exists\, \phi,\psi \\
\qquad : \quad \phi\circ\phi^{\cup} = R_{<} \;\wedge\; \phi^{\cup}\circ\phi = S_{<} \;\wedge\; \psi\circ\psi^{\cup} = R_{>} \;\wedge\; \psi^{\cup}\circ\psi = S_{>} \\
\qquad : \quad R = \phi\circ S\circ\psi^{\cup} \\
\rangle \quad .
\end{array}
$$

□

Lemma 83 The relation $\cong$ is reflexive, transitive and symmetric. That is, $\cong$ is an equivalence relation.

Proof This is very straightforward. For example, here is how symmetry is proved.

$$
\begin{array}{ll}
 & R = \phi \circ S \circ \psi^{\cup} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & \phi^{\cup} \circ R \circ \psi \;=\; \phi^{\cup} \circ \phi \circ S \circ \psi^{\cup} \circ \psi \\
= & \{ \quad \text{assume: } \phi^{\cup} \circ \phi = S{<} \text{ and } \psi^{\cup} \circ \psi = S{>} \text{, domains} \quad \} \\
 & \phi^{\cup} \circ R \circ \psi \;=\; S \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & \phi \circ \phi^{\cup} \circ R \circ \psi \circ \psi^{\cup} \;=\; \phi \circ S \circ \psi^{\cup} \\
= & \{ \quad \text{ssume: } \phi \circ \phi^{\cup} = R{<} \text{ and } \psi \circ \psi^{\cup} = R{>} \text{, domains} \quad \} \\
 & R = \phi \circ S \circ \psi^{\cup} \quad .
\end{array}
$$

That is, for all ϕ , ψ , R and S ,

$$
\begin{array}{ll}
 & (R = \phi \circ S \circ \psi^{\cup} \;\equiv\; \phi^{\cup} \circ R \circ \psi = S) \\
\Leftarrow & \phi \circ \phi^{\cup} = R{<} \;\wedge\; \phi^{\cup} \circ \phi = S{<} \;\wedge\; \psi \circ \psi^{\cup} = R{>} \;\wedge\; \psi^{\cup} \circ \psi = S{>} \quad .
\end{array}
$$

Symmetry of $\cong$ follows by definition of $\cong$, properties of converse, and Leibniz's rule.
□

The task of proving that two relations are isomorphic involves constructing ϕ and ψ that satisfy the conditions of the existential quantification in definition 82; we call the constructed values *witnesses* to the isomorphism.

Note that the requirement on ϕ in definition 82 is that it is both functional and injective; thus it is required to "witness" a (1–1) correspondence between the points in the left domain of R and the points in the left domain of S . Similarly, the requirement on ψ is that it "witnesses" a (1–1) correspondence between the points in the right domain of R and the points in the right domain of S . Formally, $R{<}$ and $S{<}$ are isomorphic as "witnessed" by ϕ and $R{>}$ and $S{>}$ are isomorphic as "witnessed" by ψ :

Lemma 84 Suppose R and S are relations such that $R \cong S$. Then $R{<} \cong S{<}$ and $R{>} \cong S{>}$.

Proof Suppose ϕ and ψ are such that

$$\phi\circ\phi^{\cup} = R{<} \;\wedge\; \phi^{\cup}\circ\phi = S{<} \;\wedge\; \psi\circ\psi^{\cup} = R{>} \;\wedge\; \psi^{\cup}\circ\psi = S{>} \quad .$$

Then

$$\begin{array}{ll}
& R{<} \\
= & \{ \quad R{<} \text{ is coreflexive} \quad \} \\
& R{<}\circ R{<} \\
= & \{ \quad \text{assumption} \quad \} \\
& \phi\circ\phi^{\cup}\circ\phi\circ\phi^{\cup} \\
= & \{ \quad \text{assumption} \quad \} \\
& \phi\circ S{<}\circ\phi^{\cup} \quad .
\end{array}$$

That is $R{<} = \phi\circ S{<}\circ\phi^{\cup}$. Similarly, $R{>} = \psi\circ S{>}\circ\psi^{\cup}$. But also (because the domain operators are closure operators),

$$\phi\circ\phi^{\cup} = (R{<}){<} \;\wedge\; \phi^{\cup}\circ\phi = (S{<}){<} \;\wedge\; \psi\circ\psi^{\cup} = (R{>}){>} \;\wedge\; \psi^{\cup}\circ\psi = (S{>}){>} \quad .$$

Applying definition 82 with $R,S,\phi,\psi := R{<}\,,S{<}\,,\phi\,,\phi$ and $R,S,\phi,\psi := R{>}\,,S{>}\,,\psi\,,\psi$, the lemma is proved.
□

3.7 Formulations of Power Transpose

Warning This section makes use of the notion of "symmetric division" as defined in [BdM97, Oli18] but *not* as defined in [Fv90]. "Symmetric division" can be defined in two non-equivalent ways which we call *symmetric left-division* and *symmetric right-division*. Given relations R of type $A{\sim}B$ and S of type $A{\sim}C$, the symmetric *right-division* is a relation of type $B{\sim}C$ defined in terms of *right* factors as

$$R\backslash S \;\cap\; (S\backslash R)^{\cup} \quad .$$

Dually, given relations R of type $B{\sim}A$ and S of type $C{\sim}A$, the symmetric *left-division* is a relation of type $B{\sim}C$ defined in terms of left factors as

$$R/S \;\cap\; (S/R)^{\cup} \quad .$$

Clearly, just from their types, neither the "symmetric" left-division nor the "symmetric" right-division is a symmetric relation. Possibly the justification for the use of the word "symmetric" is that, for homogeneous relation R, $R\cap R^{\cup}$ is a symmetric relation (indeed the largest symmetric relation included in R). Both [BdM97, Oli18] and [Fv90] use the notation $\frac{R}{S}$ (in the case of [BdM97, Oli18] to denote symmetric right-division and in the case of [Fv90] to denote symmetric left-division). The motivation for this is that the notation suggests a number of cancellation rules similar to the ones used in ordinary arithmetic. Great care must be taken, however, because —unlike in ordinary arithmetic— the cancellation rules are one-sided. For example, for symmetric right-division, we have the rule

$$R \;=\; R\circ\frac{R}{R}$$

but this is *not* valid if $\frac{R}{S}$ is defined to be symmetric left-division. Even worse, the expression $\frac{R}{R}\circ R$ does not even make sense (if $\frac{R}{S}$ is defined to be symmetric right-division) if R is a truly heterogeneous relation —with unequal source and target— purely on type grounds! For this reason, the notation $R\backslash\!\!\backslash S$ will be used here to denote the symmetric right-division. The reader should take great care when comparing formulae with those in [Fv90]. **End of Warning**

Given a relation R of type $A{\sim}B$, the *(left) power transpose* [Fv90, BdM97] of R is a total function, denoted in this paper[2] by $\ulcorner R$, of type $2^A{\leftarrow}B$. A pointwise definition of the (left) power transpose (using traditional set notation) is

$$\ulcorner R.b \;=\; \{a\,|\,a\;R\;b\} \;\;.$$

As discussed in section 3.3, there are three different but isomorphic mechanisms for representing sets in relation algebra: as coreflexives, (left or right) conditionals and squares. Using coreflexives, the power transpose $\ulcorner R$ of R is represented by the function

$$\langle b :: (R\circ b)_< \rangle \;\;.$$

It has type $\mathsf{Cor}.A \leftarrow B$ where $\mathsf{Cor}.A$ denotes the type of coreflexives of type $A{\sim}A$.

Rather than use coreflexives to define power transpose, Freyd and Ščedrov [Fv90] postulate a number of axioms that define $\ulcorner R$ in terms of set membership. Their approach is followed by Bird and De Moor [BdM97]. For our purposes, only two properties are needed. The first is that $\ulcorner R$ is a total function. That is, for all R, S and T of appropriate type,

$$(85) \qquad \ulcorner R\circ S \subseteq T \quad\equiv\quad S \subseteq (\ulcorner R)^\cup\circ T \;\;.$$

[2] Freyd and Ščedrov [Fv90] use the symbol “ Λ ” rather than “ $\ulcorner$ ”. In just the same way that we prefer the symbols “ $\backslash\!\!\backslash$ ” and “ $/\!\!/$ ” for asymmetric, but dual, operators, we prefer to use an asymmetric symbol for left power transpose, thus opening the possibility of using its mirror image for right power transpose.

This is the Galois connection (73) with $f := \Gamma R$ and specialised to the case that $f{>} = I$ (i.e. f is total); in line with our common policy when using well-known Galois connections, we refer to the rule as a "shunting rule". The second property of ΓR that we use is

(86) $$(\Gamma R)^{\cup} \circ \Gamma S \;\; = \;\; R\backslash S \cap (S\backslash R)^{\cup} \;\; .$$

From a calculational viewpoint, the two rules together enable reasoning about power transpose on the smaller and larger side of a set inclusion, respectively.

The property (86) can be derived from the definition of ΓR in our axiom system. Here is the proof.

Lemma 87 For all relations R and S,

$$(\Gamma R)^{\cup} \circ \Gamma S \;\; = \;\; R\backslash S \cap (S\backslash R)^{\cup} \;\; .$$

Proof We use indirect equality. For all relations X, R and S, we have

$$X \subseteq (\Gamma R)^{\cup} \circ \Gamma S$$

$= \quad \{ \quad$ saturation property: (19) $\quad \}$

$$\langle \forall a,b : a \circ \top\!\top \circ b \subseteq X : a \circ \top\!\top \circ b \subseteq (\Gamma R)^{\cup} \circ \Gamma S \rangle$$

$= \quad \{ \quad$ (69) with $f,g := \Gamma R, \Gamma S$ and definition of Γ $\quad \}$

$$\langle \forall a,b : a \circ \top\!\top \circ b \subseteq X : (R \circ a){<} = (S \circ b){<} \rangle$$

$= \quad \{ \quad$ corollary 61 $\quad \}$

$$\langle \forall a,b : a \circ \top\!\top \circ b \subseteq X : a \circ \top\!\top \circ b \subseteq R\backslash S \cap (S\backslash R)^{\cup} \rangle$$

$= \quad \{ \quad$ saturation property: (19) $\quad \}$

$$X \subseteq R\backslash S \cap (S\backslash R)^{\cup} \;\; .$$

Summarising, for all X, R and S,

$$X \subseteq (\Gamma R)^{\cup} \circ \Gamma S \;\; \equiv \;\; X \subseteq R\backslash S \cap (S\backslash R)^{\cup} \;\; .$$

That is, by indirect equality,

$$(\Gamma R)^{\cup} \circ \Gamma S \;\; = \;\; R\backslash S \cap (S\backslash R)^{\cup} \;\; .$$

□

Abbreviating the right side of lemma 87 to $R\backslash\!\!\backslash S$, viz.

(88) $$R\backslash\!\!\backslash S \;\; = \;\; R\backslash S \cap (S\backslash R)^{\cup} \;\; ,$$

the lemma becomes, for all R and S,

(89) $$(\Gamma R)^{\cup} \circ \Gamma S \;\; = \;\; R\backslash\!\!\backslash S \;\; .$$

We use both forms of the lemma below.

3.8 Pers and Per Domains

The relation $R\backslash\!\backslash R$ is an equivalence relation[3]. Voermans [Voe99] calls it the "greatest right domain" of R. Riguet [Rig48] calls $R\backslash\!\backslash R$ the "noyau" of R (but defines it using nested complements). Others (see [Oli18] for references) call it the "kernel" of R.

As remarked elsewhere [Oli18], the *symmetric left division* inherits a number of (left) cancellation properties from the properties of factorisation in terms of which it is defined. For our purposes, the only cancellation property we use is the following (inherited from the property $R\circ R\backslash R = R$).

Lemma 90 For all R,

$$R\circ R\backslash\!\backslash R = R \enspace .$$

Proof By mutual inclusion:

$$\begin{array}{ll}
 & R\circ R\backslash\!\backslash R \\
= & \{ \quad \text{definition: (88) with } R{,}S := R{,}R \quad \} \\
 & R\circ(R\backslash R \cap (R\backslash R)^{\cup}) \\
\subseteq & \{ \quad \text{monotonicity} \quad \} \\
 & R\circ R\backslash R \\
= & \{ \quad \text{cancellation } [\, R\circ R\backslash S \subseteq S \,] \text{ (with } R{,}S := R{,}R\text{) and } [\, I\subseteq R\backslash R \,] \quad \} \\
 & R \\
\subseteq & \{ \quad [\, I\subseteq S\backslash\!\backslash S \,] \text{ with } S := R \quad \} \\
 & R\circ R\backslash\!\backslash R \enspace .
\end{array}$$

□

Voermans [Voe99] emphasises the importance of the relation $R{>}\circ R\backslash\!\backslash R$, which is a partial equivalence relation that better reflects the right (per-)domain of R. (In accordance with his thesis, "domains" are pers rather than coreflexives.) Unlike Riguet and others, Voermans gives equal importance to the dual equivalence relation $R/\!/R$ and the left (per-)domain $R/\!/R\circ R{<}$. The combination of the two per-domains enables the definition of what we call the "core" of a relation. The "core" of a relation is important to understanding the nature of difunctional relations and block-ordered relations. See theorems 205 and 207 in section 7.3. See also section 12 for further discussion.

[3] This is a well-known fact: the relation $R\backslash\!\backslash R$ is the symmetric closure of the preorder $R\backslash R$. The easy proof is left to the reader.

Definition 91 (Partial Equivalence Relation (per)) A relation is a *partial equivalence relation* iff it is symmetric and transitive. That is, R is a partial equivalence relation iff

$$R = R^{\cup} \quad \wedge \quad R{\circ}R \subseteq R \quad .$$

Henceforth we abbreviate partial equivalence relation to *per*.
□

An equivalence relation is a reflexive, symmetric and transitive relation. Reflexivity means that the left domain, the right domain, the source and the target of the relation are all the same. A *partial* equivalence relation is not necessarily reflexive;, the absence of the reflexivity property is, however, of no consequence. Its rôle is taken by the following lemma.

Lemma 92 Suppose R is a per. Then

$$R{<} \;=\; R{>} \;\subseteq\; R \quad .$$

Proof The equality $R{<} = R{>}$ is immediate from the definition of the domain operators and the fact that a per is symmetric. Also,

$$
\begin{array}{ll}
 & R{>} \;\subseteq\; R \\
\Leftarrow & \{ \qquad R{>} \;=\; I \cap R^{\cup} \circ R \text{, transitivity of subset relation} \quad \} \\
 & R^{\cup} \circ R \;\subseteq\; R \\
= & \{ \qquad \text{assumption: } R \text{ is a per, definition 91 and Leibniz} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

□

Because the left and right domain of a per are equal, we refer to its *domain*, omitting the adjective left or right.

Definition 91 is the standard definition of a partial equivalence relation. A better definition —because it is just one equation— is expressed by the following theorem.

Theorem 93 For all relations R, R is a per equivales $R = R \circ R^{\cup}$. Symmetrically, for all relations R, R is a per equivales $R = R^{\cup} \circ R$.

Proof By mutual implication. First, suppose R is a per. Then

$$
\begin{array}{ll}
 & R{\circ}R \\
\subseteq & \{ \qquad \text{assumption: } R \text{ is transitive} \quad \} \\
 & R
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{domains} \quad \} \\
& R \circ R_{>} \\
\subseteq & \{ \quad \text{assumption: } R \text{ is a per, lemma 92} \quad \} \\
& R \circ R \quad .
\end{array}
$$

That is, by the anti-symmetry of the subset relation, $R = R \circ R$. But R is symmetric. That is, $R = R^{\cup}$. So, by Leibniz's rule, $R = R \circ R^{\cup}$.

For the follows-from, we have:

$$
\begin{array}{ll}
& R = R \circ R^{\cup} \\
= & \{ \quad (R \circ R^{\cup})^{\cup} = R \circ R^{\cup} \quad \} \\
& R = R \circ R^{\cup} = R^{\cup} \\
\Rightarrow & \{ \quad \text{subset relation is reflexive, Leibniz} \quad \} \\
& R \circ R \subseteq R \ \wedge \ R = R^{\cup} \\
= & \{ \quad \text{definition} \quad \} \\
& \text{per}.R \quad .
\end{array}
$$

□

The following lemma is a straightforward consequence of theorem 93.

Lemma 94 Suppose f is a functional relation. Then $f^{\cup} \circ f$ is a per.

□

Pers are studied in more detail in section 5. In this section the focus is on the left and right "per-domains" introduced by Voermans [Voe99].

Definition 95 (Right and Left Per Domains) The *right per-domain* of relation R, denoted $R_{\succ}$, is defined by the equation

(96) $\quad R_{\succ} \ = \ R_{>} \circ R \backslash\!\backslash R \quad .$

Dually, the *left per-domain* of relation R, denoted $R_{\prec}$, is defined by the equation

(97) $\quad R_{\prec} \ = \ R /\!/ R \circ R_{<} \quad .$

□

Although the theorems below focus on the properties of $R_{\succ}$, each can, of course, be dualised to properties of $R_{\prec}$.

The left and right per-domains are called "domains" because, like the coreflexive domains, we have the properties:

(98) $\quad R_{\prec} \circ R \ = \ R \ = \ R \circ R_{\succ} \quad .$

(The second of these equalities is an immediate consequence of lemma 90 and the properties of (coreflexive-) domains; the first is symmetric.) Indeed, $R_{\prec}$ and $R_{\succ}$ are the "least" pers that satisfy these equalities. (See [Voe99] for details of the ordering relation on pers.)

That $R_{\prec}$ and $R_{\succ}$ are indeed pers is a direct consequence of the symmetry and transitivity of $R\backslash\!\backslash R$. For example, the transitivity of $R_{\succ}$ is inherited from the transitivity of $R\backslash\!\backslash R$:

$$
\begin{array}{ll}
 & R_{\succ} \circ R_{\succ} \\
= & \{ \quad (96) \text{ and } (100) \quad \} \\
 & R_{>} \circ R\backslash\!\backslash R \circ R\backslash\!\backslash R \circ R_{>} \\
\subseteq & \{ \quad R\backslash\!\backslash R \text{ is transitive} \quad \} \\
 & R_{>} \circ R\backslash\!\backslash R \circ R_{>} \\
= & \{ \quad \text{lemma 99 and (96)} \quad \} \\
 & R_{\succ} \quad .
\end{array}
$$

The symmetry of $R_{\succ}$ (i.e. $R_{\succ} = (R_{\succ})^{\cup}$) is a similar combination of (96), (100) and the symmetry of $R\backslash\!\backslash R$. Thus $R_{\succ}$ is a per. Dually $R_{\prec}$ is also a per.

In order to prove additional properties, it is useful to record the left and right domains of the relation $R\backslash\!\backslash R \circ R_{>}$:

Lemma 99 For all R,

$$(R\backslash\!\backslash R \circ R_{>})_{>} \;=\; R_{>} \;=\; (R_{>} \circ R\backslash\!\backslash R)_{<} \quad ,$$

$$(R\backslash\!\backslash R \circ R_{>})_{<} \;=\; R_{>} \;=\; (R_{>} \circ R\backslash\!\backslash R)_{>} \quad ,$$

$$R\backslash\!\backslash R \circ R_{>} \;=\; R_{>} \circ R\backslash\!\backslash R \circ R_{>} \;=\; R_{>} \circ R\backslash\!\backslash R \quad .$$

Proof The first two equations follow from the fact that

$$(R\backslash\!\backslash R)_{<} \;=\; I \;=\; (R\backslash\!\backslash R)_{>}$$

(because $I \subseteq R\backslash R$ and $R\backslash\!\backslash R$ is the symmetric closure of $R\backslash R$). For example:

$$
\begin{array}{ll}
 & (R_{>} \circ R\backslash\!\backslash R)_{<} \\
= & \{ \quad \text{domains} \quad \} \\
 & (R_{>} \circ (R\backslash\!\backslash R)_{<})_{<} \\
= & \{ \quad (R\backslash\!\backslash R)_{<} = I \quad \}
\end{array}
$$

$$\begin{array}{ll} & (R_{>})_{<} \\ = & \{ \quad R_{>} \text{ is a coreflexive, domains} \quad \} \\ & R_{>} \ . \end{array}$$

The second two equations follow from lemma 90.

$$\begin{array}{ll} & (R\backslash\!\backslash R \circ R_{>})_{<} \\ = & \{ \quad \text{domains} \quad \} \\ & (R \circ (R\backslash\!\backslash R)^{\cup})_{>} \\ = & \{ \quad R\backslash\!\backslash R \text{ is symmetric} \quad \} \\ & (R \circ R\backslash\!\backslash R)_{>} \\ = & \{ \quad \text{lemma 90} \quad \} \\ & R_{>} \ , \end{array}$$

and

$$\begin{array}{ll} & (R_{>} \circ R\backslash\!\backslash R)_{>} \\ = & \{ \quad \text{domains} \quad \} \\ & (R \circ R\backslash\!\backslash R)_{>} \\ = & \{ \quad \text{lemma 90} \quad \} \\ & R_{>} \ . \end{array}$$

Combining the domain equations, we have

$$\begin{array}{ll} & R\backslash\!\backslash R \circ R_{>} \\ = & \{ \quad (R_{>} \circ R\backslash\!\backslash R)_{<} \ = \ R_{>} \text{, domains} \quad \} \\ & R_{>} \circ R\backslash\!\backslash R \circ R_{>} \\ = & \{ \quad (R_{>} \circ R\backslash\!\backslash R)_{>} \ = \ R_{>} \text{, domains} \quad \} \\ & R_{>} \circ R\backslash\!\backslash R \ . \end{array}$$

□

Lemma 99 has the consequence that $R_{\succ}$ can be defined equivalently by the equation

(100) $\quad R_{\succ} \ = \ R\backslash\!\backslash R \circ R_{>}$

and, moreover,

(101) $\quad (R_{\succ})_{<} \ = \ R_{>} \ = \ (R_{\succ})_{>} \ .$

A property that we need later is

Lemma 102 For all relations R,

$$R\backslash R\circ R_{\succ} \;=\; R\backslash R\circ R_{>} \;\; .$$

Proof By anti-symmetry of the subset relation:

$$
\begin{array}{cl}
 & R\backslash R\circ R_{\succ} \\
\subseteq & \{ \quad \text{by (88), (100) and monotonicity, } R_{\succ} \subseteq R\backslash R\circ R_{>} \quad \} \\
 & R\backslash R\circ R\backslash R\circ R_{>} \\
\subseteq & \{ \quad \text{by cancellation, } R\backslash R\circ R\backslash R \subseteq R\backslash R \quad \} \\
 & R\backslash R\circ R_{>} \\
\subseteq & \{ \quad I\subseteq R\backslash\!\!\backslash R\text{, so by (100) and montonicity, } R_{>}\subseteq R_{\succ} \quad \} \\
 & R\backslash R\circ R_{\succ} \;\; .
\end{array}
$$

□

The pointwise interpretations of the left and right per domains are given by the following lemma.

Lemma 103 For all relations R of type $A{\sim}B$ and all points a and a' of type A,

$$a\circ\top\!\!\top\circ a' \subseteq R_{\prec} \quad\equiv\quad a\subseteq R_{<} \;\;\wedge\;\; (a\circ R)_{>} = (a'\circ R)_{>} \;\;\wedge\;\; a'\subseteq R_{<} \;\; .$$

Dually, for all relations R of type $A{\sim}B$ and all points b and b' of type B,

$$b\circ\top\!\!\top\circ b' \subseteq R_{\succ} \quad\equiv\quad b\subseteq R_{>} \;\;\wedge\;\; (R\circ b)_{<} = (R\circ b')_{<} \;\;\wedge\;\; b'\subseteq R_{>} \;\; .$$

Proof Assume that b and b' are points. Then

$$
\begin{array}{cl}
 & b\circ\top\!\!\top\circ b' \subseteq R_{\succ} \\
= & \{ \quad \text{definition (96) and lemma 99} \quad \} \\
 & b\circ\top\!\!\top\circ b' \subseteq R_{>}\circ R\backslash\!\!\backslash R\circ R_{>} \\
= & \{ \quad \text{domains (using mutual implication)} \quad \} \\
 & b\subseteq R_{>} \;\;\wedge\;\; b\circ\top\!\!\top\circ b' \subseteq R\backslash\!\!\backslash R \;\;\wedge\;\; b'\subseteq R_{>} \\
= & \{ \quad \text{corollary 61, with } R,S := R,R \quad \} \\
 & b\subseteq R_{>} \;\;\wedge\;\; (R\circ b)_{<} = (R\circ b')_{<} \;\;\wedge\;\; b'\subseteq R_{>} \;\; .
\end{array}
$$

The dual property follows from the distributivity properties of converse.

□

Given relation R, the relation $R^{\cup}\circ R$ is symmetric but not necessarily transitive. However, it is an upper bound on the right per domain of R. That is,

(104) $\quad R^{\cup}\circ R \supseteq R_{\succ} \;\; .$

The proof is as follows:

$$
\begin{array}{cl}
 & R^{\cup}\circ R \supseteq R\succ \\
= & \{ \quad \text{definition: (96)} \quad \} \\
 & R^{\cup}\circ R \supseteq R{>}\circ R\backslash\!\backslash R \\
= & \{ \quad \text{cancellation: (90)} \quad \} \\
 & R^{\cup}\circ R\circ R\backslash\!\backslash R \supseteq R{>}\circ R\backslash\!\backslash R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R^{\cup}\circ R \supseteq R{>} \\
\Leftarrow & \{ \quad \text{definition 42} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

Dually, of course, we have:

(105) $R\circ R^{\cup} \supseteq R\prec$.

It is useful to investigate the circumstances in which the inclusions in (104) and (105) become equalities.

Lemma 106 For all relations R,

$$(R\prec \ = \ R\circ R^{\cup}) \ = \ (R \ = \ R\circ R^{\cup}\circ R) \ = \ (R^{\cup}\circ R = R\succ) \quad .$$

(As usual, we overload the equality symbol: its usage here alternates between equality of relations and equality of booleans. As always, continued equalities should be read conjunctionally.)

Proof We have:

$$
\begin{array}{cl}
 & R^{\cup}\circ R = R\succ \\
= & \{ \quad \text{(104) and anti-symmetry} \quad \} \\
 & R^{\cup}\circ R \subseteq R\succ \\
= & \{ \quad \text{definition: (96)} \quad \} \\
 & R^{\cup}\circ R \subseteq R{>}\circ R\backslash\!\backslash R \\
\Leftarrow & \{ \quad R{>}\circ R^{\cup} = R^{\cup} \text{ and monotonicity} \quad \} \\
 & R^{\cup}\circ R \subseteq R\backslash\!\backslash R \\
= & \{ \quad R^{\cup}\circ R \text{ is symmetric, } R\backslash\!\backslash R = R\backslash R\cap(R\backslash R)^{\cup} \quad \} \\
 & R^{\cup}\circ R \subseteq R\backslash R
\end{array}
$$

$$\Leftarrow \quad \{ \quad \text{factors} \quad \}$$

$$R \circ R^{\cup} \circ R \subseteq R$$

$$\Leftarrow \quad \{ \quad (98) \quad \}$$

$$R^{\cup} \circ R = R{\succ} \ .$$

We have thus proved (by mutual implication), that

$$(R \circ R^{\cup} \circ R \subseteq R) \ = \ (R^{\cup} \circ R = R{\succ}) \ .$$

But,

$$R \circ R^{\cup} \circ R \subseteq R$$

$$= \quad \{ \quad (54) \quad \}$$

$$R \circ R^{\cup} \circ R \subseteq R \ \wedge \ R \subseteq R \circ R^{\cup} \circ R$$

$$= \quad \{ \quad \text{anti-symmetry} \quad \}$$

$$R = R \circ R^{\cup} \circ R$$

Combining the two calculations (using the transitivity of boolean equality),

$$(R = R \circ R^{\cup} \circ R) \ = \ (R^{\cup} \circ R = R{\succ}) \ .$$

The dual property,

$$(R{\prec} = R \circ R^{\cup}) \ = \ (R = R \circ R^{\cup} \circ R)$$

follows by symmetry.
□

Two special cases of lemma 106:

Lemma 107 For all functional relations f (that is, for all f such that $f \circ f^{\cup} = f{<}$),

$$f{\succ} = f^{\cup} \circ f \ .$$

Proof

$$f{\succ} = f^{\cup} \circ f$$

$$= \quad \{ \quad \text{lemma 106 with } R{:=}f \quad \}$$

$$f = f \circ f^{\cup} \circ f$$

$$= \quad \{ \quad \text{assumption: } f \text{ is functional, i.e. } f \circ f^{\cup} = f{<} \quad \}$$

$$f = f{<} \circ f$$

$$= \quad \{ \quad \text{domains} \quad \}$$

$$\text{true} \ .$$

□

The following lemma extends [Rig48, Corollaire, p.134] from equivalence relations to pers.

Lemma 108 For all relations R, the following statements are all equivalent.

(i) R is a per (i.e. $R = R^{\cup} \wedge R \circ R \subseteq R$) ,

(ii) $R = R^{\cup} \circ R$,

(iii) $R = R{\prec}$,

(iv) $R = R{\succ}$.

Proof The equivalence of (i) and (ii) was shown in theorem 93. It remains to prove the equivalence of (ii) and (iii); the equivalence of (ii) and (iv) is the dual proposition.

$$R = R{\succ}$$

$$= \quad \{ \quad [\, R = R \circ R{\succ} \,] \text{ and } [\, R{\succ} = (R{\succ})^{\cup} \,] \quad \}$$

$$R = R \circ R^{\cup} = R{\succ}$$

$$= \quad \{ \quad R = R \circ R^{\cup} \;\equiv\; R = R \circ R^{\cup} = R \circ R^{\cup} \circ R$$
$$\text{(by Leibniz and predicate calculus)} \quad \}$$

$$R = R \circ R^{\cup} = R \circ R^{\cup} \circ R = R{\succ}$$

$$= \quad \{ \quad \text{lemma 106} \quad \}$$

$$R = R \circ R^{\cup} = R \circ R^{\cup} \circ R$$

$$= \quad \{ \quad \text{see above} \quad \}$$

$$R = R \circ R^{\cup} \ .$$

□

Coreflexives are, of course, pers. This implies that they are closed under the left and right per-domain operators:

Lemma 109 For all coreflexives p,

$$p{\prec} \;=\; p \;=\; p{\succ} \ .$$

Proof The lemma follows from lemma 108 since, for all coreflexives p,

$$p \;=\; p^{\cup} \;=\; p \circ p \ .$$

□

The dual of lemma 107 is that, for all injective relations f (that is, for all f such that $f^{\cup}\circ f \;=\; f_{>}$),

$$f_{\prec} \;=\; f\circ f^{\cup} \quad .$$

Noting that f is injective equivales $f^{\cup}$ is functional, we seek a convenient way of combining the two properties. Such is the following.

Lemma 110 Suppose that f and g are functional relations and R is an arbitrary relation such that

$$f\circ f^{\cup} \;=\; f_{<} \;=\; R_{<} \quad\wedge\quad g\circ g^{\cup} \;=\; g_{<} \;=\; R_{>} \quad .$$

Then

$$(f^{\cup}\circ R\circ g)_{\prec} \;=\; f\circ R_{\prec}\circ f^{\cup} \quad\wedge\quad (f^{\cup}\circ R\circ g)_{\succ} \;=\; g^{\cup}\circ R_{\succ}\circ g$$

Proof First note that

$$
\begin{array}{ll}
 & ((f^{\cup}\circ R\circ g)_{\succ})_{<} \\
= & \{\qquad (101) \quad \} \\
 & (f^{\cup}\circ R\circ g)_{>} \\
= & \{\qquad \text{assumption: } f_{<} = R_{<}\text{, domains} \quad \} \\
 & (R\circ g)_{>} \\
= & \{\qquad \text{assumption: } g_{<} \;=\; R_{>}\text{, domains} \quad \} \\
 & g_{>} \quad .
\end{array}
$$

That is,

$$(111)\qquad ((f^{\cup}\circ R\circ g)_{\succ})_{<} \;=\; g_{>} \quad .$$

Now,

$$
\begin{array}{ll}
 & g_{>}\circ(f^{\cup}\circ R\circ g)\backslash(f^{\cup}\circ R\circ g)\circ g_{>} \\
= & \{\qquad \text{lemma 79 with } S := R\circ g \quad \} \\
 & g^{\cup}\circ R\backslash(R\circ g)\circ g_{>} \\
= & \{\qquad \text{lemma 77 with } R,S,f := R,R,g \quad \} \\
 & g^{\cup}\circ R\backslash R\circ g \quad .
\end{array}
$$

That is

$$(112) \quad g_{>} \circ (f^{\cup} \circ R \circ g)\backslash(f^{\cup} \circ R \circ g) \circ g_{>} \;=\; g^{\cup} \circ R\backslash R \circ g \;\; .$$

Thus

$$\begin{array}{ll}
 & g_{>} \circ (f^{\cup} \circ R \circ g)_{\succ} \circ g_{>} \\
= & \{ \quad \text{definition 95, lemma 99 and (111)} \quad \} \\
 & g_{>} \circ ((f^{\cup} \circ R \circ g)\backslash(f^{\cup} \circ R \circ g) \,\cap\, ((f^{\cup} \circ R \circ g)\backslash(f^{\cup} \circ R \circ g))^{\cup}) \circ g_{>} \\
= & \{ \quad \text{distributivity of coreflexives over intersection} \quad \} \\
 & g_{>} \circ (f^{\cup} \circ R \circ g)\backslash(f^{\cup} \circ R \circ g) \circ g_{>} \;\cap\; (g_{>} \circ (f^{\cup} \circ R \circ g)\backslash(f^{\cup} \circ R \circ g) \circ g_{>})^{\cup} \\
= & \{ \quad (112) \quad \} \\
 & g^{\cup} \circ R\backslash R \circ g \;\;\cap\;\; (g^{\cup} \circ R\backslash R \circ g)^{\cup} \\
= & \{ \quad \text{distributivity (} g \text{ is functional)} \quad \} \\
 & g^{\cup} \circ (R\backslash R \cap (R\backslash R)^{\cup}) \circ g \\
= & \{ \quad \text{definition 95} \quad \} \\
 & g^{\cup} \circ R_{\succ} \circ g \;\; .
\end{array}$$

□

Lemma 107 is an instance of lemma 110 (obtained by instantiating both R and g to $f_{<}$ and using lemma 109 to eliminate $R_{\prec}$). Similarly, the dual of lemma 107 is also an instance. Another instance is:

Lemma 113 For all relations f and g such that

$$f \circ f^{\cup} \;=\; f_{<} \;=\; g \circ g^{\cup} \;=\; g_{<}$$

we have

$$(f^{\cup} \circ g)_{\succ} \;=\; g_{\succ} \quad \wedge \quad (f^{\cup} \circ g)_{\prec} \;=\; f_{\succ} \;\; .$$

Proof

$$\begin{array}{ll}
 & (f^{\cup} \circ g)_{\succ} \\
= & \{ \quad \text{heading for lemma 110, domains} \quad \} \\
 & (f^{\cup} \circ g_{<} \circ g)_{\succ} \\
= & \{ \quad \text{domains and lemma 110 with } R,f,g := g_{<} \,,\, f \,,\, g \quad \} \\
 & g^{\cup} \circ (g_{<})_{\succ} \circ g
\end{array}$$

$$
\begin{array}{ll}
= & \{ \quad \text{lemma 109 and domains} \quad \} \\
& g^{\cup} \circ g \\
= & \{ \quad \text{lemma 107 with } f{:=}g \quad \} \\
& g_{\succ} \ .
\end{array}
$$

The second equality is now straightforward:

$$
\begin{array}{ll}
& (f^{\cup} \circ g)_{\prec} \\
= & \{ \quad \text{converse} \quad \} \\
& ((g^{\cup} \circ f)^{\cup})_{\prec} \\
= & \{ \quad \text{definitions : (100) and (97)} \quad \} \\
& (g^{\cup} \circ f)_{\succ} \\
= & \{ \quad [\ (f^{\cup} \circ g)_{\succ} = g_{\succ} \Leftarrow f \circ f^{\cup} = f_{<} = g \circ g^{\cup} = g_{<}\ ] \\
& \quad\ \text{(just proved) with } f,g{:=}g,f \quad \} \\
& f_{\succ} \ .
\end{array}
$$

□

3.9 Provisional Orderings

There are various well-known notions of ordering: preorder, partial and linear (aka total) ordering. For our purposes all of these are too strict. So, in this section, we introduce the notion of a "provisional ordering". The adjective "provisional" has been chosen because the notion "provides" just what we need.

The standard definition of an ordering is an anti-symmetric preorder whereby a preorder is required to be reflexive and transitive. It is the reflexivity requirement that is too strict for our purposes. So, with the intention of weakening the standard definition of a preorder to requiring reflexivity of a relation over some superset of its left and right domains, we propose the following definition.

Definition 114 Suppose T is a homogeneous relation. Then T is said to be a *provisional preorder* if

$$
T_{<} \subseteq T \ \wedge \ T_{>} \subseteq T \ \wedge \ T{\circ}T \subseteq T \ .
$$

□

Fig. 4 depicts a provisional preorder on a set of eight elements as a directed graph. The blue squares should be ignored for the moment. (See the discussion following lemma

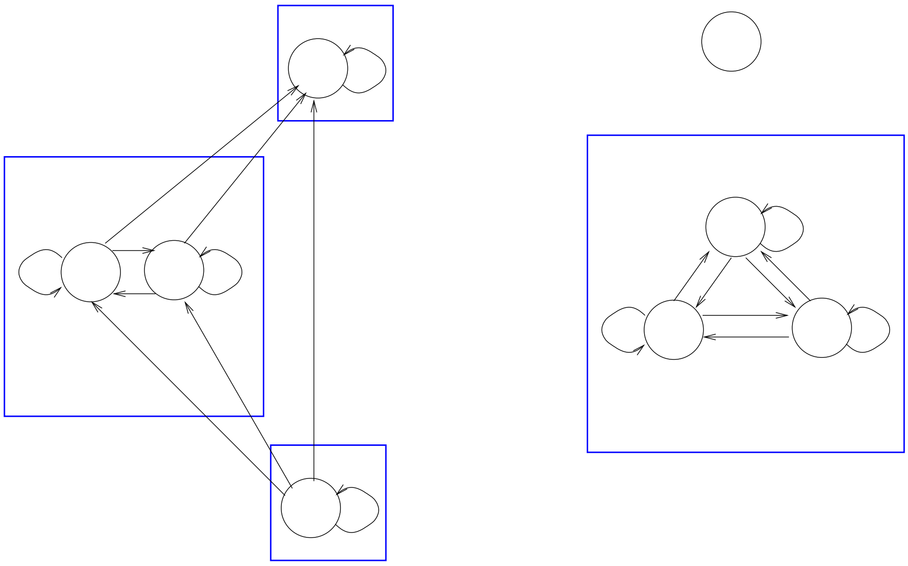


Figure 4: A Provisional Preorder

120.) Note that the relation depicted is not a preorder because it is not reflexive: the top-right node depicts an element that is not in the left or right domain of the relation.

An immediate consequence of the definition is that the left and right domains of a provisional preorder must be equal:

Lemma 115 If T is a provisional preorder then

$$T_{<} = T_{>} \quad .$$

Proof Suppose T is a provisional preorder. Then

$$
\begin{array}{ll}
 & T_{>} \subseteq T_{<} \\
= & \{ \quad \text{domains} \quad \} \\
 & (T_{>})_{<} \subseteq T_{<} \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & T_{>} \subseteq T \\
= & \{ \quad \text{assumption: } T_{>} \subseteq T \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

That is, $T_{>} \subseteq T_{<}$. Dually, $T_{<} \subseteq T_{>}$. Thus, by anti-symmetry, $T_{<} = T_{>}$.
□

A trivial property that is nevertheless used frequently:

Lemma 116 T is a provisional preorder equivales $T^{\cup}$ is a provisional preorder.

Proof Immediate from the definition and properties of converse.
□

A preorder is a provisional preorder with left (equally right) domain equal to the identity relation. In other words, a preorder is a total provisional preorder. It is easy to show that, for any relation R, the relations $R\backslash R$ and R/R are preorders. It is also easy to show that R is a preorder if and only if $R = R\backslash R$ (or equivalently if and only if $R = R/R$). These properties generalise to provisional preorders.

Lemma 117 For all relations R, the relations $R_{>} \circ R\backslash R$ and $R/R \circ R_{<}$ are provisional preorders.

Proof The proof is very straightforward. First,

$$
\begin{array}{ll}
 & (R_{>} \circ R\backslash R)_{<} \\
= & \{ \quad I \subseteq R\backslash R\text{, so } (R\backslash R)_{<} = I\text{; domains} \quad \} \\
 & (R_{>})_{<} \\
= & \{ \quad R_{>} \text{ is a coreflexive} \quad \} \\
 & R_{>} \\
\subseteq & \{ \quad I \subseteq R\backslash R\text{, monotonicity} \quad \} \\
 & R_{>} \circ R\backslash R \quad .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & (R_{>} \circ R\backslash R)_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & (R \circ R\backslash R)_{>} \\
= & \{ \quad \text{cancellation} \quad \} \\
 & R_{>} \\
\subseteq & \{ \quad I \subseteq R\backslash R\text{, monotonicity} \quad \} \\
 & R_{>} \circ R\backslash R \quad .
\end{array}
$$

Third,

$$
\begin{array}{ll}
 & R_{>} \circ R\backslash R \circ R_{>} \circ R\backslash R \\
\subseteq & \{ \quad R_{>} \subseteq I\text{, monotonicity} \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & R{>} \circ R\backslash R \circ R\backslash R \\
\subseteq & \{ \quad R\backslash R \circ R\backslash R \subseteq R\backslash R \\
 & \qquad \text{(easy use of definition of factors and cancellation)} \quad \} \\
 & R{>} \circ R\backslash R \quad .
\end{array}
$$

Comparing the above properties with definition 114, we have shown that $R{>} \circ R\backslash R$ is a provisional preorder. The dual property, $R/R \circ R{<}$ is a provisional preorder, is obtained by the instantiation $R := R^{\cup}$ and application of distributivity properties of converse.
□

Lemma 118 T is a provisional preorder equivales

$$T \;=\; T{<} \circ T\backslash T \;=\; T/T \circ T{>} \;=\; T{<} \circ T\backslash T/T \circ T{>} \quad .$$

Proof Follows-from is a straightforward consequence of the fact that $T\backslash T$ is a preorder for arbitrary T.

Implication is also straightforward. Assume that T is a provisional preorder. The proof of the leftmost equality is by mutual inclusion. First

$$
\begin{array}{ll}
 & T \subseteq T{<} \circ T\backslash T \\
\Leftarrow & \{ \quad T = T{<} \circ T \text{ and monotonicity} \quad \} \\
 & T \subseteq T\backslash T \\
= & \{ \quad \text{factors} \quad \} \\
 & T \circ T \subseteq T \\
= & \{ \quad \text{assumption: } T \text{ is transitive} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

For the opposite inclusion we have

$$
\begin{array}{ll}
 & T{<} \circ T\backslash T \subseteq T \\
\Leftarrow & \{ \quad \text{assumption: } T{<} \subseteq T \text{, monotonicity} \quad \} \\
 & T \circ T\backslash T \subseteq T \\
= & \{ \quad \text{cancellation} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

Thus $T = T{<} \circ T\backslash T$ by anti-symmetry. That $T = T/T \circ T{>}$ follows from lemma 116 and the properties of converse. Finally,

$$T$$

$$= \quad \{ \quad T = T\circ T_{>} \text{ and } T = T_{<}\circ T\backslash T \text{ (proved above)} \quad \}$$

$$T_{<}\circ T\backslash T\circ T_{>}$$

$$= \quad \{ \quad T = T/T\circ T_{>} \text{ (proved above)} \quad \}$$

$$T_{<}\circ T\backslash (T/T\circ T_{>})\circ T_{>}$$

$$= \quad \{ \quad [\ R\backslash (S\circ R_{>})\circ R_{>} = R\backslash S\circ R_{>}\ ] \text{ with } R,S := T,T \quad \}$$

$$T_{<}\circ T\backslash T/T\circ T_{>} \quad .$$

□

Lemma 118 is sometimes used in a form where the domains are replaced by per domains.

Lemma 119 Suppose T is a provisional preorder. Then

$$T \;=\; T_{\prec}\circ T\backslash T \;=\; T/T\circ T_{\succ} \;=\; T_{\prec}\circ T\backslash T/T\circ T_{\succ} \quad .$$

Proof Immediate from lemma 118 and the per domain equations, for all R,

$$R \;=\; R_{\prec}\circ R \;=\; R_{\prec}\circ R_{<}\circ R \;=\; R\circ R_{\succ} \;=\; R\circ R_{>}\circ R_{\succ} \quad .$$

For example,

$$T$$

$$= \quad \{ \quad [\ R \;=\; R_{\prec}\circ R\ ] \text{ with } R := T \quad \}$$

$$T_{\prec}\circ T$$

$$= \quad \{ \quad \text{lemma 118} \quad \}$$

$$T_{\prec}\circ T_{<}\circ T\backslash T$$

$$= \quad \{ \quad [\ R_{\prec}\circ R_{<} \;=\; R_{\prec}\ ] \text{ with } R := T \quad \}$$

$$T_{\prec}\circ T\backslash T \quad .$$

□

Lemma 120 Suppose T is a provisional preorder. Then

$$T_{\prec} \;=\; T\cap T^{\cup} \;=\; T_{\succ} \quad .$$

Hence $T\cap T^{\cup}$ is a per.

Proof We exploit lemma 118:

$$
\begin{array}{ll}
 & T_{\succ} \\
= & \{ \quad \text{definition: (96) and (88), lemma 99} \quad \} \\
 & T_{>} \circ (T\backslash T \cap (T\backslash T)^{\cup}) \circ T_{>} \\
= & \{ \quad \text{distributivity } (T_{>} \text{ is coreflexive}) \quad \} \\
 & T_{>} \circ T\backslash T \circ T_{>} \;\; \cap \;\; (T^{\cup})_{<} \circ T^{\cup} / T^{\cup} \circ (T^{\cup})_{<} \\
= & \{ \quad \text{lemma 115} \\
 & \qquad \text{(twice, once with } T{:=}T^{\cup} \text{ using lemma 116)} \quad \} \\
 & T_{<} \circ T\backslash T \circ T_{>} \;\; \cap \;\; (T^{\cup})_{<} \circ T^{\cup} / T^{\cup} \circ (T^{\cup})_{>} \\
= & \{ \quad \text{lemma 118} \quad \} \\
 & T \circ T_{>} \; \cap \; (T^{\cup})_{<} \circ T^{\cup} \\
= & \{ \quad \text{domains} \quad \} \\
 & T \cap T^{\cup} \quad .
\end{array}
$$

The dual property $T_{\prec} = T \cap T^{\cup}$ is immediate from the properties of converse.

□

Referring back to fig. 4, the blue squares depict the equivalence classes of the symmetric closure of a provisional preorder. As remarked earlier, the depicted relation is not a preorder; correspondingly, the blue squares depict a truly *partial* equivalence relation.

We assume the reader is familiar with the notions of an ordering and a linear (or total) ordering. We now extend these notions to provisional orderings. (The at-most relation on the integers is both anti-symmetric and linear. The at-most relation restricted to some arbitrary subset of the integers is an example of a linear provisional ordering according to the definition below.)

Definition 121 Suppose T is a homogeneous relation of type $A{\sim}A$, for some A. Then T is said to be *provisionally anti-symmetric* if

$$T \cap T^{\cup} \subseteq I_A \quad .$$

Also, T is said to be a *provisional ordering* if T is provisionally anti-symmetric and T is a provisional preorder. Finally, T is said to be a *linear provisional ordering* if T is a provisional ordering and

$$T \cup T^{\cup} \;=\; (T \cap T^{\cup}) \circ \top\!\!\top \circ (T \cap T^{\cup}) \quad .$$

□

Definition 121 weakens the equality in the standard notion of anti-symmetry to an inclusion. The standard definition of a partial ordering —an anti-symmetric preorder— is weakened accordingly (as mentioned earlier, in order to "provide" for our needs).

The following lemma anticipates the use of provisional preorders/orderings in examples presented later.

Lemma 122 Suppose T is a provisional ordering. Then

$$\mathsf{T}_< \;=\; \mathsf{T} \cap \mathsf{T}^\cup \;=\; \mathsf{T}_> \enspace .$$

Proof For the first equality, we have

$$\begin{array}{ll}
 & \mathsf{T} \cap \mathsf{T}^\cup \subseteq \mathsf{T}_< \\
= & \{ \quad \mathrm{I} \text{ is unit of composition, definition of } \mathsf{T}_< \quad \} \\
 & (\mathsf{T} \cap \mathsf{T}^\cup) \circ \mathrm{I} \;\subseteq\; \mathrm{I} \cap \mathsf{T} \circ \top\!\!\top \\
= & \{ \quad \text{assumption: } \mathsf{T} \cap \mathsf{T}^\cup \subseteq \mathrm{I} \text{ ; infimum and monotonicity} \quad \} \\
 & \mathsf{true} \enspace .
\end{array}$$

Also,

$$\begin{array}{ll}
 & \mathsf{T}_< \subseteq \mathsf{T} \cap \mathsf{T}^\cup \\
= & \{ \quad \text{infimum} \quad \} \\
 & \mathsf{T}_< \subseteq \mathsf{T} \;\wedge\; \mathsf{T}_< \subseteq \mathsf{T}^\cup \\
= & \{ \quad \mathsf{T} \text{ is a provisional preorder, so } \mathsf{T}_< \subseteq \mathsf{T} \text{; } (\mathsf{T}_<)^\cup = \mathsf{T}_< \quad \} \\
 & \mathsf{true} \enspace .
\end{array}$$

The second equality is obtained by instantiating T to $\mathsf{T}^\cup$.
□

4 Squares and Rectangles

Squares are by definition homogeneous relations. We now introduce the notion of a "rectangle"; rectangles are typically heterogeneous. Squares are rectangles; properties of squares are typically obtained by specialising properties of rectangles. (For example, lemma 127 shows that the intersection of two rectangles is a rectangle by giving an explicit construction; the same construction applies to squares from which it is easily shown that the intersection of two squares is a square.)

Definition 123 (Rectangle) A relation R is a *rectangle* iff $R = R{\circ}\top\!\!\top{\circ}R$.
□

An example of a rectangle is the "pair" $a{\circ}\top\!\!\top{\circ}b$ where a and b are points. More generally, we have:

Lemma 124 For all relations R and S, $R{\circ}\top\!\!\top{\circ}S$ is a rectangle. It follows that $R{\circ}T{\circ}S$ is a rectangle if T is a rectangle. In particular, if R has type $A{\sim}B$, S has type $B{\sim}C$, and b is a point of type B, the relation $R{\circ}b{\circ}S$ is a rectangle.

Proof Because the proof is based on the cone rule, a case analysis is necessary. In the case that either R or S is the empty relation, the lemma clearly holds (because $R{\circ}\top\!\!\top{\circ}S$ is the empty relation, and the empty relation is a rectangle). Suppose now that both R and S are non-empty. Then

$$
\begin{array}{ll}
 & R{\circ}\top\!\!\top{\circ}S{\circ}\top\!\!\top{\circ}R{\circ}\top\!\!\top{\circ}S \\
= & \{ \quad \text{cone rule: (5) (applied twice), assumption: } R \neq \perp\!\!\perp \text{ and } S \neq \perp\!\!\perp \quad \} \\
 & R{\circ}\top\!\!\top{\circ}S \quad .
\end{array}
$$

If T is a rectangle, $R{\circ}T{\circ}S = R{\circ}T{\circ}\top\!\!\top{\circ}T{\circ}S$; thus $R{\circ}T{\circ}S$ is a rectangle. That $R{\circ}b{\circ}S$ is a rectangle is an instance since, by (15), b is a rectangle if b is a point.
□

The type information in the statement of lemma 124 provides a useful guide when introducing definitions of particular rectangles.

4.1 Inclusion and Intersection

Using colloquial terminology, the left and right domain of a rectangle are the "sides" of the rectangle. In general, a rectangle is defined by its two sides. More precisely:

Lemma 125 Suppose R and S are rectangles of the same type. Then

$$R\subseteq S \quad\equiv\quad R< \subseteq S< \;\wedge\; R> \subseteq S> \;\;.$$

It follows that

$$R=S \quad\equiv\quad R< = S< \;\wedge\; R> = S> \;\;.$$

Proof By mutual implication:

$$\begin{array}{ll}
& R\subseteq S \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
& R< \subseteq S< \;\wedge\; R> \subseteq S> \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
& R<\circ\top\!\!\top\circ R> \;\subseteq\; S<\circ\top\!\!\top\circ S> \\
= & \{ \quad \text{domains} \quad \} \\
& R\circ\top\!\!\top\circ R \subseteq S\circ\top\!\!\top\circ S \\
= & \{ \quad \text{assumption: } R \text{ and } S \text{ are rectangles, definition 123} \quad \} \\
& R\subseteq S \;\;.
\end{array}$$

The second property follows straightforwardly from the anti-symmetry of the subset relation.
□

For squares R and S, lemma 125 simplifies the check for equality to checking that their included points are the same:

Corollary 126 If R and S are both squares then

$$R=S \;\equiv\; \langle\forall a :: a\subseteq R \;\equiv\; a\subseteq S\rangle \;\;.$$

Proof

$$\begin{array}{ll}
& R=S \\
= & \{ \quad \text{lemma 125 and assumption: } R \text{ and } S \text{ are squares} \quad \} \\
& R< = S< \\
\Leftarrow & \{ \quad \text{saturation axiom: (16)} \quad \} \\
& \langle\forall a :: a\subseteq R< \;\equiv\; a\subseteq S<\rangle \\
= & \{ \quad \text{lemma 125 and assumption: } R \text{ and } S \text{ are squares} \quad \}
\end{array}$$

$$\langle\forall a :: a\subseteq R \;\equiv\; a\subseteq S\rangle$$

$\Leftarrow$ { Leibniz }

$$R=S \quad .$$

□

Lemma 127 The intersection of two rectangles is a rectangle. Specifically, for all rectangles R and S,

$$R\cap S \;=\; (R_{<}\cap S_{<})\circ\top\!\!\top\circ(R_{>}\cap S_{>}) \quad .$$

Proof We have, for all R, S, T and U,

$$R\circ\top\!\!\top\circ S \;\cap\; T\circ\top\!\!\top\circ U$$

$=$ { property of conditionals }

$$R\circ\top\!\!\top \;\cap\; \top\!\!\top\circ S \;\cap\; T\circ\top\!\!\top \;\cap\; \top\!\!\top\circ U$$

$=$ { property of conditionals }

$$(R\cap T)\circ\top\!\!\top \;\cap\; \top\!\!\top\circ(S\cap U)$$

$=$ { property of conditionals }

$$(R\cap T)\circ\top\!\!\top\circ(S\cap U) \quad .$$

(The properties of conditionals used above are not shown in this paper but easily proven. Hint: use the modularity rule (3).) Also, for all R and S, $R\circ\top\!\!\top\circ S = R_{<}\circ\top\!\!\top\circ S_{>}$. So

$$R\cap S$$

$=$ { assumption: R and S are rectangles }

$$R\circ\top\!\!\top\circ R \cap S\circ\top\!\!\top\circ S$$

$=$ { $[\; R\circ\top\!\!\top\circ S = R_{<}\circ\top\!\!\top\circ S_{>} \;]$ with $R,S := R,R$ and $R,S := S,S$ }

$$R_{<}\circ\top\!\!\top\circ R_{>} \;\cap\; S_{<}\circ\top\!\!\top\circ S_{>}$$

$=$ { above with $R,S,T,U := R_{<},R_{>},S_{<},S_{>}$ }

$$(R_{<}\cap S_{<})\circ\top\!\!\top\circ(R_{>}\cap S_{>}) \quad .$$

□

Lemma 128 If U is a rectangle then, for all points b (of appropriate type)

$$(U\circ b)_{<} = U_{<} \quad\vee\quad (U\circ b)_{<} = \bot\!\!\bot \quad .$$

Proof

$$
\begin{array}{ll}
 & (\mathsf{U}{\circ}\mathsf{b}){<} \\
= & \{ \quad \text{assumption: } \mathsf{U} \text{ is a rectangle} \quad \} \\
 & (\mathsf{U}{\circ}\top\!\!\top{\circ}\mathsf{U}{\circ}\mathsf{b}){<} \\
= & \{ \quad \text{domains} \quad \} \\
 & (\mathsf{U}\circ\top\!\!\top\circ\mathsf{U}{>}\circ\mathsf{b}){<} \\
= & \{ \quad \text{assumption: } \mathsf{b} \text{ is a point. So } \mathsf{U}{>}\circ\mathsf{b} = \mathsf{b} \vee \mathsf{U}{>}\circ\mathsf{b} = \bot\!\!\bot \quad \} \\
 & \textsf{if } \mathsf{U}{>}\circ\mathsf{b} = \mathsf{b} \rightarrow (\mathsf{U}{\circ}\top\!\!\top{\circ}\mathsf{b}){<} \ \square\ \mathsf{U}{>}\circ\mathsf{b} = \bot\!\!\bot \rightarrow \bot\!\!\bot \textsf{ fi} \\
= & \{ \quad \text{assumption: } \mathsf{b} \text{ is a point. So } (\top\!\!\top{\circ}\mathsf{b}){<} = \mathrm{I} \quad \} \\
 & \textsf{if } \mathsf{U}{>}\circ\mathsf{b} = \mathsf{b} \rightarrow \mathsf{U}{<} \ \square\ \mathsf{U}{>}\circ\mathsf{b} = \bot\!\!\bot \rightarrow \bot\!\!\bot \textsf{ fi} \ .
\end{array}
$$

□

4.2 Completely Disjoint Rectangles

As is well-known, an equivalence relation *partitions* its domain into a set of disjoint *classes*. Also well-known is that the existence of such a partitioning is precisely formulated by the function that maps an element of the domain to its *equivalence class*: two elements are equivalent if and only if their equivalence classes are equal. When represented by relations, equivalence classes are squares. The theory of difunctional relations generalises this partitioning property to "completely disjoint" rectangles. This section lays the foundations for this theory. Specifically, theorem 141 formulates a correspondence between pairs of functional relations and sets of completely disjoint rectangles.

Definition 129 (Indexed Bag/Set) Suppose $\mathcal{R}$ is a function with source K. Then $\mathcal{R}$ is said to be a *bag indexed by* K. The values $\mathcal{R}.\mathsf{k}$, where k ranges over K, are said to be the *elements* of $\mathcal{R}$. In the case that $\mathcal{R}$ is injective, it is said to be an *indexed set*.
□

The distinction between "bag" and "set" in definition 129 emphasises the fact that the same element may occur repeatedly in an indexed bag whereas each element occurs exactly once in an indexed set. That is, an indexed set $\mathcal{R}$ has the property that, for all j and k in K,

$$\mathcal{R}.\mathsf{j} = \mathcal{R}.\mathsf{k} \ \equiv\ \mathsf{j} = \mathsf{k} \ .$$

We normally apply definition 129 to bags/sets of rectangles. Specifically, suppose A, B and K are types and $\mathcal{R}$ is a function with source K and target rectangles of type $\mathsf{A}{\sim}\mathsf{B}$.

Then $\mathcal{R}$ is said to be an *indexed bag of rectangles*; it is an indexed *set* of rectangles if it is injective.

Two relations R and S are *disjoint* if $R \cap S = \perp\!\!\perp$. One can show that, for all rectangles R and S ,

$$R \cap S = \perp\!\!\perp \quad \equiv \quad R_< \cap S_< = \perp\!\!\perp \;\vee\; R_> \cap S_> = \perp\!\!\perp \quad .$$

(This is a consequence of lemma 127.) The definition of "completely" disjoint strengthens the disjunction to a conjunction. Note that we don't use continued equality because the symbol " $\perp\!\!\perp$ " is overloaded.

Definition 130 (Completely Disjoint) Two rectangles R and S are said to be *completely disjoint* iff

$$R_< \cap S_< = \perp\!\!\perp \quad \wedge \quad R_> \cap S_> = \perp\!\!\perp \quad .$$

Suppose $\mathcal{R}$ is an indexed bag of rectangles. Then $\mathcal{R}$ is said to be a *completely disjoint bag of rectangles* iff, for all j and k in the index set of $\mathcal{R}$,

$$\mathcal{R}.j \neq \mathcal{R}.k \quad \equiv \quad (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \perp\!\!\perp \;\wedge\; (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \perp\!\!\perp \quad .$$

$\mathcal{R}$ is said to be a completely disjoint *set* of rectangles iff in addition it is injective. That is, $\mathcal{R}$ is a *completely disjoint set of rectangles* iff, for all j and k in the index set of $\mathcal{R}$,

$$j \neq k \quad \equiv \quad (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \perp\!\!\perp \;\wedge\; (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \perp\!\!\perp \quad .$$

□

We give several constructions of bags/sets of rectangles. When we do so, the verification that the bag/sets are completely disjoint is achieved by mutual implication. The "if" part is established by proving its contrapositive. That is, the proof obligation becomes to show that, for all indices j and k ,

$$\mathcal{R}.j = \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< \neq \perp\!\!\perp \;\wedge\; (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> \neq \perp\!\!\perp$$

which simplifies to, for all j ,

$$\mathcal{R}.j \neq \perp\!\!\perp \quad .$$

(The same simplification is valid whether the construction yields a bag or a set.) Thus the first step is to show that the construction yields non-empty elements. The "only-if" part is to show that, for all indices j and k ,

$$\mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \perp\!\!\perp \;\wedge\; (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \perp\!\!\perp \quad .$$

For this part, the following lemma is exploited.

Lemma 131 For all relations R and S,

$$R{<} \cap S{<} \;=\; \perp\!\!\perp \quad\equiv\quad R^{\cup} \circ S \;=\; \perp\!\!\perp \quad .$$

Symmetrically,

$$R{>} \cap S{>} \;=\; \perp\!\!\perp \quad\equiv\quad R \circ S^{\cup} \;=\; \perp\!\!\perp \quad .$$

Proof First note that

$$R{<} \cap S{<} \;=\; \perp\!\!\perp \quad\equiv\quad R{<} \circ S{<} \;=\; \perp\!\!\perp$$

since the intersection of coreflexives is the same as their composition. Then

$$\begin{array}{ll}
 & R{<} \circ S{<} \;=\; \perp\!\!\perp \\
\Rightarrow & \{ \quad \perp\!\!\perp \text{ is zero of composition} \quad \} \\
 & R^{\cup} \circ R{<} \circ S{<} \circ S \;=\; \perp\!\!\perp \\
= & \{ \quad \text{domains: (45)} \quad \} \\
 & R^{\cup} \circ S \;=\; \perp\!\!\perp \\
\Rightarrow & \{ \quad \perp\!\!\perp \text{ is zero of composition} \quad \} \\
 & R \circ R^{\cup} \circ S \circ S^{\cup} \;=\; \perp\!\!\perp \\
\Rightarrow & \{ \quad \text{monotonicity, } [\, R = \perp\!\!\perp \equiv R \subseteq \perp\!\!\perp \,] \text{ (applied twice)} \quad \} \\
 & (I \cap R \circ R^{\cup}) \circ (I \cap S \circ S^{\cup}) = \perp\!\!\perp \\
= & \{ \quad \text{domains: definition 42} \quad \} \\
 & R{<} \circ S{<} \;=\; \perp\!\!\perp \quad .
\end{array}$$

The lemma follows by mutual implication.
□

The foregoing discussion is formalised in the following lemma.

Lemma 132 Suppose $\mathcal{R}$ is an indexed bag of rectangles. Then $\mathcal{R}$ is completely disjoint iff

$$\begin{array}{ll}
 & \langle \forall j :: \mathcal{R}.j \neq \perp\!\!\perp \rangle \\
\wedge & \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \;=\; \perp\!\!\perp \;\wedge\; \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \perp\!\!\perp \rangle \quad .
\end{array}$$

Also, $\mathcal{R}$ is completely disjoint and injective —i.e. an indexed set— iff

$$\begin{array}{ll}
 & \langle \forall j :: \mathcal{R}.j \neq \perp\!\!\perp \rangle \\
\wedge & \langle \forall j,k \;::\; j \neq k \;\Rightarrow\; (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \;=\; \perp\!\!\perp \;\wedge\; \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \perp\!\!\perp \rangle \quad .
\end{array}$$

Proof

$$
\begin{array}{ll}
 & \mathcal{R} \text{ is completely disjoint} \\
= & \{ \quad \text{definition 130} \quad \} \\
 & \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \equiv (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \perp\!\!\perp \wedge (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \perp\!\!\perp \rangle \\
= & \{ \quad \text{mutual implication} \quad \} \\
 & \quad \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Leftarrow (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \perp\!\!\perp \wedge (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \perp\!\!\perp \rangle \\
 & \wedge \quad \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Rightarrow (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \perp\!\!\perp \wedge (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \perp\!\!\perp \rangle \\
= & \{ \quad \text{contrapositive; lemma 131} \quad \} \\
 & \quad \langle \forall j,k :: \mathcal{R}.j = \mathcal{R}.k \Rightarrow (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< \neq \perp\!\!\perp \vee (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> \neq \perp\!\!\perp \rangle \\
 & \wedge \quad \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Rightarrow \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \perp\!\!\perp \rangle \\
= & \{ \quad \text{Leibniz, reflexivity of equality, idempotence of intersection} \quad \} \\
 & \quad \langle \forall j :: (\mathcal{R}.j)_< \neq \perp\!\!\perp \vee (\mathcal{R}.j)_> \neq \perp\!\!\perp \rangle \\
 & \wedge \quad \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Rightarrow \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \perp\!\!\perp \rangle \\
= & \{ \quad \text{domains} \\
 & \quad\quad (\; [\, (R_< = \perp\!\!\perp) = (R = \perp\!\!\perp) = (R_> = \perp\!\!\perp) \,] \text{ with } R := \mathcal{R}.j \,)) \quad \} \\
 & \quad \langle \forall j :: \mathcal{R}.j \neq \perp\!\!\perp \rangle \\
 & \wedge \quad \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Rightarrow \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \perp\!\!\perp \rangle \; .
\end{array}
$$

Injectivity of $\mathcal{R}$ is the property that $\langle \forall j,k :: \mathcal{R}.j = \mathcal{R}.k \equiv j = k \rangle$. The characterisation of completely disjoint and injective thus follows by the use of Leibniz's rule.
□

Here is the first example of such a construction.

Lemma 133 Suppose f and g are relations with common target C such that

$$f \circ f^{\cup} = f_< = g \circ g^{\cup} = g_< \; .$$

Then the relation $f^{\cup} \circ g$ is the supremum of an indexed set of completely disjoint rectangles. Specifically, with dummy c ranging over points of type C,

$$f^{\cup} \circ g = \langle \cup c : c \subseteq g_< : f^{\cup} \circ c \circ g \rangle \; .$$

Proof As remarked in lemma 124, the relation $R \circ c \circ S$ is a rectangle, for all points c and all relations R and S; so this is also true of $f^{\cup} \circ c \circ g$. This collection of rectangles covers $f^{\cup} \circ g$ since

$$
\begin{array}{ll}
 & f^{\cup} \circ g \\
= & \{ \quad g = g_{<} \circ g \ \text{ and saturation axiom: (16)} \quad \} \\
 & f^{\cup} \circ \langle \cup c \ : \ c \subseteq g_{<} \ : \ c \rangle \circ g \\
= & \{ \quad \text{distributivity} \quad \} \\
 & \langle \cup c : c \subseteq g_{<} : f^{\cup} \circ c \circ g \rangle \quad .
\end{array}
$$

To show that the function $\langle c : c \subseteq g_{<} : f^{\cup} \circ c \circ g \rangle$ is an indexed set of completely disjoint rectangles, we apply lemma 132. First, if $c \subseteq g_{<}$, the rectangle $f^{\cup} \circ c \circ g$ is non-empty since

$$
\begin{array}{ll}
 & f^{\cup} \circ c \circ g \ = \ \perp\!\!\perp \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & (f^{\cup} \circ c \circ g)_{>} \ = \ \perp\!\!\perp \\
= & \{ \quad \text{domains} \quad \} \\
 & (f_{<} \circ c \circ g)_{>} \ = \ \perp\!\!\perp \\
= & \{ \quad f_{<} = g_{<} \ \text{ and } \ c \subseteq g_{<} \quad \} \\
 & (c \circ g)_{>} = \perp\!\!\perp \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & ((c \circ g)_{>} \circ g^{\cup})_{>} = \perp\!\!\perp \\
= & \{ \quad \text{domains} \quad \} \\
 & (c \circ g \circ g^{\cup})_{>} = \perp\!\!\perp \\
= & \{ \quad g \circ g^{\cup} \ = \ g_{<} \ \text{ and } \ c \subseteq g_{<} \quad \} \\
 & c = \perp\!\!\perp \\
= & \{ \quad c \ \text{ is a point} \quad \} \\
 & \mathsf{false} \quad .
\end{array}
$$

That is,

$$(134) \quad \langle \forall c \ : \ c \subseteq g_{<} \ : \ f^{\cup} \circ c \circ g \ \neq \ \perp\!\!\perp \rangle \quad .$$

Also, assuming that $c \subseteq g_{<}$ and $c \neq c'$, we have:

$$
\begin{array}{ll}
 & (f^{\cup} \circ c \circ g)^{\cup} \circ (f^{\cup} \circ c' \circ g) \\
= & \{ \quad \text{distributivity, } \ c = c^{\cup} \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & g^{\cup}\circ c\circ f\circ f^{\cup}\circ c'\circ g \\
= & \{\quad \text{assumption:}\quad f\circ f^{\cup} \;=\; g_{<} \quad\} \\
 & g^{\cup}\circ c\circ g_{<}\circ c'\circ g \\
= & \{\quad c\subseteq g_{<} \quad\} \\
 & g^{\cup}\circ c\circ c'\circ g \\
= & \{\quad \text{assumption:}\; c\neq c',\ (17)\ \text{with}\ a,a':=c,c' \quad\} \\
 & \perp\!\!\perp\ .
\end{array}
$$

That is,

(135) $\langle\forall c,c' : c\subseteq g_{<} : c\neq c' \Rightarrow (f^{\cup}\circ c\circ g)^{\cup}\circ(f^{\cup}\circ c'\circ g)=\perp\!\!\perp\rangle$.

An almost identical argument shows that

(136) $\langle\forall c,c' : c\subseteq g_{<} : c\neq c' \Rightarrow (f^{\cup}\circ c\circ g)\circ(f^{\cup}\circ c'\circ g)^{\cup}=\perp\!\!\perp\rangle$.

Applying lemma 132 with $\mathcal{R}:=\langle c : c\subseteq g_{<} : f^{\cup}\circ c\circ g\rangle$, properties (134), (135) and (136) establish that $f^{\cup}\circ g$ is indeed an indexed set of completely disjoint rectangles.
□

We now establish the converse of lemma 133. (The proof is quite long because of all the details that need to be checked.)

Lemma 137 Suppose relation R is the supremum of a completely disjoint set of rectangles. Then

$$\langle\exists f,g \;:\; f\circ f^{\cup} \;=\; f_{<} \;=\; g\circ g^{\cup} \;=\; g_{<} \;:\; R = f^{\cup}\circ g\rangle\ .$$

Proof Suppose $\mathcal{R}$ is a completely disjoint set of rectangles indexed by the set K. Suppose also that $R=\cup\mathcal{R}$. Define the relations f and g by, for all k in K and all points a such that $a\subseteq R_{<}$,

(138) $k\circ\top\!\!\top\circ a \subseteq f \;\;\equiv\;\; a\circ(\mathcal{R}.k)_{<} = a$,

and, for all k in K and all points b such that $b\subseteq R_{>}$

(139) $k\circ\top\!\!\top\circ b \subseteq g \;\;\equiv\;\; (\mathcal{R}.k)_{>}\circ b = b$.

Both f and g are functional. For example, here is the proof that f is functional: for all j and k in K,

$$j{\circ}\top\!\!\top{\circ}k \;\subseteq\; f\circ f^{\cup}$$

$=\quad\{\quad$ saturation axiom: (16) and irreducibility: (20) $\quad\}$

$$\langle\exists a :: j{\circ}\top\!\!\top{\circ}a \subseteq f \wedge a{\circ}\top\!\!\top{\circ}j \subseteq f^{\cup}\rangle$$

$=\quad\{\quad$ (138) and converse $\quad\}$

$$\langle\exists a :: a\circ(\mathcal{R}.j)_{<} = a \;\wedge\; a\circ(\mathcal{R}.k)_{<} = a\rangle$$

$\Rightarrow\quad\{\quad$ coreflexives $\quad\}$

$$(\mathcal{R}.j)_{<} \cap (\mathcal{R}.k)_{<} \;\neq\; \bot\!\!\bot \quad .$$

So

$$j{\circ}\top\!\!\top{\circ}k \;\subseteq\; f\circ f^{\cup}$$

$=\quad\{\quad$ $f\circ f^{\cup}$ is symmetric (i.e. $j{\circ}\top\!\!\top{\circ}k \subseteq f\circ f^{\cup} \;\equiv\; k{\circ}\top\!\!\top{\circ}j \subseteq f\circ f^{\cup}$) $\quad\}$

$$j{\circ}\top\!\!\top{\circ}k \subseteq f\circ f^{\cup} \;\;\wedge\;\; k{\circ}\top\!\!\top{\circ}j \subseteq f\circ f^{\cup}$$

$\Rightarrow\quad\{\quad$ above (applied twice, once with $j,k := k,j$) $\quad\}$

$$(\mathcal{R}.j)_{<} \cap (\mathcal{R}.k)_{<} \neq \bot\!\!\bot \quad\wedge\quad (\mathcal{R}.k)_{<} \cap (\mathcal{R}.j)_{<} \neq \bot\!\!\bot$$

$=\quad\{\quad$ $\mathcal{R}$ is a completely disjoint set of rectangles, definition 130 $\quad\}$

$$j = k \quad .$$

That is, by the saturation axiom and the definition of I_K , $f\circ f^{\cup} \subseteq I_K$.

Both f and g are also surjective. For suppose k is in K . Then

$$\mathsf{true}$$

$=\quad\{\quad$ definition 130 with $j := k$ $\quad\}$

$$\mathcal{R}.k \neq \bot\!\!\bot$$

$=\quad\{\quad$ saturation axiom: (16) $\quad\}$

$$\langle\exists a :: a\circ(\mathcal{R}.k)_{<} = a\rangle$$

$=\quad\{\quad$ (138) $\quad\}$

$$\langle\exists a :: k{\circ}\top\!\!\top{\circ}a \subseteq f\rangle$$

$\Rightarrow\quad\{\quad$ a and k are points, so $k = k{\circ}\top\!\!\top{\circ}k = k{\circ}\top\!\!\top{\circ}a{\circ}\top\!\!\top{\circ}k$ $\quad\}$

$$k \subseteq f\circ f^{\cup} \quad .$$

That is, by the saturation axiom, $I_K \subseteq f\circ f^{\cup}$.

Combining the functionality of f with its surjectivity, we conclude that $f\circ f^{\cup} = I_K$. Similarly, $g\circ g^{\cup} = I_K$. So we have constructed relations f and g such that

$$(140)\qquad f\circ f^{\cup} \;=\; f_{<} = I_K \;=\; g\circ g^{\cup} \;=\; g_{<} \quad .$$

We now have to show that $R = f^{\cup} \circ g$. A first step is to show that $f{>} = R{<}$ and $g{>} = R{>}$. We have, for all points a,

$$
\begin{array}{ll}
 & a \subseteq R{<} \\
= & \{ \quad R = \cup\mathcal{R} \quad \} \\
 & a \subseteq (\cup\mathcal{R}){<} \\
= & \{ \quad \text{distributivity} \quad \} \\
 & a \subseteq \langle \cup k :: (\mathcal{R}.k){<} \rangle \\
= & \{ \quad \text{irreducibility of points} \quad \} \\
 & \langle \exists k :: a \subseteq (\mathcal{R}.k){<} \rangle \\
= & \{ \quad \text{coreflexives} \quad \} \\
 & \langle \exists k :: a \circ (\mathcal{R}.k){<} = a \rangle \\
= & \{ \quad (138) \quad \} \\
 & \langle \exists k :: k \circ \top\!\!\top \circ a \subseteq f \rangle \\
= & \{ \quad \text{domains} \quad \} \\
 & a \subseteq f{<} \quad .
\end{array}
$$

We conclude by the saturation axiom (16) that $f{>} = R{<}$. Again, the property $g{>} = R{>}$ is proved similarly. It follows that

$$
\begin{array}{ll}
 & (f^{\cup} \circ g){>} \\
= & \{ \quad \text{domains} \quad \} \\
 & (f{<} \circ g){>} \\
= & \{ \quad (140) \text{ (specifically, } f{<} = g{<}) \quad \} \\
 & g{>} \\
= & \{ \quad \text{above} \quad \} \\
 & R{>} \quad .
\end{array}
$$

Similarly, $(f^{\cup} \circ g){<} = R{<}$. So, for all points a and b such that $a \subseteq R{<}$ and $b \subseteq R{>}$,

$$
\begin{array}{ll}
 & a \circ f^{\cup} \circ g \circ b \\
= & \{ \quad \text{saturation axiom: (16) and distributivity} \quad \} \\
 & \langle \cup k : k \subseteq f{<} \wedge k \subseteq g{<} : a \circ f^{\cup} \circ k \circ g \circ b \rangle \\
= & \{ \quad (140) \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & \langle \cup k : k{\in}K : a\circ f^{\cup}\circ k\circ g\circ b\rangle \\
= & \{ \quad \text{all-or-nothing: theorem 11} \quad \} \\
 & \langle \cup k \,:\, a{\circ}\top\!\!\top{\circ}k \subseteq f^{\cup} \wedge k{\circ}\top\!\!\top{\circ}b \subseteq g \,:\, a{\circ}\top\!\!\top{\circ}k{\circ}k{\circ}\top\!\!\top{\circ}b\rangle \\
= & \{ \quad \text{assumption: } a\subseteq R_{<} \text{ and } b\subseteq R_{>}\text{; (138) and (139), and } k \text{ is a point} \quad \} \\
 & \langle \cup k \,:\, a\circ(\mathcal{R}.k)_{<} = a \wedge (\mathcal{R}.k)_{>}\circ b = b \,:\, a{\circ}\top\!\!\top{\circ}b\rangle \\
= & \{ \quad a \text{ is a point, so } a\circ(\mathcal{R}.k)_{<} = a \vee a\circ(\mathcal{R}.k)_{<} = \bot\!\!\bot \\
 & \qquad b \text{ is a point, so } (\mathcal{R}.k)_{>}\circ b = b \vee (\mathcal{R}.k)_{>}\circ b = \bot\!\!\bot \\
 & \qquad \text{range disjunction and } \bot\!\!\bot \text{ is least} \quad \} \\
 & \langle \cup k :: a\circ(\mathcal{R}.k)_{<}\circ\top\!\!\top\circ(\mathcal{R}.k)_{>}\circ b\rangle \\
= & \{ \quad \text{domains and } \mathcal{R}.k \text{ is a rectangle: definition 123} \quad \} \\
 & \langle \cup k :: a\circ\mathcal{R}.k\circ b\rangle \\
= & \{ \quad R = \langle \cup k :: \mathcal{R}.k\rangle \text{ and distributivity} \quad \} \\
 & a{\circ}R{\circ}b \quad .
\end{array}
$$

We conclude that $R = f^{\cup}\circ g$ by the saturation property (19).
□

Theorem 141 A relation R is the supremum of a set of completely disjoint rectangles if and only if

$$\left\langle \exists f,g \,:\, f\circ f^{\cup} = f_{<} = g\circ g^{\cup} = g_{<} \,:\, R = f^{\cup}\circ g\right\rangle \quad .$$

Proof "If" is lemma 133 and "only-if" is lemma 137.
□

In terms of the mental picture of a relation R as the supremum of a set of completely disjoint rectangles, the set of vertical and the set of horizontal sides each defines a per on the source and target of the relation. These two pers are the relations $R_{\prec}$ and $R_{\succ}$ (defined by (96) and (97)). Formally, we have:

Lemma 142 Suppose R, f and g are relations such that

$$f\circ f^{\cup} = f_{<} = g\circ g^{\cup} = g_{<} \quad \wedge \quad R = f^{\cup}\circ g \quad .$$

Then

$$R_{\prec} = f_{\succ} = f^{\cup}\circ f = R\circ R^{\cup} \quad \wedge \quad R_{\succ} = g_{\succ} = g^{\cup}\circ g = R^{\cup}\circ R \quad .$$

Proof Immediate application of lemmas 113 and 107.
□

5 Characterisations of Partial Equivalence Relations

The theorem we prove in this section is that every partial equivalence relation is the supremum of a set of disjoint squares. Specifically, the goal of this section is the proof of the following characterisation of pers:

Theorem 143 For all relations R, the following statements are equivalent:

(i) R is a per,

(ii) R is the supremum of an indexed set of disjoint squares,

(iii) $\langle \exists f \; : \; f \circ f^{\cup} = f_{<} \; : \; R = f^{\cup} \circ f \rangle$.

□

An informal understanding of theorem 143 is that a per *partitions* its domain into disjoint sets — commonly called *equivalence classes*. Two ways of representing the equivalence classes are given by either —theorem 143(ii)— a set of disjoint squares or —theorem 143(iii)— a functional relation f whereby two points in the domain of a per are in the same equivalence class iff they are mapped to the same value by f. (There are, of course, other ways of representing the classes.)

The proof that 143(iii) implies 143(i) is straightforward. See lemma 94. The converse (143(i) implies 143(iii)) is also easy to prove. Thus 143(i) is equivalent to 143(iii). See theorem 144.

To prove that both 143(i) and 143(iii) are equivalent to 143(ii), we first show that 143(ii) implies 143(i). See lemma 145. We complete the proof by showing that 143(iii) implies 143(ii). See lemma 150. (The equivalence then follows from the equivalence of 143(i) and 143(iii).)

5.1 Proof of the Characterisation Theorem

As outlined above, we begin with the proof that 143(i) is equivalent to 143(iii). Note that $\Gamma R \circ R_{>}$ is a functional relation and thus witnesses the existential quantification in 143(iii).

Theorem 144 A relation R is a per iff $R = (\Gamma R \circ R_{>})^{\cup} \circ (\Gamma R \circ R_{>})$.

Proof By mutual implication. First, assume that R is a per. Then

$$R$$

$$= \quad \{ \quad \text{assumption: } R = R^{\cup} \text{, domains} \quad \}$$

$$
\begin{array}{ll}
 & R{>}\circ R \\
= & \{ \quad \text{assumption: } R \text{ is a per; theorem 108 and definition 95} \quad \} \\
 & R{>}\circ R\backslash\!\backslash R\circ R{>} \\
= & \{ \quad \text{(89) with } R,S := R,R \quad \} \\
 & R{>}\circ(\Gamma R)^{\cup}\circ\Gamma R\circ R{>} \\
= & \{ \quad \text{converse} \quad \} \\
 & (\Gamma R\circ R{>})^{\cup}\circ(\Gamma R\circ R{>}) \quad .
\end{array}
$$

The converse is immediate from lemma 94.

□

The next step is to show that 143(ii) implies 143(i).

Lemma 145 Suppose $\mathcal{R}$ is a bag of disjoint squares. Then $\cup\mathcal{R}$ is a per.

Proof We aim to apply theorem 93 with $R := \cup\mathcal{R}$.

$$
\begin{array}{ll}
 & \cup\mathcal{R}\circ(\cup\mathcal{R})^{\cup} \\
= & \{ \quad \text{distributivity} \quad \} \\
 & \langle\cup j,k :: \mathcal{R}.j\circ(\mathcal{R}.k)^{\cup}\rangle \\
= & \{ \quad \mathcal{R} \text{ is a bag of disjoint squares, so} \\
 & \qquad \mathcal{R}.j\circ(\mathcal{R}.k)^{\cup} = \perp\!\!\perp \;\equiv\; \mathcal{R}.j \neq \mathcal{R}.k \quad \} \\
 & \langle\cup j :: \mathcal{R}.j\circ(\mathcal{R}.j)^{\cup}\rangle \\
= & \{ \quad \text{for all } j\,,\ \mathcal{R}.j \text{ is a square} \quad \} \\
 & \langle\cup j :: \mathcal{R}.j\circ\top\!\!\top\circ(\mathcal{R}.j)^{\cup}\circ\mathcal{R}.j\circ\top\!\!\top\circ(\mathcal{R}.j)^{\cup}\rangle \\
= & \{ \quad \text{for all } j\,,\ \mathcal{R}.j \neq \perp\!\!\perp\,;\ \text{cone rule} \quad \} \\
 & \langle\cup j :: \mathcal{R}.j\circ\top\!\!\top\circ(\mathcal{R}.j)^{\cup}\rangle \\
= & \{ \quad \text{for all } j\,,\ \mathcal{R}.j \text{ is a square} \quad \} \\
 & \langle\cup j :: \mathcal{R}.j\rangle \\
= & \{ \quad \text{definition} \quad \} \\
 & \cup\mathcal{R} \quad .
\end{array}
$$

That is, $\cup\mathcal{R} = \cup\mathcal{R}\circ(\cup\mathcal{R})^{\cup}$. Applying theorem 93, we conclude that $\cup\mathcal{R}$ is a per.

□

The final step is to show that 143(iii) implies 143(ii). We aim to use lemma 133. In order to do so, we make use of the fact that 143(iii) and 143(i) are equivalent.

Lemma 146 Suppose R is a per and suppose f and g are such that

$$f \circ f^{\cup} \;=\; f{<} \;=\; g \circ g^{\cup} \;=\; g{<} \quad \wedge \quad R = f^{\cup} \circ g \quad .$$

Then $f^{\cup} \circ f = R = g^{\cup} \circ g$.

Proof

$$
\begin{array}{ll}
 & f^{\cup} \circ f \\
= & \{ \quad \text{domains} \quad \} \\
 & f^{\cup} \circ f{<} \circ f \\
= & \{ \quad \text{assumption: } f{<} \;=\; g \circ g^{\cup} \quad \} \\
 & f^{\cup} \circ g \circ g^{\cup} \circ f \\
= & \{ \quad \text{converse} \quad \} \\
 & f^{\cup} \circ g \circ (f^{\cup} \circ g)^{\cup} \\
= & \{ \quad \text{assumption: } R = f^{\cup} \circ g \quad \} \\
 & R \circ R^{\cup} \\
= & \{ \quad \text{assumption: } R \text{ is a per, theorem 93} \quad \} \\
 & R \quad .
\end{array}
$$

Thus $f^{\cup} \circ f = R$. The dual statement $R = g^{\cup} \circ g$ is proved similarly.
□

Lemma 147 Suppose $R = \cup\mathcal{R}$ where $\mathcal{R}$ is an indexed bag of completely disjoint rectangles and suppose R is a per. Then $\mathcal{R}$ is an indexed bag of disjoint squares.

Proof We exploit theorem 93. That is, we assume that $R = R^{\cup} \circ R$. Then

$$
\begin{array}{ll}
 & R^{\cup} \circ R \\
= & \{ \quad R = \cup\mathcal{R} \quad \} \\
 & (\cup\mathcal{R})^{\cup} \circ \cup\mathcal{R} \\
= & \{ \quad \text{distributivity} \quad \} \\
 & \langle \cup j,k :: (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \rangle \\
= & \{ \quad \text{domains} \quad \} \\
 & \langle \cup j,k :: (\mathcal{R}.j)^{\cup} \circ (\mathcal{R}.j){<} \circ (\mathcal{R}.k){<} \circ \mathcal{R}.k \rangle \\
= & \{ \quad \mathcal{R} \text{ is a bag of completely disjoint rectangles}
\end{array}
$$

$$
\begin{array}{ll}
 & \qquad \text{so } (\mathcal{R}.j){<}\circ(\mathcal{R}.k){<} \;=\; \perp\!\!\perp \;\Leftarrow\; \mathcal{R}.j \neq \mathcal{R}.k\,; \\
 & \qquad \text{range splitting (on } \mathcal{R}.j = \mathcal{R}.k \text{ and } \mathcal{R}.j \neq \mathcal{R}.k\text{)} \quad \} \\
 & \langle \cup j,k : \mathcal{R}.j = \mathcal{R}.k : (\mathcal{R}.j)^{\cup} \circ (\mathcal{R}.j){<} \circ (\mathcal{R}.k){<} \circ \mathcal{R}.k \rangle \\
= & \quad \{ \quad \text{Leibniz, idempotency of set union} \quad \} \\
 & \langle \cup k :: (\mathcal{R}.k)^{\cup} \circ (\mathcal{R}.k){<} \circ (\mathcal{R}.k){<} \circ \mathcal{R}.k \rangle \\
= & \quad \{ \quad \text{domains} \quad \} \\
 & \langle \cup k :: (\mathcal{R}.k)^{\cup} \circ \mathcal{R}.k \rangle \quad .
\end{array}
$$

That is,

(148) $\quad R^{\cup} \circ R \;=\; \langle \cup k :: (\mathcal{R}.k)^{\cup} \circ \mathcal{R}.k \rangle \quad .$

Also, for all k,

$$
\begin{array}{ll}
 & (\mathcal{R}.k){<} \circ R \\
= & \quad \{ \quad R = \cup\mathcal{R} \text{ and distributivity} \quad \} \\
 & \langle \cup j :: (\mathcal{R}.k){<} \circ \mathcal{R}.j \rangle \\
= & \quad \{ \quad (\mathcal{R}.k){<} \circ (\mathcal{R}.j){<} \;=\; \perp\!\!\perp \;\Leftarrow\; \mathcal{R}.j \neq \mathcal{R}.k \\
 & \qquad \text{range splitting (see above)} \quad \} \\
 & (\mathcal{R}.k){<} \circ \mathcal{R}.k \\
= & \quad \{ \quad \text{domains} \quad \} \\
 & \mathcal{R}.k \quad .
\end{array}
$$

Together with its dual, we thus have, for all k,

(149) $\quad (\mathcal{R}.k){<} \circ R \;=\; \mathcal{R}.k \;=\; R \circ (\mathcal{R}.k){>} \quad .$

Hence, for all k,

$$
\begin{array}{ll}
 & \mathcal{R}.k \\
= & \quad \{ \quad (149) \quad \} \\
 & R \circ (\mathcal{R}.k){>} \\
= & \quad \{ \quad R = R^{\cup} \circ R \quad \} \\
 & R^{\cup} \circ R \circ (\mathcal{R}.k){>} \\
= & \quad \{ \quad (148) \quad \} \\
 & \langle \cup j :: (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.j \rangle \circ (\mathcal{R}.k){>}
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{distributivity, } (\mathcal{R}.k)\!<\circ(\mathcal{R}.j)\!< \;=\; \bot\!\!\bot \;\Leftarrow\; \mathcal{R}.j \neq \mathcal{R}.k \\
 & \quad\;\; \text{range splitting (see above)} \quad \} \\
 & (\mathcal{R}.k)^{\cup}\circ\mathcal{R}.k\circ(\mathcal{R}.k)\!> \\
= & \{ \quad \text{domains} \quad \} \\
 & (\mathcal{R}.k)^{\cup}\circ\mathcal{R}.k \quad .
\end{array}
$$

That is, for all k, $\mathcal{R}.k = (\mathcal{R}.k)^{\cup}\circ\mathcal{R}.k$. Applying theorem 93, for all k, $\mathcal{R}.k$ is a per, and hence symmetric. It is also a rectangle and a symmetric rectangle is a square. We conclude that $\mathcal{R}$ is a bag of disjoint squares.
□

Lemma 150 Suppose f is such that

$$f\circ f^{\cup} \;=\; f\!< \quad .$$

Then the relation $f^{\cup}\circ f$ is the supremum of an indexed set of disjoint squares.

Proof This an instance of lemmas 133 and 147. From lemma 133 (with $g:=f$), $f^{\cup}\circ f$ is the supremum of a set of completely disjoint rectangles. But $f^{\cup}\circ f$ is a per. (See lemma 94.) So, by lemma 147, $f^{\cup}\circ f$ is the supremum of a set of completely disjoint squares.
□

This completes the proof of theorem 143. We have shown that 143(i) and 143(iii) are equivalent (theorem 144), that 143(ii) implies 143(i) (lemma 145) and 143(iii) implies 143(ii) (lemma 150).

5.2 Unicity of Characterisations

The characterisation of a per in the form $f^{\cup}\circ f$ where f is a functional relation is not unique. The characterisation is sometimes described as being "essentially" unique or sometimes as unique "up to isomorphism". This is made precise by theorem 151:

Theorem 151 Suppose R is a per and suppose f and g are functional relations such that $R \;=\; f^{\cup}\circ f \;=\; g^{\cup}\circ g$. Then $f\cong g$.

Proof We have

$$
\begin{array}{ll}
 & f\circ g^{\cup}\circ(f\circ g^{\cup})^{\cup} \\
= & \{ \quad \text{converse} \quad \} \\
 & f\circ g^{\cup}\circ g\circ f^{\cup}
\end{array}
$$

$$= \quad \{ \quad \text{assumption: } f^{\cup} \circ f \;=\; g^{\cup} \circ g \quad \}$$

$$f \circ f^{\cup} \circ f \circ f^{\cup}$$

$$= \quad \{ \quad \text{assumption: } f \text{ is functional, i.e. } f \circ f^{\cup} = f< \quad \}$$

$$f< \quad .$$

That is,

(152) $\quad f \circ g^{\cup} \circ (f \circ g^{\cup})^{\cup} \;=\; f< \quad .$

Similarly,

(153) $\quad (f \circ g^{\cup})^{\cup} \circ f \circ g^{\cup} \;=\; g< \quad .$

Also,

$$g>$$

$$= \quad \{ \quad \text{domains} \quad \}$$

$$(g^{\cup} \circ g)>$$

$$= \quad \{ \quad \text{assumption: } f^{\cup} \circ f \;=\; g^{\cup} \circ g \quad \}$$

$$(f^{\cup} \circ f)>$$

$$= \quad \{ \quad \text{domains} \quad \}$$

$$f> \quad .$$

That is,

(154) $\quad f> \;=\; g> \quad .$

Hence,

$$f$$

$$= \quad \{ \quad \text{domains} \quad \}$$

$$f< \circ f$$

$$= \quad \{ \quad (152) \quad \}$$

$$f \circ g^{\cup} \circ (f \circ g^{\cup})^{\cup} \circ f$$

$$= \quad \{ \quad \text{properties of converse} \quad \}$$

$$f \circ g^{\cup} \circ g \circ f^{\cup} \circ f$$

$$= \quad \{ \quad \text{assumption: } f^{\cup} \circ f \;=\; g^{\cup} \circ g \quad \}$$

$$f \circ g^{\cup} \circ g \circ g^{\cup} \circ g$$

$$= \quad \{ \quad \text{assumption: } g \text{ is functional, i.e. } g \circ g^{\cup} = g_{<} \quad \}$$

$$f \circ g^{\cup} \circ g \quad .$$

Applying definition 82 with $R,S,\phi,\psi := f\ ,\ g\ ,\ f \circ g^{\cup}\ ,\ g_{>}$, we conclude that $f \cong g$. (Properties (152) and (153) are the required properties of ϕ; property (154) together with straightforward properties of the right-domain operator establish the required properties of ψ .)

□

It is important to note that theorem 151 assumes that there is at least one characterisation of per R by a functional relation; it thus establishes that there is *at most one* such characterisation ("up to isomorphism").

Uniqueness "up to isomorphism" is a common phenomenon. We see it again, for example, in the characterisation of difunctional relations by means of a pair of functional relations: section 6.2 shows that there is at most one characterisation whilst section 6.3 shows that there is at least one (in fact, that there are several). Dealing with this phenomenon can be awkward. See the definition of the "core" of a relation in section 7.3.

5.3 Decomposition of Provisional Preorders

In this section, we exploit the characterisation of pers, in particular the equivalence of theorem 143(i) and 143(iii), to show how a provisional preorder is decomposed into a per and a provisional ordering of the per's equivalence classes. (This generalises the well-known decomposition of a preorder into an equivalence relation and an ordering on the equivalence classes.)

We assume that T is a provisional preorder. That is, by definition 114 and lemma 118,

(155) $\quad T_{<} = T_{>} \ \wedge\ T_{<} \subseteq T \ \wedge\ T_{>} \subseteq T \ \wedge\ T \circ T \subseteq T \quad .$

Also, by lemma 120,

(156) $\quad T \cap T^{\cup} \ =\ T_{\prec} \ =\ T_{\succ} \quad .$

Theorem 157 Suppose T is a provisional preorder and assume that f partitions $T \cap T^{\cup}$ as prescribed by theorem 143(iii). ($T \cap T^{\cup}$ is a per by (156).) That is,

(158) $\quad f \circ f^{\cup} \ =\ f_{<} \ \wedge\ f^{\cup} \circ f \ =\ T \cap T^{\cup} \quad .$

Then the relation $f \circ T \circ f^{\cup}$ is a provisional ordering and $T = f^{\cup} \circ (f \circ T \circ f^{\cup}) \circ f$.

Proof The equation $\mathsf{T} = \mathsf{f}^{\cup} \circ (\mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup}) \circ \mathsf{f}$ is easily proved:

$$
\begin{array}{ll}
 & \mathsf{T} \\
= & \{ \quad \text{per domains} \quad \} \\
 & \mathsf{T}_{\prec} \circ \mathsf{T} \circ \mathsf{T}_{\succ} \\
= & \{ \quad (156) \quad \} \\
 & (\mathsf{T} \cap \mathsf{T}^{\cup}) \circ \mathsf{T} \circ (\mathsf{T} \cap \mathsf{T}^{\cup}) \\
= & \{ \quad (158) \quad \} \\
 & \mathsf{f}^{\cup} \circ \mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \circ \mathsf{f} \quad .
\end{array}
$$

We now prove that $\mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup}$ is a provisional ordering. It is transitive:

$$
\begin{array}{ll}
 & \mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \circ \mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \\
= & \{ \quad (156),\ (158) \text{ and per domains} \quad \} \\
 & \mathsf{f} \circ \mathsf{T} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \\
\subseteq & \{ \quad \text{assumption: } \mathsf{T} \text{ is transitive; monotonicity} \quad \} \\
 & \mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \quad .
\end{array}
$$

It is provisionally reflexive:

$$
\begin{array}{ll}
 & (\mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup})_{<} \ \subseteq \ \mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \\
\Leftarrow & \{ \quad [\, (\mathsf{R} \circ \mathsf{S})_{<} \subseteq \mathsf{R}_{<} \,] \text{ with } \mathsf{R},\mathsf{S} := \mathsf{f} \,,\, \mathsf{T} \circ \mathsf{f}^{\cup} \quad \} \\
 & \mathsf{f}_{<} \ \subseteq \ \mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \\
= & \{ \quad (158) \quad \} \\
 & \mathsf{f} \circ \mathsf{f}^{\cup} \ \subseteq \ \mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \\
= & \{ \quad \text{domains} \quad \} \\
 & \mathsf{f} \circ \mathsf{f}_{>} \circ \mathsf{f}^{\cup} \ \subseteq \ \mathsf{f} \circ \mathsf{T} \circ \mathsf{f}^{\cup} \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & \mathsf{f}_{>} \subseteq \mathsf{T} \\
\Leftarrow & \{ \quad [\ \mathsf{R}_{>} \ = \ \mathrm{I} \cap \mathsf{R}^{\cup} \circ \mathsf{R} \ ] \quad \} \\
 & \mathsf{f}^{\cup} \circ \mathsf{f} \subseteq \mathsf{T} \\
= & \{ \quad \text{by (158), } \mathsf{f}^{\cup} \circ \mathsf{f} = \mathsf{T} \cap \mathsf{T}^{\cup} \text{ ; infima} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

Finally, it is anti-symmetric:

$$
\begin{array}{ll}
 & f \circ T \circ f^{\cup} \cap (f \circ T \circ f^{\cup})^{\cup} \\
= & \{ \quad \text{converse} \quad \} \\
 & f \circ T \circ f^{\cup} \cap f \circ T^{\cup} \circ f^{\cup} \\
\subseteq & \{ \quad \text{modularity rules: (3) and (4)} \quad \} \\
 & f \circ (f^{\cup} \circ f \circ T \circ f^{\cup} \circ f \ \cap \ T^{\cup}) \circ f^{\cup} \\
= & \{ \quad \text{(156), (158) and per domains} \quad \} \\
 & f \circ (T \cap T^{\cup}) \circ f^{\cup} \\
= & \{ \quad \text{(158)} \quad \} \\
 & f \circ f^{\cup} \circ f \circ f^{\cup} \\
\subseteq & \{ \quad \text{(158) and domains} \quad \} \\
 & I \ .
\end{array}
$$

□

Fig. 4 (page 65) illustrates theorem 157: as mentioned earlier, the square boxes depict the equivalence classes and the arrows connecting the boxes depict the provisional ordering.

As we shall see, theorem 157 establishes that all provisional preorders are “block-ordered”. See example 228.

6 Difunctional Relations

This section is where our study of difunctional relations and block-ordered relations begins.

As Riguet remarked, difunctional relations generalise both functional relations [Rig48] and pers [Rig50, "quasi-equivalences"] in the sense that a difunctional relation is characterised by a pair of functional relations whilst a per is characterised by a single functional relation (theorem 143); equivalently, a difunctional relation is a union of completely disjoint rectangles whilst a per is the union of disjoint squares (theorem 143). See theorems 161 and 163. We present several different calculational proofs of theorem 161 in section 6.3 using both point-free and pointwise calculations, with a view to gaining insight into the efficacy and aesthetics of the calculational method. Note that, although the proofs are quite different, the constructed characterisations are essentially the same, as is made precise in section 6.2. Theorem 163 is a straightforward combination of theorem 161 and the (already-proven) theorem 141.

The "difunctional closure" of a relation is the smallest difunctional relation that is a superset of a given relation. Its definition and properties, given in section 6.4, involve the application of standard techniques of Galois connections and fixed-point calculus; as such, it is included here for completeness.

Whereas the "difunctional closure" of a relation is a superset of the relation, the "diagonal" of a relation is a subset of the relation. The "diagonal" of a relation is introduced in section 7. (Recall the mental picture, depicted in fig. 2, of the "diagonal" of the "staircase" relation depicted in fig. 1.)

Both the "diagonal" and the "difunctional closure" ("fermeture difonctionelle") are due to Riguet [Rig50, Rig51]; our contribution is partly historical —giving true credit to the original publications— , partly to make the constructions more accessible to modern readers, but primarily as an application of the calculational method.

6.1 Formal Definition and Characterisation

In this subsection we give the formal definition of a "difunctional relation" and state the theorem (theorem 161) that we prove in subsection 6.3. Theorem 161 uses the notion of a "characterisation" of a difunctional relation; this notion is also introduced in this subsection.

Formally, relation R is *difunctional* equivales

(159) $R \circ R^{\cup} \circ R \subseteq R$.

As for pers, there are several equivalent definitions of "difunctional". We begin with the simplest:

Theorem 160 For all R, the following statements are all equivalent.

(i) R is difunctional (i.e. $R \circ R^{\cup} \circ R \subseteq R$) ,

(ii) $R = R \circ R^{\cup} \circ R$,

(iii) $R{\succ} = R^{\cup} \circ R$,

(iv) $R{\prec} = R \circ R^{\cup}$,

(v) $R = R \cap (R\backslash R/R)^{\cup}$.

Proof For the equivalence of (i) and (ii), we first observe that, for all R,

$$R \subseteq R \circ R^{\cup} \circ R$$

since

$$
\begin{array}{ll}
 & R \subseteq R \circ R^{\cup} \circ R \\
\Leftarrow & \{ \quad R{>} \subseteq R^{\cup} \circ R \text{ and monotonicity} \quad \} \\
 & R = R \circ R{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

That (i) and (ii) are equivalent thus follows from the anti-symmetry of the subset relation.

For the equivalence of (i) and (iii), we again begin by observing a property that holds for all R, namely

$$R^{\cup} \circ R \supseteq R{\succ} \ .$$

The proof is as follows:

$$
\begin{array}{ll}
 & R^{\cup} \circ R \supseteq R{\succ} \\
= & \{ \quad \text{definition: (96)} \quad \} \\
 & R^{\cup} \circ R \supseteq R{>} \circ R\backslash\backslash R \\
= & \{ \quad \text{cancellation: (90)} \quad \} \\
 & R^{\cup} \circ R \circ R\backslash\backslash R \supseteq R{>} \circ R\backslash\backslash R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R^{\cup} \circ R \supseteq R{>} \\
\Leftarrow & \{ \quad \text{definition 42} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

We now prove that the opposite inclusion follows from (i).

$$
\begin{array}{ll}
 & R^{\cup}\circ R \;\subseteq\; R{\succ} \\
= & \{\quad \text{definition: (96)} \quad\} \\
 & R^{\cup}\circ R \;\subseteq\; R{>}\circ R\backslash\!\!\backslash R \\
\Leftarrow & \{\quad R{>}\circ R^{\cup} = R^{\cup} \ \text{ and monotonicity} \quad\} \\
 & R^{\cup}\circ R \;\subseteq\; R\backslash\!\!\backslash R \\
= & \{\quad R^{\cup}\circ R \ \text{ is symmetric, } \ R\backslash\!\!\backslash R = R\backslash R\cap(R\backslash R)^{\cup} \quad\} \\
 & R^{\cup}\circ R \;\subseteq\; R\backslash R \\
\Leftarrow & \{\quad \text{factors} \quad\} \\
 & R\circ R^{\cup}\circ R \;\subseteq\; R \ \ .
\end{array}
$$

Thus, by anti-symmetry, (iii) follows from (i). But

$$
\begin{array}{ll}
 & R{\succ} \;=\; R^{\cup}\circ R \\
\Rightarrow & \{\quad \text{Leibniz} \quad\} \\
 & R\circ R{\succ} \;=\; R\circ R^{\cup}\circ R \\
= & \{\quad \text{per domains} \quad\} \\
 & R \;=\; R\circ R^{\cup}\circ R \ \ .
\end{array}
$$

That is, (iii) implies (ii) which, as we have shown, is equivalent to (i). We conclude, by mutual implication, that (iii) and (i) are equivalent.

The equivalence of (i) and (iv) is obtained by instantiating R to $R^{\cup}$.

The proof that (v) is equivalent to (159) is straightforward:

$$
\begin{array}{ll}
 & R \;=\; R\cap(R\backslash R/R)^{\cup} \\
= & \{\quad \text{definition of infimum} \quad\} \\
 & R\subseteq(R\backslash R/R)^{\cup} \\
= & \{\quad \text{converse and factors} \quad\} \\
 & R\circ R^{\cup}\circ R \;\subseteq\; R \ \ .
\end{array}
$$

□

The equivalence of 160(i) and 160(ii) is well-known and due to Riguet [Rig48]; the equivalence of 160(i), (iii) and (iv) is due to Voermans [Voe99]. Definition (159) is the most useful when it is required to establish that a particular relation is difunctional, whereas properties 160(ii)-(iv) are more useful when it is required to exploit the fact that a particular relation is difunctional.

In order to relate this formal definition to the informal mental picture, an important step on the way is to characterise difunctional relations via a pair of functional relations. Recall that a relation R is said to be *functional* iff $R \circ R^{\cup} = R{<}$ (where $R{<}$ denotes the left domain of R: see definition 42). We use lower case letters f, g, h and k to denote functional relations. The theorem is the following.

Theorem 161 (Characterisation Theorem) For all relations R,

$$R \text{ is difunctional} \;\equiv\; \left\langle \exists f,g \;:\; f \circ f^{\cup} \;=\; f{<} \;=\; g \circ g^{\cup} \;=\; g{<} \;:\; R = f^{\cup} \circ g \right\rangle \;\; .$$

□

Theorem 161 —which is due to Riguet [Rig50]— is key to establishing the property that difunctional relations are exactly the relations that fit the mental picture shown in fig. 2 of a collection of completely disjoint rectangles. Later, we say that difunctional relations are "characterised" by a pair of functional relations. The formal definition is as follows.

Definition 162 A *characterisation (of a difunctional relation)* is a pair of functional relations with the same target (but possibly different sources). A *minimal characterisation (of a difunctional relation)* is a pair of relations f and g with the same target such that

$$f \circ f^{\cup} \;=\; f{<} \;=\; g \circ g^{\cup} \;=\; g{<} \;\; .$$

That is, a minimal characterisation is a pair of functional relations with equal left domains.

□

The mental picture of a difunctional relation (fig. 2) is a set of completely disjoint rectangles. We can now make the picture precise.

Recall the definition of minimal characterisations, definition 162. Theorem 141 expresses the equivalence of minimal characterisations with sets of completely disjoint rectangles. So, by combining theorems 161 and 141, we have:

Theorem 163 A relation R is difunctional if and only if it is the supremum of a set of completely disjoint rectangles.

□

The "minimality" requirement —the domain restrictions on f and g— may be omitted ("without loss of generality" in mathematical jargon). It is necessary, however, to establishing the "essential" uniqueness of the characterisation. (See theorem 166.) Formally we have:

Lemma 164 Suppose f and g are functional relations with the same target. Then

$$f^{\cup}\circ g \quad = \quad (g{<}\circ f)^{\cup}\circ(f{<}\circ g) \quad .$$

Moreover, $g{<}\circ f$ and $f{<}\circ g$ are functional relations and

$$(g{<}\circ f)\circ(g{<}\circ f)^{\cup} \quad = \quad (g{<}\circ f){<} \quad = \quad (f{<}\circ g)\circ(f{<}\circ g)^{\cup} \quad = \quad (f{<}\circ g){<} \quad .$$

That is, the pair $g{<}\circ f$ and $f{<}\circ g$ is a minimal characterisation.

Proof We show that $g{<}\circ f$ is functional as follows.

$$
\begin{array}{cl}
 & (g{<}\circ f)\circ(g{<}\circ f)^{\cup} \\
= & \{ \quad \text{associativity and converse} \quad \} \\
 & g{<}\circ f\circ f^{\cup}\circ g{<} \\
= & \{ \quad f \text{ is functional, so } f\circ f^{\cup} = f{<} \quad \} \\
 & g{<}\circ f{<}\circ g{<} \\
= & \{ \quad \text{coreflexives commute and are idempotent} \quad \} \\
 & f{<}\circ g{<} \quad .
\end{array}
$$

Symmetrically,

$$(f{<}\circ g)\circ(f{<}\circ g)^{\cup} \;=\; g{<}\circ f{<} \quad .$$

That is, $f{<}\circ g$ is functional. The lemma follows immediately from the fact that composition of coreflexives is symmetric and yields a coreflexive.
□

The characterisation theorem for difunctional relations (theorem 161) has the consequence that a difunctional relation divides its left and right domains into classes that are in (1–1) correspondence.

Lemma 165 Suppose f and g are relations with common target C such that

$$f\circ f^{\cup} \;=\; f{<} \;=\; g\circ g^{\cup} \;=\; g{<} \quad .$$

Then the functions $\langle X :: g^{\cup}\circ f\circ X\circ f^{\cup}\circ g\rangle$ and $\langle Y :: f^{\cup}\circ g\circ Y\circ g^{\cup}\circ f\rangle$ define a (1–1) correspondence between the classes of the partial equivalence relations $f^{\cup}\circ f$ and $g^{\cup}\circ g$. That is, for all c,

$$\big\langle X :: g^{\cup}\circ f\circ X\circ f^{\cup}\circ g\big\rangle.(f^{\cup}\circ c\circ f) \quad = \quad g^{\cup}\circ c\circ g$$

and

$$\big\langle Y :: f^{\cup}\circ g\circ Y\circ g^{\cup}\circ f\big\rangle.(g^{\cup}\circ c\circ g) \quad = \quad f^{\cup}\circ c\circ f \quad .$$

Proof The verification of the first equality is as follows.

$$
\begin{array}{ll}
 & \langle X :: g^{\cup} \circ f \circ X \circ f^{\cup} \circ g \rangle \, . \, (f^{\cup} \circ c \circ f) \\
= & \{ \quad \text{definition of function application} \quad \} \\
 & g^{\cup} \circ f \circ f^{\cup} \circ c \circ f \circ f^{\cup} \circ g \\
= & \{ \quad \text{assumption: } f \circ f^{\cup} = f< = g \circ g^{\cup} = g< \quad \} \\
 & g^{\cup} \circ g< \circ c \circ g< \circ g \\
= & \{ \quad \text{domains} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

The second equality is verified in the same way.
□

See also section 7.3 for a more explicit formulation of lemma 165.

Warning Symmetry places a major rôle in reasoning about difunctional relations. (Obviously, R is difunctional equivales $R^{\cup}$ is difunctional.) But our definition of "functional" is asymmetric and reflects a right-to-left bias in our interpretation of relations as having inputs and outputs. Jaoua et al [JMBD91] choose a left-to-right interpretation: they use the term "deterministic" to mean $R^{\cup} \circ R \subseteq I$. Their formulation of theorem 161 is correspondingly different. See also our earlier warning on "symmetric division". **End of Warning**

The name "difunctional" is suggestive of theorem 161; Riguet's 1948 paper [Rig48, Proposition 11] introduces the notion and gives a (natural-language-based) proof. Riguet's 1950 paper [Rig50] states that it is a generalisation of the theorem that a relation R is a partial equivalence relation equivales $R = f^{\cup} \circ f$ for some functional relation f. Since then it appears to have become a folklore theorem. Hutton and Voermans [GE92, lemma 39], for example, state the theorem but do not provide a proof nor an attribution. The English text of [SS93, p.75] suggests that Schmidt and Ströhlein may be aware of the theorem but they also do not provide a proof. (They prove the easy "if" part of the theorem but not the converse; [SS93, Proposition 4.4.10] states that the characterisation "may be achieved *in essentially one* fashion" (their emphasis) but the accompanying proof actually establishes that the characterisation can be achieved in *at most* one fashion. That is, if such a characterisation exists, it is unique "up to a bijection".)

A theme of this section is how to formalise different proofs of theorem 161. One issue is whether or not the so-called "power transpose" of a relation, espoused by Freyd and Ščedrov [Fv90] and Bird and De Moor [BdM97], is sufficiently expressive. A second issue is the extent to which pointwise (as opposed to point-free) reasoning is desirable.

Section 6.2 sets the scene. The proof of theorem 161 is an "if-and-only-if" proof and the section begins with the (trivial) proof of the "if" part. The main task is thus to give an explicit construction of a characterisation of a given difunction (the “only-if” part). A formal theorem —theorem 166— states that although the details of the proof may be different, the constructed characterisations are formally equivalent (in a way made precise by the theorem). A very informal outline of several different ways of making the construction is then given.

The informal account in section 6.2 is made precise in sections 6.3.1 and 6.3.2; the former proves theorem 161 by showing how to construct a set of “rectangles” that “covers” a given difunctional relation whilst the latter presents a construction in terms of the “power transpose” of the given relation. Section 6.3.3 gives a third method of proving theorem 161 that exploits theorem 143. As already remarked —see theorem 163— no matter how a characterisation is constructed, it defines a “completely disjoint covering” of the given difunction.

6.2 Different Proofs, Identical Characterisations

The proof of theorem 161 is by mutual implication. Follows-from is straightforward. Assume

$$\left\langle \exists f,g : f\circ f^{\cup} \;=\; f_{<} \;=\; g\circ g^{\cup} \;=\; g_{<} : R = f^{\cup}\circ g \right\rangle \;\; .$$

Then

$$
\begin{array}{ll}
 & R\circ R^{\cup}\circ R \\
= & \{ \quad \text{assumption and converse} \quad \} \\
 & f^{\cup}\circ g\circ g^{\cup}\circ f\circ f^{\cup}\circ g \\
= & \{ \quad \text{assumption: } f\circ f^{\cup} = g_{<} = g\circ g^{\cup} \quad \} \\
 & f^{\cup}\circ g_{<}\circ g_{<}\circ g \\
= & \{ \quad g_{<}\circ g = g \text{, and } R = f^{\cup}\circ g \quad \} \\
 & R \;\; .
\end{array}
$$

The much more demanding task —which occupies all of subsection 6.3— is to establish the existence of a (minimal) characterisation of a given difunction. The theorem that there is *at most one* (up to isomorphism) is the following.

Theorem 166 Suppose f and g are relations such that

$$f\circ f^{\cup} \;=\; f_{<} \;=\; g\circ g^{\cup} \;=\; g_{<} \;\; .$$

Suppose also that h and k are relations such that

$$h\circ h^{\cup} \;=\; h_{<} \;=\; k\circ k^{\cup} \;=\; k_{<} \;\; .$$

Suppose further that

$$f^{\cup}\circ g \;=\; h^{\cup}\circ k \quad .$$

Then

$$f \cong h \;\wedge\; g \cong k \quad .$$

Proof Our task is to construct witnesses ϕ and ψ satisfying definition 82 (with $R{,}S := f{,}h$ and $R{,}S := g{,}k$). Define ϕ by $\phi = f\circ h^{\cup}$. We prove that

$$(167)\quad \phi\circ\phi^{\cup} \;=\; f{<} \;\wedge\; \phi^{\cup}\circ\phi \;=\; h{<} \quad .$$

(In words, ϕ is a bijection with left domain the common left domain of f and g, and right domain the common left domain of h and k.) The proof is as follows.

$$\begin{array}{ll}
 & \phi\circ\phi^{\cup} \\
= & \quad\{\quad \text{definition, converse} \quad\} \\
 & f\circ h^{\cup}\circ h\circ f^{\cup} \\
= & \quad\{\quad \text{assumption: } h{<} \;=\; k\circ k^{\cup} \quad\} \\
 & f\circ h^{\cup}\circ k\circ k^{\cup}\circ h\circ f^{\cup} \\
= & \quad\{\quad \text{assumption: } f^{\cup}\circ g \;=\; h^{\cup}\circ k \quad\} \\
 & f\circ f^{\cup}\circ g\circ g^{\cup}\circ f\circ f^{\cup} \\
= & \quad\{\quad \text{assumption: } f\circ f^{\cup} \;=\; f{<} \;=\; g\circ g^{\cup} \quad\} \\
 & f{<}
\end{array}$$

and

$$\begin{array}{ll}
 & \phi^{\cup}\circ\phi \\
= & \quad\{\quad \text{definition, converse} \quad\} \\
 & h\circ f^{\cup}\circ f\circ h^{\cup} \\
= & \quad\{\quad \text{assumption: } f{<} \;=\; g\circ g^{\cup} \quad\} \\
 & h\circ f^{\cup}\circ g\circ g^{\cup}\circ f\circ h^{\cup} \\
= & \quad\{\quad \text{assumption: } f^{\cup}\circ g \;=\; h^{\cup}\circ k \text{ (used twice)} \quad\} \\
 & h\circ h^{\cup}\circ k\circ k^{\cup}\circ h\circ h^{\cup} \\
= & \quad\{\quad \text{assumption: } h\circ h^{\cup} \;=\; h{<} \;=\; k\circ k^{\cup} \quad\} \\
 & h{<} \quad .
\end{array}$$

We now prove that $f = \phi \circ h$.

$$
\begin{array}{ll}
 & \phi \circ h \\
= & \{ \quad \text{definition} \quad \} \\
 & f \circ h^{\cup} \circ h \\
= & \{ \quad \text{assumption: } h< \;=\; k \circ k^{\cup} \quad \} \\
 & f \circ h^{\cup} \circ k \circ k^{\cup} \circ h \\
= & \{ \quad \text{assumption: } f^{\cup} \circ g \;=\; h^{\cup} \circ k \text{ (used twice)} \quad \} \\
 & f \circ f^{\cup} \circ g \circ g^{\cup} \circ f \\
= & \{ \quad \text{assumption: } f \circ f^{\cup} \;=\; f< \;=\; g \circ g^{\cup} \quad \} \\
 & f \;\; .
\end{array}
$$

It follows that

(168) $\quad f \;=\; \phi \circ h \circ h> \quad \wedge \quad h> \;=\; f> \;\; .$

The combination of (167) and (168) (together with straightforward properties of $h>$) establishes that ϕ and $h>$ witness the isomorphism $f \cong h$. The property $g \cong k$ is proved similarly.

□

As the name “functional” suggests, the only-if part of theorem 161 is established by defining a type C, for each a in the left domain of R, a point $f.a$ in C, and, for each point b in the right domain of R, a point $g.b$ in C. The requirement is that, $f.a$ and $g.b$ are equal exactly when a and b are related by R. Fig. 5 shows three different but isomorphic (in the sense of theorem 166) characterisations of the relation depicted in fig. 2.

In the top-left figure, the type C is the set of rectangles (relations of type $A{\sim}B$) defined by the relation R: the functional relation f maps a point a in the left domain of R to the rectangle defined by a and, similarly, the functional relation g maps a point b in the right domain of R to the rectangle defined by b. If a and b are points related by R, the rectangles $f.a$ and $g.b$ are equal; if a and b are not related by R, the rectangles $f.a$ and $g.b$ are not equal (and, in fact, they are “completely disjoint” in the sense that there are no points common to their sides).

In the top-right figure, the type C is a set of squares of type $B{\sim}B$ and, in the bottom-left figure the type C is a set of squares of type $A{\sim}A$. In the case of the top-right figure, the functional relation g maps point b to the square defined by b. The definition of f is more complicated: for a point a in the left domain of R, the value of $f.a$ is the square defined by some point b such that a and b are points related by

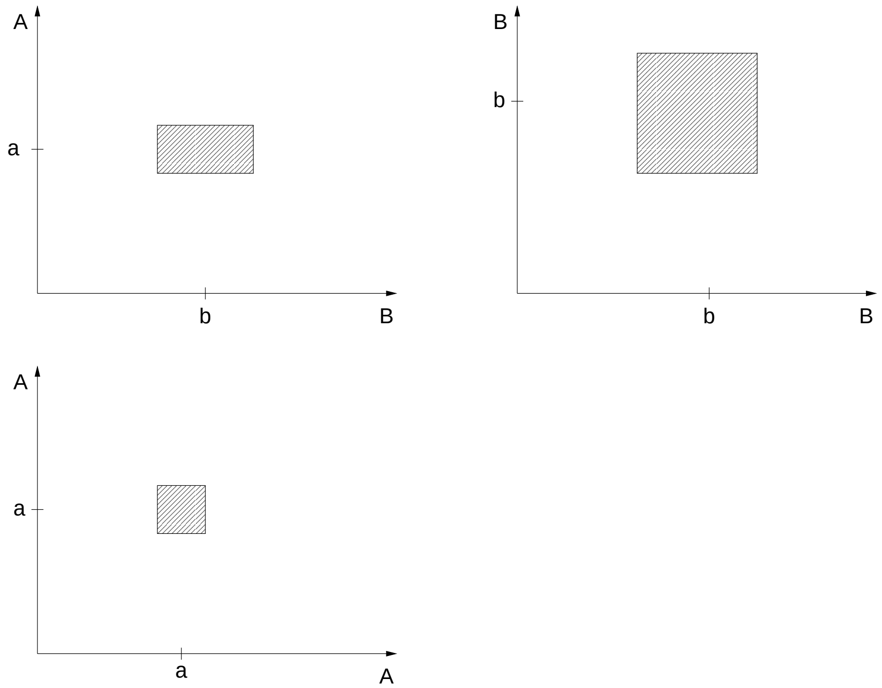


Figure 5: Three Different (but Isomorphic) Characterisations

R. The definitions of f and g are similar in the case of the bottom-left figure. (Just interchange the rôles of a and b.)

Of course, a "square" is defined by a "side" of the square. So there is a fourth and a fifth way of representing a difunctional relation as a pair of functional relations: the type C can be defined to be the set of subsets of the left domain of R or the set of subsets of the right domain of R and, in each case, appropriate definitions of f and g must be constructed.

As mentioned earlier, all of these characterisations are the same — in the sense made precise by theorem 166.

6.3 The Characterisation Theorem

As illustrated by fig. 5, there are three different ways to approach the proof[4] of theorem 161. The top-right and bottom-left figures are "dual" in the sense that one depicts a homogeneous relation on the target of the given relation whilst the other depicts a homogeneous relation on the source of the given relation. The top-left figure is more attractive because it does not exhibit any bias towards the source or target of the given relation. Section 6.3.1 presents such an unbiased proof of theorem 161 whilst section

[4]Strictly, the "only-if" part of the proof. Recall from section 6.2 that the "if" part is trivial.

6.3.2 presents the dual proofs. Section 6.3.3 gives yet another proof based on exploiting theorem 143.

6.3.1 The Rectangle Proof

A relation R is a partial equivalence relation exactly when $R \circ R^{\cup} = R$; the "classes" of R are the squares $R \circ a \circ R^{\cup}$ where a is a point such that $a \subseteq R$. A relation R is a difunction exactly when $R \circ R^{\cup} \circ R = R$. By analogy and type considerations, this suggests that, if $a \subseteq R<$, the rectangle defined by a is given by $R \circ R^{\cup} \circ a \circ R$; similarly, if $b \subseteq R>$, the rectangle defined by b is given by $R \circ b \circ R^{\cup} \circ R$. This is the key to the proof.

Lemma 169 Suppose R of type $A{\sim}B$ is difunctional. Then, for all points a and b,

$$a \circ \top\top \circ b \subseteq R \quad \Rightarrow \quad R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R \quad .$$

Proof Assume R is difunctional. Assume also that $a \circ \top\top \circ b \subseteq R$. Then

$$
\begin{array}{ll}
 & R \circ b \circ R^{\cup} \circ R \\
= & \{ \quad b \text{ is a point} \quad \} \\
 & R \circ b \circ b \circ R^{\cup} \circ R \\
\subseteq & \{ \quad \text{assumption: } a \circ \top\top \circ b \subseteq R \text{ , lemma 57} \quad \} \\
 & R \circ b \circ R^{\cup} \circ a \circ R \circ R^{\cup} \circ R \\
\subseteq & \{ \quad R \text{ is difunctional} \quad \} \\
 & R \circ b \circ R^{\cup} \circ a \circ R \quad .
\end{array}
$$

That is,

$$R \circ b \circ R^{\cup} \circ R \;\subseteq\; R \circ b \circ R^{\cup} \circ a \circ R \quad .$$

By a symmetric argument

$$R \circ R^{\cup} \circ a \circ R \;\subseteq\; R \circ b \circ R^{\cup} \circ a \circ R \quad .$$

But, since a is a point (and thus coreflexive),

$$R \circ b \circ R^{\cup} \circ a \circ R \;\subseteq\; R \circ b \circ R^{\cup} \circ R \quad .$$

Symmetrically,

$$R \circ b \circ R^{\cup} \circ a \circ R \;\subseteq\; R \circ R^{\cup} \circ a \circ R \quad .$$

The lemma follows by the anti-symmetry of equality.

□

The "only-if" part of theorem 161 is a consequence of lemma 169. Specifically, suppose R is difunctional. Let C be the set of subsets of the relation R defined as follows:

$$C \;=\; \{a : a \subseteq R< : R \circ R^{\cup} \circ a \circ R\} \;\;.$$

(The dummy a ranges over points.) Note that $C = C'$ where

$$C' \;=\; \{b : b \subseteq R> : R \circ b \circ R^{\cup} \circ R\}$$

since

$$
\begin{array}{ll}
 & \{a : a \subseteq R< : R \circ R^{\cup} \circ a \circ R\} \\
= & \{ \quad \text{domains} \quad \} \\
 & \{a \;:\; \langle \exists b :: a{\circ}R{\circ}b = a{\circ}\top\!\top{\circ}b \rangle \;:\; R \circ R^{\cup} \circ a \circ R\} \\
= & \{ \quad \text{range disjunction} \quad \} \\
 & \{a,b : a{\circ}R{\circ}b = a{\circ}\top\!\top{\circ}b : R \circ R^{\cup} \circ a \circ R\} \\
= & \{ \quad \text{assumption: } R \text{ is difunctional; lemma 169} \quad \} \\
 & \{a,b : a{\circ}R{\circ}b = a{\circ}\top\!\top{\circ}b : R \circ b \circ R^{\cup} \circ R\} \\
= & \{ \quad \text{range disjunction and domains (as in first two steps)} \quad \} \\
 & \{b : b \subseteq R> : R \circ b \circ R^{\cup} \circ R\} \;\;.
\end{array}
$$

Define f and g by, for all points a such that $a \subseteq R<$ and all points b such that $b \subseteq R>$,

$$(170) \quad f.a \;=\; R \circ R^{\cup} \circ a \circ R \quad \wedge \quad g.b \;=\; R \circ b \circ R^{\cup} \circ R \;\;.$$

Then, by definition, f and g are both functional, and surjective onto C and C', respectively. That is —exploiting the fact that C and C' are equal—

$$f \circ f^{\cup} = I_C = g \circ g^{\cup} \;\;.$$

We must now show that $R = f^{\cup} \circ g$. Guided by the definitions of f and g, we calculate that:

$$
\begin{array}{ll}
 & R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & R \circ R^{\cup} \circ a \circ R \circ R^{\cup} \;=\; R \circ b \circ R^{\cup} \circ R \circ R^{\cup} \\
\Rightarrow & \{ \quad \text{assumption: } R \text{ is difunctional (thus so too is } R^{\cup}\text{)}, \\
 & \qquad R< \subseteq R \circ R^{\cup} \quad \} \\
 & R< \circ a \circ R< \subseteq R \circ b \circ R^{\cup}
\end{array}
$$

$$\begin{array}{ll}
= & \{ \quad \text{assumption: } a \subseteq R_{<} \quad \} \\
& a \subseteq R \circ b \circ R^{\cup} \\
= & \{ \quad \text{lemma 57} \quad \} \\
& a \circ \top\!\top \circ b \subseteq R \\
\Rightarrow & \{ \quad \text{assumption: } R \text{ is difunctional; lemma 169} \quad \} \\
& R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R \quad .
\end{array}$$

We conclude (by mutual implication) that

$$R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R \quad \equiv \quad a \circ \top\!\top \circ b \subseteq R \quad .$$

But, by the definitions of f and g and the definition of function application,

$$R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R \quad \equiv \quad a \circ \top\!\top \circ b \;\subseteq\; f^{\cup} \circ g \quad .$$

Thus $R = f^{\cup} \circ g$ by the saturation axiom: (16).

6.3.2 The Power-Transpose Construction

Recalling fig. 5 once again, two alternative —but dual— ways of proving theorem 161 are to construct functional relations that return square relations. Equivalently, one can construct functional relations that return the "side" of such a square, i.e. a subset of the source or, dually, a subset of the target of the given difunctional relation. In this section, we present such a construction using the power transpose function. The proof was obtained by revising the proof given by Jaoua et al [JMBD91] in a way that eliminated the unnecessary assumption that R is homogeneous. One component of the characterisation is the relation $\Gamma R \circ R^{\cup}$. Since this is not obviously functional, we need a lemma to show that it is.

Lemma 171 For all relations R,

$$R \text{ is difunctional} \quad \equiv \quad \Gamma R \circ R^{\cup} \;\subseteq\; \Gamma(R \circ R^{\cup}) \circ R_{<} \quad .$$

Proof

$$\begin{array}{ll}
& \Gamma R \circ R^{\cup} \;\subseteq\; \Gamma(R \circ R^{\cup}) \circ R_{<} \\
= & \{ \quad \text{domains (specifically, } R^{\cup} \circ R_{<} = R^{\cup}\text{)} \quad \} \\
& \Gamma R \circ R^{\cup} \;\subseteq\; \Gamma(R \circ R^{\cup}) \\
= & \{ \quad \Gamma R \text{ is a total function; shunting rule} \quad \} \\
& R^{\cup} \;\subseteq\; (\Gamma R)^{\cup} \circ \Gamma(R \circ R^{\cup})
\end{array}$$

$$= \quad \{ \quad \text{lemma 87} \quad \}$$

$$R^{\cup} \;\subseteq\; R\backslash(R\circ R^{\cup}) \,\cap\, ((R\circ R^{\cup})\backslash R)^{\cup}$$

$$= \quad \{ \quad \text{converse is an order isomorphism, factors} \quad \}$$

$$R\circ R^{\cup} \subseteq R\circ R^{\cup} \quad\wedge\quad R\circ R^{\cup}\circ R \subseteq R$$

$$= \quad \{ \quad \text{definition} \quad \}$$

$$R \text{ is difunctional} \quad .$$

□

Corollary 172 For all difunctional relations R,

$$(\Gamma R\circ R^{\cup}) \;\circ\; (\Gamma R\circ R^{\cup})^{\cup} \quad=\quad \Gamma R \circ R{>}\circ(\Gamma R)^{\cup} \quad .$$

In particular, if R is difunctional, $\Gamma R\circ R^{\cup}$ is functional.

Proof The proof is by mutual inclusion. First, for all relations R,

$$(\Gamma R\circ R^{\cup}) \;\circ\; (\Gamma R\circ R^{\cup})^{\cup}$$

$$= \quad \{ \quad \text{converse} \quad \}$$

$$\Gamma R \circ R^{\cup}\circ R\circ(\Gamma R)^{\cup}$$

$$\supseteq \quad \{ \quad R^{\cup}\circ R \supseteq R{>} \text{ , monotonicity} \quad \}$$

$$\Gamma R \circ R{>}\circ(\Gamma R)^{\cup} \quad .$$

Second, for all difunctional relations R,

$$\Gamma R \circ R^{\cup}\circ R\circ(\Gamma R)^{\cup} \;\subseteq\; \Gamma R \circ R{>}\circ(\Gamma R)^{\cup}$$

$$\Leftarrow \quad \{ \quad \text{assumption: } R \text{ is difunctional; lemma 171} \quad \}$$

$$\Gamma(R\circ R^{\cup})\circ R{<}\circ(\Gamma(R\circ R^{\cup}))^{\cup} \;\subseteq\; \Gamma R \circ R{>}\circ(\Gamma R)^{\cup}$$

$$= \quad \{ \quad \Gamma(R\circ R^{\cup}) \text{ is a total function, shunting : (85), and (89)} \quad \}$$

$$R{<} \;\subseteq\; (R\circ R^{\cup})\backslash\!\backslash R \circ R{>}\circ((R\circ R^{\cup})\backslash\!\backslash R)^{\cup}$$

$$\Leftarrow \quad \{ \quad \text{domains (specifically } R{<} \subseteq R\circ R^{\cup} \text{ and } R = R\circ R{>}\text{)} \quad \}$$

$$R\circ R{>}\circ R^{\cup} \;\subseteq\; (R\circ R^{\cup})\backslash\!\backslash R \circ R{>}\circ((R\circ R^{\cup})\backslash\!\backslash R)^{\cup}$$

$$\Leftarrow \quad \{ \quad \text{monotonicity and converse} \quad \}$$

$$R \;\subseteq\; (R\circ R^{\cup})\backslash\!\backslash R$$

$$= \quad \{ \quad \text{assumption: } R \text{ is difunctional}$$
$$\text{as in last two steps of proof of lemma 171} \quad \}$$

$$\text{true} \quad .$$

□

Theorem 173 Suppose R is a difunctional relation. Then the relations $\Gamma R \circ R^{\cup}$ and $\Gamma R \circ R_{>}$ are both functional. Moreover,

$$(\Gamma R \circ R^{\cup}) \circ (\Gamma R \circ R^{\cup})^{\cup} \quad = \quad (\Gamma R \circ R_{>}) \circ (\Gamma R \circ R_{>})^{\cup}$$

and

$$R \quad = \quad (\Gamma R \circ R^{\cup})^{\cup} \circ (\Gamma R \circ R_{>}) \quad .$$

That is, these two relations fulfill the requirements of f and g in theorem 161.

Dually, the relations $\Gamma(R^{\cup}) \circ R$ and $\Gamma(R^{\cup}) \circ R_{<}$ are both functional. Moroever,

$$(\Gamma(R^{\cup}) \circ R_{<}) \circ (\Gamma(R^{\cup}) \circ R_{<})^{\cup} \quad = \quad (\Gamma(R^{\cup}) \circ R) \circ (\Gamma(R^{\cup}) \circ R)^{\cup}$$

and

$$R \quad = \quad (\Gamma(R^{\cup}) \circ R_{<})^{\cup} \circ (\Gamma(R^{\cup}) \circ R) \quad .$$

That is, these two functions also fulfill the requirements of f and g theorem 161.

Proof That $\Gamma R \circ R_{>}$ is functional is immediate from the fact that ΓR is a total function (by definition) and $R_{>}$ is a subset of the identity relation. That $\Gamma R \circ R^{\cup}$ is functional was shown in corollary 172. It remains to prove the final equation.

$$
\begin{array}{ll}
 & (\Gamma R \circ R^{\cup})^{\cup} \circ (\Gamma R \circ R_{>}) \\
= & \{ \quad \text{converse} \quad \} \\
 & R \circ (\Gamma R)^{\cup} \circ \Gamma R \circ R_{>} \\
= & \{ \quad (89) \quad \} \\
 & R \circ R \backslash\backslash R \circ R_{>} \\
= & \{ \quad \text{lemma 90} \quad \} \\
 & R \circ R_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & R \quad .
\end{array}
$$

The dual theorem is obtained by instantiating R to $R^{\cup}$ (and noting that R is difunctional equivales $R^{\cup}$ is difunctional) and simplifying.

□

Theorem 144 is an instance of theorem 173. In order to show that this is the case, it is necessary to prove that, for a per R,

$$\Gamma R \circ R^{\cup} \;=\; \Gamma R \circ R{>} \quad .$$

This is done as follows:

$$\begin{array}{ll}
 & \Gamma R \circ R^{\cup} \;=\; \Gamma R \circ R{>} \\
= & \{ \quad R \text{ is a per, so } R^{\cup} = R\,;\text{ lemma 92} \quad \} \\
 & \Gamma R \circ R \;\subseteq\; \Gamma R \circ R{>} \\
\Leftarrow & \{ \quad \Gamma R \text{ is functional} \quad \} \\
 & R \;\subseteq\; (\Gamma R)^{\cup} \circ \Gamma R \circ R{>} \\
= & \{ \quad \text{lemma 87} \quad \} \\
 & R \;\subseteq\; R\backslash\!\backslash R \circ R{>} \\
= & \{ \quad \text{definition 95 and theorem 108} \quad \} \\
 & \mathsf{true} \quad .
\end{array}$$

6.3.3 The Per Construction

The third method of proving theorem 161 exploits theorem 144. We owe the construction to Winter [Win04].

The basis for the construction is the construction of a per from a difunctional relation:

Lemma 174 For all relations R, $R \circ R^{\cup}$ is a per if R is difunctional.

Proof Suppose R is difunctional. We exploit theorem 93:

$$\begin{array}{ll}
 & R \circ R^{\cup} \text{ is a per} \\
= & \{ \quad \text{theorem 93 with } R := R \circ R^{\cup} \text{ and converse} \quad \} \\
 & R \circ R^{\cup} \;=\; R \circ R^{\cup} \circ R \circ R^{\cup} \\
\Leftarrow & \{ \quad \text{Leibniz} \quad \} \\
 & R \;=\; R \circ R^{\cup} \circ R \\
= & \{ \quad \text{theorem 160} \quad \} \\
 & R \text{ is difunctional.}
\end{array}$$

□

Suppose now that R is difunctional. Exploiting lemma 174 combined with theorem 143,

(175) $\langle \exists f : f \circ f^{\cup} = f_{<} : R \circ R^{\cup} = f^{\cup} \circ f \rangle$.

Suppose therefore that $f \circ f^{\cup} = f_{<}$ and $R \circ R^{\cup} = f^{\cup} \circ f$. Define the relation g by

(176) $g = f \circ R$.

Then

$$
\begin{array}{ll}
 & g \circ g^{\cup} \\
= & \{ \quad (176) \text{ and converse} \quad \} \\
 & f \circ R \circ R^{\cup} \circ f^{\cup} \\
= & \{ \quad (175) \quad \} \\
 & f \circ f^{\cup} \circ f \circ f^{\cup} \\
= & \{ \quad (175) \quad \} \\
 & f_{<} \circ f_{<} \\
= & \{ \quad f_{<} \text{ is a coreflexive} \quad \} \\
 & f_{<} \ .
\end{array}
$$

It follows that $g_{<} = g \circ g^{\cup}$. Thus

(177) $f \circ f^{\cup} = f_{<} = g_{<} = g \circ g^{\cup}$.

Moreover,

$$
\begin{array}{ll}
 & f^{\cup} \circ g \\
= & \{ \quad (176) \quad \} \\
 & f^{\cup} \circ f \circ R \\
= & \{ \quad R \circ R^{\cup} = f^{\cup} \circ f \quad \} \\
 & R \circ R^{\cup} \circ R \\
= & \{ \quad R \text{ is difunctional: theorem 160} \quad \} \\
 & R \ .
\end{array}
$$

Combined with (177), we have thus shown that

(178) $\langle \exists f,g : f \circ f^{\cup} = f_{<} = g \circ g^{\cup} = g_{<} : R = f^{\cup} \circ g \rangle$

as required to prove the only-if part of theorem 161.

Winter does not prove theorem 144; instead he assumes the theorem is valid. It is interesting to compare the details of Winter's construction with the functionals constructed in theorem 173. Applying the instantiation $R := R \circ R^{\cup}$ in theorem 144 and simplifying, Winter's construction yields

$$R \quad = \quad (\Gamma(R \circ R^{\cup}) \circ R_{<})^{\cup} \circ (\Gamma(R \circ R^{\cup}) \circ R) \ .$$

This is, of course, an isomorphic characterisation of R in the sense of theorem 166. Recalling our earlier informal account of how to prove the theorem, the construction corresponds in essence to the bottom-left figure of fig. 5.

6.4 Difunctional Closure

Because a difunctional relation is a prefix point of a monotonic function (the function $\langle X :: X \circ X^{\cup} \circ X \rangle$) fixed-point calculus predicts that the least prefix point

$$\langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle$$

is the least difunctional relation that includes R — the *difunctional closure* of R. More precisely,

$$\langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \text{ is difunctional}$$

and

$$\langle \forall S : S \circ S^{\cup} \circ S \subseteq S : R \subseteq S \equiv \langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \subseteq S \rangle \ .$$

(The general theorem is that, if f is a monotonic endofunction on a complete lattice, the function $f^{\star}$ defined by

$$f^{\star}.x = \langle \mu y :: x \sqcup f.y \rangle$$

has the property that

$$\langle \forall y : f.y \sqsubseteq y : x \sqsubseteq y \equiv f^{\star}.x \sqsubseteq y \rangle \ .$$

The straightforward proof is left to the reader. Examples include the transitive closure and the reflexive-transitive closure of a relation. See [Bac02] for an exposition of the techniques involved.)

In this section, we explore simplifications of the definition of difunctional closure.

The following theorem expresses the same result but in more familiar terms (specifically in terms of the reflexive-transitive closure operator).

Theorem 179 (Difunctional Closure) For all relations R,

$$\langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \;=\; \langle \mu X :: R \cup X \circ R^{\cup} \circ X \rangle \;\; .$$

Hence,

$$\langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \;=\; R \circ (R^{\cup} \circ R)^{*} \;\; .$$

Also,

$$R \circ (R^{\cup} \circ R)^{*} \text{ is difunctional}$$

and

$$\langle \forall S : S \circ S^{\cup} \circ S \subseteq S : R \subseteq S \;\equiv\; R \circ (R^{\cup} \circ R)^{*} \subseteq S \rangle \;\; .$$

(Thus $\langle R :: R \circ (R^{\cup} \circ R)^{*} \rangle$ is the upper adjoint in a Galois connection (of the relations of a given type and the difunctional relations of the same type) of the function that “forgets” that a difunctional relation is indeed difunctional.)

Proof We establish the equality by mutual inclusion. We begin by noting that the equality

$$\langle \mu X :: R \cup X \circ R^{\cup} \circ X \rangle \;=\; R \circ (R^{\cup} \circ R)^{*}$$

is an instance of (the possibly little known) exercise 67(c) in [Bac02]. Also

$$
\begin{array}{ll}
 & \langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \\
= & \{ \quad \text{diagonal rule of fixed-point calculus} \quad \} \\
 & \langle \mu X :: \langle \mu Y :: R \cup Y \circ X^{\cup} \circ Y \rangle \rangle \\
= & \{ \quad \text{[Bac02, exercise 67(c)]} \quad \} \\
 & \langle \mu X :: R \circ (X^{\cup} \circ R)^{*} \rangle \;\; .
\end{array}
$$

So

$$
\begin{array}{ll}
 & \langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \subseteq \langle \mu X :: R \cup X \circ R^{\cup} \circ X \rangle \\
= & \{ \quad \text{above} \quad \} \\
 & \langle \mu X :: R \circ (X^{\cup} \circ R)^{*} \rangle \subseteq R \circ (R^{\cup} \circ R)^{*} \\
\Leftarrow & \{ \quad \text{fixed-point induction} \quad \} \\
 & R \circ ((R \circ (R^{\cup} \circ R)^{*})^{\cup} \circ R)^{*} \subseteq R \circ (R^{\cup} \circ R)^{*} \\
= & \{ \quad \text{properties of converse} \quad \}
\end{array}
$$

$$R\circ((R^{\cup}\circ R)^{*}\circ R^{\cup}\circ R)^{*} \ \subseteq\ R\circ(R^{\cup}\circ R)^{*}$$

$\Leftarrow$ { Leibniz and reflexivity of the subset relation }

$$((R^{\cup}\circ R)^{*}\circ R^{\cup}\circ R)^{*} \ =\ (R^{\cup}\circ R)^{*}$$

$=$ { properties of reflexive-transitive closure }

true .

For the converse, we have:

$$\langle\mu X :: R\cup X\circ R^{\cup}\circ X\rangle \ \subseteq\ \langle\mu X :: R\cup X\circ X^{\cup}\circ X\rangle$$

$=$ { for brevity, let rhs denote $\langle\mu X :: R\cup X\circ X^{\cup}\circ X\rangle$ }

$$\langle\mu X :: R\cup X\circ R^{\cup}\circ X\rangle \ \subseteq\ rhs$$

$\Leftarrow$ { fixed-point induction }

$$R\cup rhs\circ R^{\cup}\circ rhs \ \subseteq\ rhs$$

$=$ { fixed-point computation and definition of rhs }

$$R\cup rhs\circ R^{\cup}\circ rhs \ \subseteq\ R\cup rhs\circ rhs^{\cup}\circ rhs$$

$\Leftarrow$ { monotonicity }

$$R \subseteq rhs$$

$=$ { fixed-point computation and definition of rhs }

true .

□

Theorem is observed by Jaoua et al [JMBD91, Proposition 4.12] but is expressed using the definition of S^{*} as the sum of powers of S. Their (incomplete) proof uses induction over the natural numbers. Just as the notion of the "différence" of a relation is due to Riguet [Rig51], theorem 179 is also due to Riguet [Rig50]. He calls the relation $R\circ(R^{\cup}\circ R)^{+}$ the "difunctional closure" ("fermeture difonctionelle") of R. Note the difference. This suggests that there is a mistake in Riguet's definition or in theorem 179. In fact, both are correct:

Lemma 180 For arbitrary relation R,

$$R\subseteq R\circ R^{\cup}\circ R \ .$$

It follows that, for all relations R,

$$R\circ(R^{\cup}\circ R)^{+} \ =\ R\circ(R^{\cup}\circ R)^{*} \ .$$

Proof We have:

$$
\begin{array}{ll}
& R \circ R^{\cup} \circ R \\
\supseteq & \{ \quad \text{monotonicity} \quad \} \\
& R \circ (I \cap R^{\cup} \circ R) \\
\supseteq & \{ \quad \text{modularity rule: (3)} \quad \} \\
& R \circ I \cap R \\
= & \{ \quad I \text{ is identity of composition, infimum is idempotent} \quad \} \\
& R \quad .
\end{array}
$$

So,

$$
\begin{array}{ll}
& R \circ (R^{\cup} \circ R)^{+} \;=\; R \circ (R^{\cup} \circ R)^{*} \\
= & \{ \quad \text{fixed-point computation and distributivity} \quad \} \\
& R \circ (R^{\cup} \circ R)^{+} \;=\; R \circ (R^{\cup} \circ R)^{+} \cup R \\
= & \{ \quad \text{supremum} \quad \} \\
& R \subseteq R \circ (R^{\cup} \circ R)^{+} \\
\Leftarrow & \{ \quad \text{fixed-point computation and distributivity} \quad \} \\
& R \subseteq R \circ R^{\cup} \circ R \\
= & \{ \quad \text{above} \quad \} \\
& \mathsf{true} \quad .
\end{array}
$$

□

7 The Diagonal

This section anticipates the study of block-ordered relations in section 9. We introduce the notion of the “diagonal” of a relation in section 7.1 and formulate some basic properties in section 7.2. We then introduce the notion of a “non-redundant”, “polar” covering of a relation by rectangles in section 8. We prove that every relation has a polar covering but that not every relation has a non-redundant polar covering. Our definition of “non-redundancy” does not preclude the possibility that elements of a covering are duplicated: a “polar covering” is a bag of rectangles, and not necessarily a set, in the sense of definition 129. This is remedied in section 8.1 where we show that every relation has an injective polar covering. The key to doing so is the notion of the “core” of a relation introduced in section 7.3. Finally, in section 8.2, we explore conditions under which the diagonal of the relation guarantees the non-redundancy of the covering.

The principal driving force behind the investigation reported in this section was to gain a full understanding of Riguet’s “analogie frappante” (theorem 262) which exploits polar coverings to link the notion of the diagonal of a relation with the notion of being block-ordered. However, on the way, several results were obtained that are independent of Riguet’s “analogie”. The idea of reducing a relation to its “core” discussed in section 7.3 stands out. The germs of this idea were sown by Voermans’ [Voe99] introduction of the left-per-domain $\prec$ and right-per-domain $\succ$ operators. (See definition 95.) Some of the theorems in this section, for example theorem 205, have their origins in Riguet’s study of difunctional relations [Rig50]. (See the reference to an “application biunivoque”.) However, we have to admit to being too lazy to try to properly understand Riguet’s theorems and so are unable to give a precise correspondence.

7.1 Definition and Examples

Straightforwardly from the definition of factors, properties of converse and set intersection,

(181) $\quad \mathsf{R}$ is difunctional $\quad\equiv\quad \mathsf{R} \,=\, \mathsf{R}\cap(\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup}$.

More generally, we have:

Lemma 182 For all R, $\mathsf{R}\cap(\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup}$ is difunctional.

Proof Let S denote $\mathsf{R}\cap(\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup}$. We have to prove that S is difunctional. That is, by definition,

$$\mathsf{S}\circ\mathsf{S}^{\cup}\circ\mathsf{S} \subseteq \mathsf{S} \ .$$

Since the right side is an intersection, this is equivalent to

$$S\circ S^{\cup}\circ S \;\subseteq\; R \quad\wedge\quad S\circ S^{\cup}\circ S \;\subseteq\; (R\backslash R/R)^{\cup} \quad .$$

The first is (almost) trivial:

$$\begin{array}{ll}
 & S\circ S^{\cup}\circ S \\
\subseteq & \{ \quad S\subseteq R\,,\; S\subseteq(R\backslash R/R)^{\cup}\,, \\
 & \quad\;\; \text{converse, monotonicity} \quad \} \\
 & R\circ R\backslash R/R\circ R \\
\subseteq & \{ \quad \text{cancellation} \quad \} \\
 & R \quad .
\end{array}$$

In the above calculation, the trick was to replace the outer occurrences of S on the left side by R and the middle occurrence by $(R\backslash R/R)^{\cup}$. The replacement is done the opposite way around in the second calculation.

$$\begin{array}{ll}
 & S\circ S^{\cup}\circ S \;\subseteq\; (R\backslash R/R)^{\cup} \\
\Leftarrow & \{ \quad S\subseteq(R\backslash R/R)^{\cup}\,,\; S\subseteq R\,, \text{ monotonicity and transitivity} \quad \} \\
 & (R\backslash R/R)^{\cup}\circ R^{\cup}\circ(R\backslash R/R)^{\cup} \;\subseteq\; (R\backslash R/R)^{\cup} \\
= & \{ \quad \text{converse} \quad \} \\
 & R\backslash R/R\circ R\circ R\backslash R/R \;\subseteq\; R\backslash R/R \\
= & \{ \quad \text{Galois connection} \quad \} \\
 & R\circ R\backslash R/R\circ R\circ R\backslash R/R\circ R \;\subseteq\; R \\
= & \{ \quad \text{cancellation, monotonicity and transitivity} \quad \} \\
 & \mathsf{true} \quad .
\end{array}$$

□

In order to reflect the mental picture of a difunctional relation, we call the relation $R\cap(R\backslash R/R)^{\cup}$ the *diagonal* of R; Riguet [Rig51] calls it the “différence” of the relation. (Riguet’s definition does not use factors but is equivalent.)

Definition 183 (Diagonal) The *diagonal* of relation R is the relation $R\cap(R\backslash R/R)^{\cup}$. For brevity, $R\cap(R\backslash R/R)^{\cup}$ will sometimes be denoted by ΔR.
□

Many readers will be familiar with the decomposition of a preorder into a partial ordering on a set of equivalence classes. The diagonal of a preorder T is the equivalence relation $T\cap T^{\cup}$. More generally:

Example 184 The diagonal of a provisional preorder T is $T \cap T^{\cup}$. This is because, for an arbitrary relation T,

$$T \cap (T\backslash T/T)^{\cup} \;=\; T \,\cap\, T{<}\circ(T\backslash T/T)^{\cup}\circ T{>} \;\;.$$

But, if T is a provisional preorder,

$$T{<}\circ(T\backslash T/T)^{\cup}\circ T{>} \;=\; T^{\cup} \;\;.$$

(See lemmas 115 and 118.)
□

For readers familiar with algorithmic graph theory (acyclic graphs, topological orderings, strongly connected components), we include a running example. (See examples 185, 229.) Briefly, a finite graph can be represented by a homogeneous relation G on its nodes: the relation holds between nodes a and b if there is an edge from a to b. The (reflexive, transitive) relation G^* holds between nodes a and b if there is a path from a to b. See [BDGv22, BDGv21] for full details.

Example 185 A particular instance of example 184 is if G is the edge relation of a finite graph. Then $\Delta(G^*)$ is $G^* \cap (G^{\cup})^*$, the relation that holds between nodes a and b if there is a path from a to b and a path from b to a in the graph. Thus $\Delta(G^*)$ is the equivalence relation that holds between nodes that are in the same strongly connected component of G.
□

Example 186 In this example, we consider three versions of the less-than relation: the homogeneous less-than relation on integers, which we denote by $<_{\mathbb{Z}}$, the homogeneous less-than relation on real numbers, which we denote by $<_{\mathbb{R}}$, and the heterogeneous less-than relation on integers and real numbers, which we denote by ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$. Specifically, the relation ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$ relates integer m to real number x when $m<x$ (using the conventional over-loaded notation). We also subscript the at-most symbol $\leq$ in the same way in order to indicate the type of the relation in question.

The diagonal of the less-than relation on integers is the predecessor relation (i.e. it relates integer m to integer n exactly when $n=m+1$). This is because $<_{\mathbb{Z}}\backslash<_{\mathbb{Z}} \;=\; \leq_{\mathbb{Z}}$, and $\leq_{\mathbb{Z}}/<_{\mathbb{Z}}$ relates integer m to integer n exactly when $m \leq_{\mathbb{Z}} n+1$ (where the subscript $\mathbb{Z}$ indicates the type of the ordering). The diagonal is thus functional and injective.

The diagonal of the less-than relation on real numbers is the empty relation. This is because $<_{\mathbb{R}}\backslash<_{\mathbb{R}} = \leq_{\mathbb{R}}$, $\leq_{\mathbb{R}}/<_{\mathbb{R}} = \leq_{\mathbb{R}}$ and $<_{\mathbb{R}} \cap \geq_{\mathbb{R}} = \perp\!\!\perp_{\mathbb{R}}$. (Again, the subscript indicates the type of the ordering.)

The diagonal of the heterogeneous less-than relation ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$ relates integer m to real number x when $m < x \leq m+1$. This is equivalent to $\lceil x \rceil = m+1$. The diagonal is thus a difunctional relation characterised by —in the sense of theorem 161— the ceiling function $\langle x :: \lceil x \rceil \rangle$ and the successor function $\langle m :: m+1 \rangle$.

We leave the reader to check the details of this example. See also examples 212, 244 and 315, and theorem 319.

□

The following example introduces a general mechanism for constructing illustrative examples of the concepts introduced later. The example exploits the observation that ΔR is injective if the preorder $R \backslash R$ is anti-symmetric; that is, ΔR is injective if $R \backslash R$ is a partial ordering. (Equivalently, ΔR is functional if R/R is a partial ordering.) We leave the straightforward proof to the reader. (See section 3.5 for the point-free definitions of functionality and injectivity.)

Example 187 Suppose $\mathcal{X}$ is a finite type. We use dummy x to range over elements of type $\mathcal{X}$. Let $\mathcal{S}$ denote a subset of $2^{\mathcal{X}}$. Let in denote the membership relation of type $\mathcal{X} \sim \mathcal{S}$. That is, if S is a subset of $\mathcal{S}$, $x \circ \top\!\!\top \circ S \subseteq \mathsf{in}$ exactly when x is an element of the set S . The relation $\mathsf{in} \backslash \mathsf{in}$ is the subset relation of type $\mathcal{S} \sim \mathcal{S}$.

(Conventionally, in is denoted by the symbol " $\in$ " and one writes $x \in S$ instead of $x \circ \top\!\!\top \circ S \subseteq \mathsf{in}$. Also, the relation $\mathsf{in} \backslash \mathsf{in}$ is conventionally denoted by the symbol " $\subseteq$ ". That is, if S and S' are both elements of $\mathcal{S}$, $S \circ \top\!\!\top \circ S' \subseteq \mathsf{in} \backslash \mathsf{in}$ exactly when $S \subseteq S'$. Were we to adopt conventional practice, the overloading of the notation that occurs is likely to cause confusion, so we choose to avoid it.)

The relation $\mathsf{in} \backslash \mathsf{in}$ is anti-symmetric. As a consequence, $\Delta \mathsf{in}$ is injective. (Equivalently, $(\Delta \mathsf{in})^{\cup}$ is functional.) Specifically, for all x of type $\mathcal{X}$ and S of type $\mathcal{S}$,

$$x \circ \top\!\!\top \circ S \subseteq \Delta \mathsf{in} \quad \equiv \quad x \circ \top\!\!\top \circ S \subseteq \mathsf{in} \ \wedge \ \langle \forall S' : x \circ \top\!\!\top \circ S' \subseteq \mathsf{in} : S \circ \top\!\!\top \circ S' \subseteq \mathsf{in} \backslash \mathsf{in} \rangle \quad ,$$

where dummy S' ranges over elements of $\mathcal{S}$. Using conventional notation, the right side of this equation is recognised as the definition of a minimum, and one might write

$$x \, [\![\Delta \mathsf{in}]\!] \, S \quad \equiv \quad S \doteqdot \langle \mathsf{MIN}\, S' : x \in S' : S' \rangle$$

where the venturi tube " $\doteqdot$ " indicates an equality assuming the well-definedness of the expression on its right side.

Fig. 6 shows a particular instance. The set $\mathcal{X}$ is the set of numbers from 0 to 3 . The set $\mathcal{S}$ is a subset of $2^{\{0,1,2,3\}}$; the chosen subset and the relation $\mathsf{in} \backslash \mathsf{in}$ for this choice are depicted by the directed graph forming the central portion of fig. 6. The relation $\Delta \mathsf{in}$ of type $\mathcal{X} \sim \mathcal{S}$ is depicted by the undirected graph whereby the atoms of the relation are depicted as rectangles. Note that the numbers 0 and 3 are not related by $\Delta \mathsf{in}$ to any of the elements of $\mathcal{S}$. See example 264 for further discussion of this example.

□

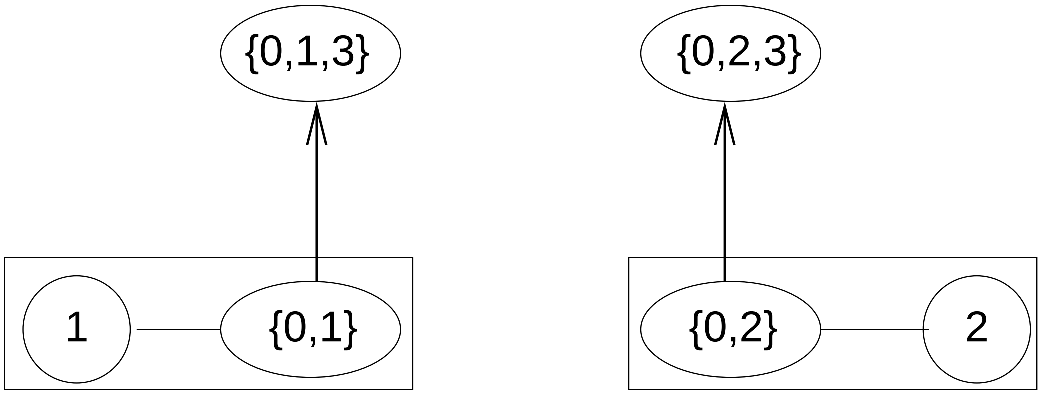


Figure 6: Diagonal of an Instance of the Membership Relation

7.2 Basic Properties

Primarily for notational convenience, we note a simple property of the diagonal:

Lemma 188

$$(\Delta R)^{\cup} \;=\; \Delta(R^{\cup}) \quad .$$

Proof

$$\begin{array}{ll} & (\Delta R)^{\cup} \\ = & \{ \quad \text{definition and distributivity} \quad \} \\ & R^{\cup} \cap R\backslash R/R \\ = & \{ \quad \text{factors} \quad \} \\ & R^{\cup} \cap (R^{\cup} \backslash R^{\cup} / R^{\cup})^{\cup} \\ = & \{ \quad \text{definition} \quad \} \\ & \Delta(R^{\cup}) \quad . \end{array}$$

□

A consequence of lemma 188 is that we can write $\Delta R^{\cup}$ without ambiguity. This we do from now on.

Very straightforwardly, the relation $R \circ R^{\cup}$ is a per if R is difunctional. For a difunctional relation R, the relation $R \circ R^{\cup}$ is the per representation of the left domain of R. Symmetrically, $R^{\cup} \circ R$ is the per representation of the right domain of R. (See theorem 160, parts (iii) and (iv).) Thus $\Delta R \circ (\Delta R)^{\cup}$ is the per representation of the left domain of the diagonal of R. The following lemma is the basis of the construction, in certain cases, of an economic representation of the diagonal of R and, hence, of R itself. See definition 209 and theorems 218 and 222.

Lemma 189 For all relations R,

$$(\Delta R)_{\prec} \;=\; (\Delta R)_{<} \circ R_{\prec} \quad .$$

Dually,

$$(\Delta R)_{\succ} \;=\; (\Delta R)_{>} \circ R_{\succ} \quad .$$

Proof We prove the first equation by mutual inclusion. First,

$$
\begin{array}{ll}
 & (\Delta R)_{\prec} \subseteq (\Delta R)_{<} \circ R_{\prec} \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 160; definition: (96)} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \subseteq (\Delta R)_{<} \circ R /\!\!/ R \\
\Leftarrow & \{ \quad \text{domains and monotonicity} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \subseteq R /\!\!/ R \\
= & \{ \quad \text{definition of } R /\!\!/ R \text{, converse and factors} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R \subseteq R \\
= & \{ \quad \Delta R \subseteq R \,;\; \Delta R^{\cup} \subseteq R \backslash R / R \text{ and cancellation} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & (\Delta R)_{<} \circ R_{\prec} \subseteq (\Delta R)_{\prec} \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 160} \quad \} \\
 & (\Delta R)_{<} \circ R_{\prec} \subseteq \Delta R \circ \Delta R^{\cup} \\
\Leftarrow & \{ \quad \text{domains and definition: (96)} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R /\!\!/ R \subseteq \Delta R \circ \Delta R^{\cup} \\
\Leftarrow & \{ \quad \text{monotonicity and converse} \quad \} \\
 & R /\!\!/ R \circ \Delta R \subseteq \Delta R \\
= & \{ \quad \text{definition of diagonal} \quad \} \\
 & R /\!\!/ R \circ \Delta R \subseteq R \quad \wedge \quad R /\!\!/ R \circ \Delta R \subseteq (R \backslash R / R)^{\cup} \\
\Leftarrow & \{ \quad \Delta R \subseteq R \text{ ; converse} \quad \} \\
 & R /\!\!/ R \circ R \subseteq R \quad \wedge \quad \Delta R^{\cup} \circ R /\!\!/ R \subseteq R \backslash R / R \\
= & \{ \quad \text{cancellation; factors} \quad \} \\
 & \mathsf{true} \;\wedge\; R \circ \Delta R^{\cup} \circ R /\!\!/ R \circ R \subseteq R
\end{array}
$$

$$\Leftarrow \quad \{ \quad \text{cancellation and } \Delta R^{\cup} \subseteq R\backslash R/R \quad \}$$

$$R \circ R\backslash R/R \circ R \subseteq R$$

$$= \quad \{ \quad \text{cancellation} \quad \}$$

$$\mathsf{true} \quad .$$

The dual properties are obtained by instantiating R to $R^{\cup}$ and simplifying using properties of converse.

□

The following corollary of lemma 189 proves to be crucial later: see the discussion following lemma 259.

Lemma 190 For all relations R,

$$(\Delta R)_{\prec} = R_{\prec} \quad \equiv \quad (\Delta R)_{<} = R_{<} \quad .$$

Dually,

$$(\Delta R)_{\succ} = R_{\succ} \quad \equiv \quad (\Delta R)_{>} = R_{>} \quad .$$

Proof The proof is by mutual implication:

$$(\Delta R)_{<} = R_{<}$$

$$\Rightarrow \quad \{ \quad \text{lemma 189 and Leibniz} \quad \}$$

$$(\Delta R)_{\prec} = R_{<} \circ R_{\prec}$$

$$= \quad \{ \quad \text{dual of (101)} \quad \}$$

$$(\Delta R)_{\prec} = R_{\prec}$$

$$\Rightarrow \quad \{ \quad \text{Leibniz} \quad \}$$

$$((\Delta R)_{\prec})_{<} = (R_{\prec})_{<}$$

$$= \quad \{ \quad \text{dual of (101) with } R{:=}\Delta R \text{ and } R{:=}R \quad \}$$

$$(\Delta R)_{<} = R_{<} \quad .$$

□

7.3 Reduction to the Core

Suppose R is an arbitrary relation. Both $R_{\prec}$ and $R_{\succ}$ are pers so can be characterised by their equivalence classes. Specifically, for a given R, suppose

$$R_{\prec} = \lambda^{\cup} \circ \lambda \quad \wedge \quad R_{\succ} = \rho^{\cup} \circ \rho$$

where λ and ρ are functional relations. (The existence of λ and ρ is guaranteed by theorem 143.) Then

$$R = \lambda^{\cup} \circ \lambda \circ R \circ \rho^{\cup} \circ \rho \quad .$$

The relation $\lambda \circ R \circ \rho^{\cup}$, which we denote by $|R|$, is a relation on the equivalence classes. For a mental picture of such a relation, refer to fig. 18 (page 199): the individual blocks of the relation R become points of the relation $|R|$.

Definition 191 (Core) Suppose R is an arbitrary relation and suppose

$$R{\prec} \;=\; \lambda^{\cup} \circ \lambda \quad \wedge \quad R{\succ} \;=\; \rho^{\cup} \circ \rho$$

where λ and ρ are functional relations. Then the *core* of R, which is denoted by $|R|$, is defined by

$$|R| \;=\; \lambda \circ R \circ \rho^{\cup} \quad .$$

□

Example 192 Fig. 7 depicts a relation (on the left) and its core (on the right). Both are depicted as bipartite graphs. The relation R is a relation on blue and red nodes. Its core $|R|$ is depicted as a relation on squares of blue nodes and squares of red nodes, each square being an equivalence class of $R{\prec}$ (on the left) or of $R{\succ}$ (on the right).

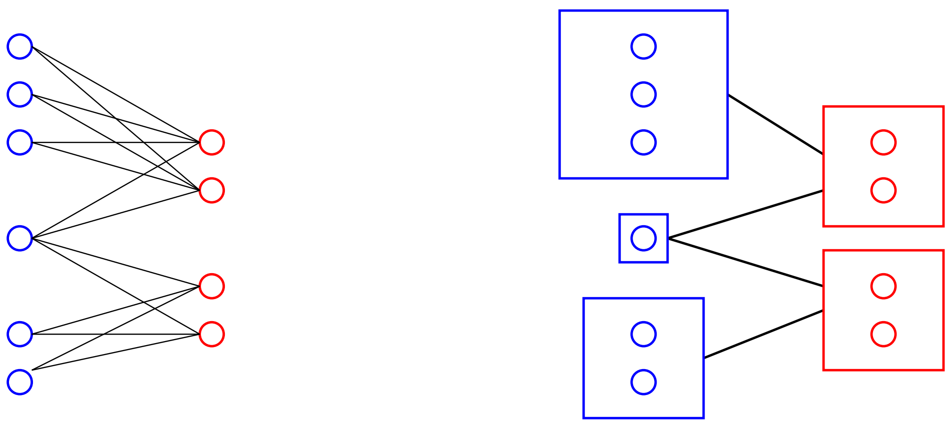


Figure 7: A Relation and Its Core

□

Generally, in order to avoid the clutter that is evident in fig. 7, examples from now on will almost invariably be of relations that are isomorphic to their cores. However, this is *not* the case for example 224 because it has been chosen to illustrate some of the limitations of the theory we develop.

Lemma 193 Suppose R, λ and ρ are as in definition 191. Then

$$R \;=\; \lambda^{\cup}\circ|R|\circ\rho \quad .$$

Proof

$$\begin{array}{cl}
 & R \\
= & \{ \quad \text{per domains: (98)} \quad \} \\
 & R{\prec}\circ R\circ R{\succ} \\
= & \{ \quad R{\prec} = \lambda^{\cup}\circ\lambda \text{ and } R{\succ} = \rho^{\cup}\circ\rho \quad \} \\
 & \lambda^{\cup}\circ\lambda\circ R\circ\rho^{\cup}\circ\rho \\
= & \{ \quad \text{definition 191} \quad \} \\
 & \lambda^{\cup}\circ|R|\circ\rho \quad .
\end{array}$$

□

As previously observed, there are several different ways in which a per can be written as $f^{\cup}\circ f$ for some functional relation f. However, all are "isomorphic". (See theorem 151.) Correspondingly, there are several different ways to construct a core of a relation, but all are "isomorphic" in the sense of definition 82:

Theorem 194 Suppose S_0 and S_1 are both cores of R. Then $S_0 \cong S_1$.

Proof Suppose, for $i=0$ and $i=1$, $S_i = \lambda_i\circ R\circ\rho_i^{\cup}$ where $R{\prec} = \lambda_i^{\cup}\circ\lambda_i$ and $R{\succ} = \rho_i^{\cup}\circ\rho_i$. (That is, S_0 and S_1 are both cores of R.) Then

$$\begin{array}{cl}
 & S_0 \\
= & \{ \quad \text{assumption} \quad \} \\
 & \lambda_0\circ R\circ\rho_0^{\cup} \\
= & \{ \quad \text{lemma 193} \quad \} \\
 & \lambda_0\circ\lambda_1^{\cup}\circ S_1\circ\rho_1\circ\rho_0^{\cup} \quad .
\end{array}$$

Applying definition 82 with $f,g := \lambda_0\circ\lambda_1^{\cup}\ ,\ \rho_1\circ\rho_0^{\cup}$ in combination with theorem 151, we conclude that $S_0 \cong S_1$.

□

For later use, we calculate the left and right domains of the core of a relation.

Lemma 195 Suppose R, λ and ρ are as in definition 191. Then

$$R{<} = \lambda{>} \;\wedge\; |R|{<} = \lambda{<} \;\wedge\; R{>} = \rho{>} \;\wedge\; |R|{>} = \rho{<} \quad .$$

Proof We prove the middle two equations. First,

$R_{>}$

$= \quad \{ \quad (101) \quad \}$

$(R_{\succ})_{<}$

$= \quad \{ \quad \text{definition 191} \quad \}$

$(\rho^{\cup} \circ \rho)_{<}$

$= \quad \{ \quad \text{domains} \quad \}$

$\rho_{>} \quad .$

The dual equation, $R_{<} = \lambda_{>}$, is proved similarly. Second,

$|R|_{<}$

$= \quad \{ \quad \text{definition 191} \quad \}$

$(\lambda \circ R \circ \rho^{\cup})_{<}$

$= \quad \{ \quad R_{>} = \rho_{>} \ \text{(just proved)} \quad \}$

$(\lambda \circ R \circ R_{>})_{<}$

$= \quad \{ \quad \text{domains} \quad \}$

$(\lambda \circ R_{<})_{<}$

$= \quad \{ \quad R_{<} = \lambda_{>} \ \text{(see above)} \quad \}$

$\lambda_{<} \quad .$

The final equation is also proved similarly.
□

A distinguishing feature of the core of a relation is that its left and right per-domains equal its left and right domains, respectively.

Theorem 196 Suppose R, λ and ρ are as in definition 191. Then

(197) $|R|_{\succ} \ = \ |R|_{>} \quad .$

Also,

(198) $|R|_{\prec} \ = \ |R|_{<} \quad .$

Proof The proof of (197) has several (non-trivial) steps. First, we show that

(199) $|R|_{\succ} \ = \ S \cap S^{\cup}$

where

(200) $S \ = \ \rho_{<} \circ (\lambda \circ R \circ \rho^{\cup}) \backslash (\lambda \circ R \circ \rho^{\cup}) \circ \rho_{<} \quad .$

Then we simplify several subcomponents of S. We have

$$|R|_{\succ}$$

$= \quad \{ \quad$ (96) and (101) $\quad \}$

$$|R|_{>} \circ |R| \mathbin{\backslash\!\backslash} |R| \circ |R|_{>}$$

$= \quad \{ \quad$ lemma 195 and definition 191 $\quad \}$

$$\rho_{<} \circ (\lambda \circ R \circ \rho^{\cup}) \mathbin{\backslash\!\backslash} (\lambda \circ R \circ \rho^{\cup}) \circ \rho_{<}$$

$= \quad \{ \quad$ (88), converse and distributivity of coreflexives over infima $\quad \}$

$$\rho_{<} \circ (\lambda \circ R \circ \rho^{\cup}) \backslash (\lambda \circ R \circ \rho^{\cup}) \circ \rho_{<} \;\; \cap \;\; (\rho_{<} \circ (\lambda \circ R \circ \rho^{\cup}) \backslash (\lambda \circ R \circ \rho^{\cup}) \circ \rho_{<})^{\cup}$$

$= \quad \{ \quad$ (200) $\quad \}$

$$S \cap S^{\cup} \quad .$$

Next we show that

$$(201) \quad (\lambda \circ R \circ \rho^{\cup}) \backslash (\lambda \circ R \circ \rho^{\cup}) \; = \; (R \circ \rho^{\cup}) \backslash (R \circ \rho^{\cup}) \quad .$$

We have

$$(\lambda \circ R \circ \rho^{\cup}) \backslash (\lambda \circ R \circ \rho^{\cup})$$

$= \quad \{ \quad$ factors $\quad \}$

$$(R \circ \rho^{\cup}) \backslash (\lambda \backslash (\lambda \circ R \circ \rho^{\cup}))$$

$= \quad \{ \quad$ in order to cancel the two occurrences of λ,

we aim to apply lemma 75

$[\, R\backslash S = R\backslash(R_{<} \circ S) \,]$ with $R,S := R \circ \rho^{\cup} \, , \, \lambda\backslash(\lambda \circ R \circ \rho^{\cup})$ $\quad \}$

$$(R \circ \rho^{\cup}) \backslash ((R \circ \rho^{\cup})_{<} \circ \lambda \backslash (\lambda \circ R \circ \rho^{\cup}))$$

$= \quad \{ \quad$ by lemma 195, $(R \circ \rho^{\cup})_{<} = R_{<} = \lambda_{>}$ $\quad \}$

$$(R \circ \rho^{\cup}) \backslash (\lambda_{>} \circ \lambda \backslash (\lambda \circ R \circ \rho^{\cup}))$$

$= \quad \{ \quad$ lemma 75 with $f,R := \lambda \, , \, \lambda \circ R \circ \rho^{\cup}$ $\quad \}$

$$(R \circ \rho^{\cup}) \backslash (\lambda^{\cup} \circ \lambda \circ R \circ \rho^{\cup})$$

$= \quad \{ \quad$ $\lambda^{\cup} \circ \lambda = R_{\prec}$ and $R_{\prec} \circ R = R$ $\quad \}$

$$(R \circ \rho^{\cup}) \backslash (R \circ \rho^{\cup}) \quad .$$

The next step is to show that

$$(202) \quad \rho_{<} \circ (R \circ \rho^{\cup}) \backslash (R \circ \rho^{\cup}) \; = \; \rho \circ R \backslash (R \circ \rho^{\cup}) \quad .$$

We have

$$
\begin{array}{ll}
 & \rho_{<} \circ (R \circ \rho^{\cup}) \backslash (R \circ \rho^{\cup}) \\
= & \{ \quad \rho_{<} = \rho \circ \rho^{\cup} \quad \} \\
 & \rho \circ \rho^{\cup} \circ (R \circ \rho^{\cup}) \backslash (R \circ \rho^{\cup}) \\
= & \{ \quad \text{lemma 75 with } f,R := \rho \ , \ (R \circ \rho^{\cup}) \backslash (R \circ \rho^{\cup}) \quad \} \\
 & \rho \circ \rho_{>} \circ \rho \backslash ((R \circ \rho^{\cup}) \backslash (R \circ \rho^{\cup})) \\
= & \{ \quad \text{domains and factors} \quad \} \\
 & \rho \circ (R \circ \rho^{\cup} \circ \rho) \backslash (R \circ \rho^{\cup}) \\
= & \{ \quad \rho^{\cup} \circ \rho = R_{\succ} \text{ and } R \circ R_{\succ} = R \quad \} \\
 & \rho \circ R \backslash (R \circ \rho^{\cup}) \quad .
\end{array}
$$

We have thus proven (202). Now we show that

$$(203) \quad R \backslash (R \circ \rho^{\cup}) \circ \rho_{<} \ = \ R \backslash R \circ \rho^{\cup} \quad .$$

We have

$$
\begin{array}{ll}
 & R \backslash (R \circ \rho^{\cup}) \circ \rho_{<} \\
= & \{ \quad \rho_{<} = \rho \circ \rho^{\cup} \quad \} \\
 & R \backslash (R \circ \rho^{\cup}) \circ \rho \circ \rho^{\cup} \\
= & \{ \quad \text{lemma 77 with } R,S,f := R \ , \ R \circ \rho^{\cup} \ , \ \rho \quad \} \\
 & R \backslash (R \circ \rho^{\cup} \circ \rho) \circ \rho_{>} \circ \rho^{\cup} \\
= & \{ \quad \rho^{\cup} \circ \rho = R_{\succ} , \\
 & \qquad [\ R \circ R_{\succ} = R \ ] \text{ and } [\ R_{>} \circ R^{\cup} = R^{\cup} \ ] \text{ with } R := \rho \quad \} \\
 & R \backslash R \circ \rho^{\cup} \quad .
\end{array}
$$

We have thus proven (203). Now we put the above steps together:

$$
\begin{array}{ll}
 & \rho_{<} \circ (\lambda \circ R \circ \rho^{\cup}) \backslash (\lambda \circ R \circ \rho^{\cup}) \circ \rho_{<} \\
= & \{ \quad (201) \quad \} \\
 & \rho_{<} \circ (R \circ \rho^{\cup}) \backslash (R \circ \rho^{\cup}) \circ \rho_{<} \\
= & \{ \quad (202) \quad \} \\
 & \rho \circ R \backslash (R \circ \rho^{\cup}) \circ \rho_{<} \\
= & \{ \quad (203) \quad \} \\
 & \rho \circ R \backslash R \circ \rho^{\cup} \quad .
\end{array}
$$

That is,

$$(204)\quad \rho_{<}\circ(\lambda\circ R\circ\rho^{\cup})\backslash(\lambda\circ R\circ\rho^{\cup})\circ\rho_{<} \;=\; \rho\circ R\backslash R\circ\rho^{\cup}\;.$$

So

$$\begin{aligned}
&|R|_{\succ}\\
=\;&\quad\{\quad (199)\text{ and }(200)\quad\}\\
&\rho_{<}\circ(\lambda\circ R\circ\rho^{\cup})\backslash(\lambda\circ R\circ\rho^{\cup})\circ\rho_{<}\;\cap\;(\rho_{<}\circ(\lambda\circ R\circ\rho^{\cup})\backslash(\lambda\circ R\circ\rho^{\cup})\circ\rho_{<})^{\cup}\\
=\;&\quad\{\quad (204)\quad\}\\
&\rho\circ R\backslash R\circ\rho^{\cup}\;\cap\;(\rho\circ R\backslash R\circ\rho^{\cup})^{\cup}\\
=\;&\quad\{\quad \text{converse}\quad\}\\
&\rho\circ R\backslash R\circ\rho^{\cup}\;\cap\;\rho\circ(R\backslash R)^{\cup}\circ\rho^{\cup}\\
=\;&\quad\{\quad \text{see below}\quad\}\\
&\rho\circ R_{\succ}\circ\rho^{\cup}\\
=\;&\quad\{\quad R_{\succ}=\rho^{\cup}\circ\rho\quad\}\\
&\rho\circ\rho^{\cup}\circ\rho\circ\rho^{\cup}\\
=\;&\quad\{\quad \rho\circ\rho^{\cup}=\rho_{<}=|R|_{>}\text{, coreflexives}\quad\}\\
&|R|_{>}\;.
\end{aligned}$$

The unproven middle step asserts that

$$\rho\circ R\backslash R\circ\rho^{\cup}\;\cap\;\rho\circ(R\backslash R)^{\cup}\circ\rho^{\cup} \;=\; \rho\circ R_{\succ}\circ\rho^{\cup}\;.$$

This is proved using the anti-symmetry of the subset relation. Note first that

$$\rho\circ R_{\succ}\circ\rho^{\cup} \;=\; \rho\circ(R\backslash R\cap(R\backslash R)^{\cup})\circ\rho^{\cup}$$

since

$$\begin{aligned}
&\rho\circ R_{\succ}\circ\rho^{\cup}\\
=\;&\quad\{\quad \text{definition 95, (96) and (101)}\quad\}\\
&\rho\circ R_{>}\circ(R\backslash R\cap(R\backslash R)^{\cup})\circ R_{>}\circ\rho^{\cup}\\
=\;&\quad\{\quad \text{lemma 195 (in particular, } R_{>}=\rho_{>})\quad\}\\
&\rho\circ(R\backslash R\cap(R\backslash R)^{\cup})\circ\rho^{\cup}\;.
\end{aligned}$$

So our task is to prove that

$$\rho\circ R\backslash R\circ\rho^{\cup}\;\;\cap\;\;\rho\circ(R\backslash R)^{\cup}\circ\rho^{\cup}\;\;=\;\;\rho\circ(R\backslash R\cap(R\backslash R)^{\cup})\circ\rho^{\cup}\;\;.$$

We begin with the right side because its inclusion in the left side is easy.

$$\begin{array}{ll}
 & \rho\circ(R\backslash R\cap(R\backslash R)^{\cup})\circ\rho^{\cup} \\
\subseteq & \{\quad \text{infima and monotonicity} \quad\} \\
 & \rho\circ R\backslash R\circ\rho^{\cup}\;\;\cap\;\;\rho\circ(R\backslash R)^{\cup}\circ\rho^{\cup} \\
\subseteq & \{\quad \text{modularity rules: (3) and (4)} \quad\} \\
 & \rho\circ(\rho^{\cup}\circ\rho\circ R\backslash R\circ\rho^{\cup}\circ\rho\;\cap\;(R\backslash R)^{\cup})\circ\rho^{\cup} \\
= & \{\quad R_{\succ}=\rho^{\cup}\circ\rho \quad\} \\
 & \rho\circ(R_{\succ}\circ R\backslash R\circ R_{\succ}\;\cap\;(R\backslash R)^{\cup})\circ\rho^{\cup} \\
\subseteq & \{\quad \text{by definition 95 and monotonicity, } R_{\succ}\subseteq R\backslash R \quad\} \\
 & \rho\circ(R\backslash R\circ R\backslash R\circ R\backslash R\;\cap\;(R\backslash R)^{\cup})\circ\rho^{\cup} \\
\subseteq & \{\quad R\backslash R\circ R\backslash R\circ R\backslash R\subseteq R\backslash R \text{ and monotonicity} \quad\} \\
 & \rho\circ(R\backslash R\cap(R\backslash R)^{\cup})\circ\rho^{\cup}\;\;.
\end{array}$$

This completes the proof of the middle step and, hence, of (197).

The proof of (198) involves instantiating (197). Since $R_{\prec}=(R^{\cup})_{\succ}$ and $R_{\succ}=(R^{\cup})_{\prec}$, we can define $|R^{\cup}|$ to be $\rho\circ R^{\cup}\circ\lambda^{\cup}$. Then

$$\begin{array}{ll}
 & \mathsf{true} \\
= & \{\quad (197) \quad\} \\
 & |R^{\cup}|_{\succ}\;=\;|R^{\cup}|_{>} \\
= & \{\quad \text{definition of } |R^{\cup}| \quad\} \\
 & (\rho\circ R^{\cup}\circ\lambda^{\cup})_{\succ}\;=\;(\rho\circ R^{\cup}\circ\lambda^{\cup})_{>} \\
= & \{\quad \text{converse} \quad\} \\
 & (\lambda\circ R\circ\rho^{\cup})_{\prec}\;=\;(\lambda\circ R\circ\rho^{\cup})_{<} \\
= & \{\quad \text{definition of } |R| \quad\} \\
 & |R|_{\prec}\;=\;|R|_{<}\;\;.
\end{array}$$

□

The diagonal of a relation is difunctional. A general property of the core of a difunction is the following.

Theorem 205 Suppose R is difunctional. Then the core of R is functional and injective. Specifically, if $R = f^{\cup} \circ g$ where $f \circ f^{\cup} = f_< = g \circ g^{\cup} = g_<$, then

$$|R| \circ |R|^{\cup} = f_< \quad \wedge \quad |R|^{\cup} \circ |R| = g_< \quad .$$

Thus, if R is difunctional, its core $|R|$ defines a (1–1) correspondence between the equivalence classes of $R\prec$ and the equivalence classes of $R\succ$.

Proof If R is difunctional, the characterisation of difunctional relations given by theorem 161 allows us to assume that $R = f^{\cup} \circ g$ where $f \circ f^{\cup} = f_< = g \circ g^{\cup} = g_<$. Then, by lemma 142,

$$R\prec \;=\; f^{\cup} \circ f \;=\; R \circ R^{\cup} \quad \wedge \quad R\succ \;=\; g^{\cup} \circ g \;=\; R^{\cup} \circ R \quad .$$

So

$$
\begin{array}{ll}
 & |R| \circ |R|^{\cup} \\
= & \{ \quad \text{definition 191} \quad \} \\
 & f \circ R \circ g^{\cup} \circ g \circ R^{\cup} \circ f^{\cup} \\
= & \{ \quad \text{definition 191} \quad \} \\
 & f \circ R \circ R\succ \circ R^{\cup} \circ f^{\cup} \\
= & \{ \quad \text{per domains: (98)} \quad \} \\
 & f \circ R \circ R^{\cup} \circ f^{\cup} \\
= & \{ \quad f^{\cup} \circ f \;=\; R \circ R^{\cup} \quad \} \\
 & f \circ f^{\cup} \circ f \circ f^{\cup} \\
= & \{ \quad f \circ f^{\cup} \;=\; f_< \quad \} \\
 & f_< \quad .
\end{array}
$$

That is, $|R|$ is functional with left domain $f_<$, (the coreflexive representation of) the set of equivalence classes of $R\prec$. By symmetry, $|R|$ is injective with right domain $g_<$, (the coreflexive representation of) the set of equivalence classes of $R\succ$.
□

A relation that is both injective and functional establishes a (1–1) correspondence between the points of its left and right domains. If these points are ordered arbitrarily but in such a way that the ordering respects the correspondence, and the relation is depicted by a graph whose axes depict the orderings of the domains, the relation will form a subdiagonal of the graph. Thus the mental picture of the core $|R|$ of a difunctional relation R is a subdiagonal of a graph; the mental picture of the (difunctional) relation

R itself is a collection of completely disjoint rectanges arranged along the diagonal of a graph. It follows from theorem 205 that the core $|\Delta R|$ of the diagonal of an arbitrary relation R is functional and injective. The mental picture we have just sketched thus applies to the diagonal ΔR; this is the motivation for our chosen terminology.

Now we turn to properties of the diagonal of the core of a relation.

Lemma 206 Suppose R, λ and ρ are as in definition 191. Then

$$R_{>}\circ R\backslash R/R\circ R_{<} \;=\; \rho^{\cup}\circ|R|\backslash|R|/|R|\circ\lambda \;\;.$$

Proof For brevity, the calculation introduces the abbreviation S for $|R|$.

$$\begin{array}{ll}
 & R_{>}\circ R\backslash R/R\circ R_{<} \\
= & \{\quad (101) \quad\} \\
 & (R_{\succ})_{>}\circ R\backslash R/R\circ(R_{\prec})_{<} \\
= & \{\quad R_{\prec} \,=\, \lambda^{\cup}\circ\lambda\,,\;\; R_{\succ} \,=\, \rho^{\cup}\circ\rho\,, \text{ and domains} \quad\} \\
 & \rho_{>}\circ R\backslash R/R\circ\lambda_{>} \\
= & \{\quad \text{lemma 193, } S=|R| \quad\} \\
 & \rho_{>}\circ(\lambda^{\cup}\circ S\circ\rho)\backslash(\lambda^{\cup}\circ S\circ\rho)/(\lambda^{\cup}\circ S\circ\rho)\circ\lambda_{>} \\
= & \{\quad \text{lemma 78 with } f,g,U,V,W := \rho,\lambda,S,S,S \quad\} \\
 & \rho^{\cup}\circ(\lambda_{<}\circ S)\backslash S/(S\circ\rho_{<})\circ\lambda \\
= & \{\quad S=|R| \quad\} \\
 & \rho^{\cup}\circ(\lambda_{<}\circ|R|)\backslash|R|/(|R|\circ\rho_{<})\circ\lambda \\
= & \{\quad |R| = \lambda\circ R\circ\rho^{\cup}\,;\text{ so } \lambda_{<}\circ|R| = |R| = |R|\circ\rho_{<} \quad\} \\
 & \rho^{\cup}\circ|R|\backslash|R|/|R|\circ\lambda \;\;.
\end{array}$$

□

Theorem 207 Suppose R, λ and ρ are as in definition 191. Then

$$\Delta R \;=\; \lambda^{\cup}\circ\Delta|R|\circ\rho \quad\wedge\quad \Delta|R| \;=\; \lambda\circ\Delta R\circ\rho^{\cup} \;\;.$$

Proof As in lemma 206, we abbreviate $|R|$ to S.

$$\begin{array}{ll}
 & \Delta R \\
= & \{\quad \text{definition} \quad\} \\
 & R\cap(R\backslash R/R)^{\cup}
\end{array}$$

$= \quad \{ \quad$ domains and converse $\quad \}$

$\quad R \cap (R_{>} \circ R\backslash R/R \circ R_{<})^{\cup}$

$= \quad \{ \quad S = |R|$, lemma 206 $\quad \}$

$\quad R \cap (\rho^{\cup} \circ S\backslash S/S \circ \lambda)^{\cup}$

$= \quad \{ \quad S = |R|$, lemma 193 $\quad \}$

$\quad \lambda^{\cup} \circ S \circ \rho \;\cap\; (\rho^{\cup} \circ S\backslash S/S \circ \lambda)^{\cup}$

$= \quad \{ \quad$ distributivity of converse and functional relations $\quad \}$

$\quad \lambda^{\cup} \circ (S \cap (S\backslash S/S)^{\cup}) \circ \rho$

$= \quad \{ \quad$ definition 183, $S = |R| \quad \}$

$\quad \lambda^{\cup} \circ \Delta|R| \circ \rho$.

Hence

$\quad \lambda \circ \Delta R \circ \rho^{\cup}$

$= \quad \{ \quad$ above $\quad \}$

$\quad \lambda \circ \lambda^{\cup} \circ \Delta|R| \circ \rho \circ \rho^{\cup}$

$= \quad \{ \quad \lambda$ and ρ are functional $\quad \}$

$\quad \lambda_{<} \circ \Delta|R| \circ \rho_{<}$

$= \quad \{ \quad \Delta|R| \subseteq |R|$; so $(\Delta|R|)_{<} \subseteq |R|_{<}$ and $(\Delta|R|)_{>} \subseteq |R|_{>}$

lemma 195 and domains $\quad \}$

$\quad \Delta|R|$.

□

Theorem 207 may have practical importance for very large datasets. In applications where computing the diagonal of a relation R is required it may be more efficient to first reduce it to its core $|R|$ instead of computing the diagonal directly. This of course requires computing partitionings of $R_{\prec}$ and $R_{\succ}$. The task of determining whether or not a given relation can be block-ordered is an example: see theorem 265.

Small examples that one encounters in the literature typically have the property that $R = |R|$, in order to avoid unnecessary clutter. The same is true for the concrete examples that we present here. See the discussion following theorem 265.

The final theorem in this section is motivated by theorem 205. The diagonal of an arbitrary relation R is difunctional, so theorem 205 (with $R := \Delta R$) states that $|\Delta R|$ —the core of the diagonal of R— defines a (1–1) connection between the equivalence classes of $(\Delta R)_{\prec}$ and $(\Delta R)_{\succ}$. Theorem 208 is a slightly weaker property of $\Delta|R|$ —the diagonal of the core of R— in relation to the per domains $R_{\prec}$ and $R_{\succ}$.

Theorem 208 Suppose R, λ and ρ are as in definition 191. Then

$$\Delta|R|\circ\Delta|R|^{\cup} \subseteq \lambda_{<} \quad\wedge\quad \Delta|R|^{\cup}\circ\Delta|R| \subseteq \rho_{<} \ .$$

That is, $\Delta|R|$ defines a (1–1) correspondence between a subset of the equivalence classes of $R_{\prec}$ (specifically, the points in $|R|_{<}$) and a subset of the equivalence classes of $R_{\succ}$ (the points in $|R|_{>}$).

Proof

$$\begin{array}{ll}
 & \Delta|R|\circ\Delta|R|^{\cup} \\
= & \{ \quad \text{theorem 207 and converse} \quad \} \\
 & \lambda\circ\Delta R\circ\rho^{\cup}\circ\rho\circ\Delta R^{\cup}\circ\lambda^{\cup} \\
= & \{ \quad \text{definition 191} \quad \} \\
 & \lambda\circ\Delta R\circ R_{\succ}\circ\Delta R^{\cup}\circ\lambda^{\cup} \\
= & \{ \quad \text{domains} \quad \} \\
 & \lambda\circ\Delta R\circ(\Delta R)_{>}\circ R_{\succ}\circ\Delta R^{\cup}\circ\lambda^{\cup} \\
= & \{ \quad \text{lemma 189 and per domains: (98)} \quad \} \\
 & \lambda\circ\Delta R\circ\Delta R^{\cup}\circ\lambda^{\cup} \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 160 with } R{:=}\Delta R \quad \} \\
 & \lambda\circ(\Delta R)_{\prec}\circ\lambda^{\cup} \\
\subseteq & \{ \quad \text{lemma 189} \quad \} \\
 & \lambda\circ R_{\prec}\circ\lambda^{\cup} \\
= & \{ \quad \text{definition 191} \quad \} \\
 & \lambda\circ\lambda^{\cup}\circ\lambda\circ\lambda^{\cup} \\
= & \{ \quad \lambda \text{ is functional, i.e. } \lambda\circ\lambda^{\cup} = \lambda_{<}\text{, domains} \quad \} \\
 & \lambda_{<} \ .
\end{array}$$

The fact that $\Delta|R|$ is functional follows from the fact that λ is functional (and, of course, the transitivity of the subset relation). The property that $\Delta|R|^{\cup}$ is injective is the converse dual.

□

8 Polar Coverings

This section is, at first sight, a detour from the study of the diagonal of a relation. We introduce the notion of a "polar covering" of a relation R and show that every relation has such a covering. See theorem 211. In a sense, theorem 211 is a generalisation of theorem 163 (the theorem that every difunctional relation is the supremum of a set of completely disjoint rectangles). The relevance to the diagonal of a relation becomes clearer when we study "non-redundant" polar coverings in section 8.2.

Definition 209 (Polar Covering) Suppose $\mathcal{R}$ is an indexed bag of rectangles. (See definition 129.) Then $\mathcal{R}$ is said to be *polar* if, for all elements U and V of $\mathcal{R}$,

$$U_< \subseteq V_< \;\equiv\; U_> \supseteq V_> \quad .$$

Also, $\mathcal{R}$ is said to be *linear* if, for all elements U and V of $\mathcal{R}$,

$$U_< \subseteq V_< \quad \vee \quad V_< \subseteq U_< \quad .$$

(Equivalently,

$$U_> \subseteq V_> \quad \vee \quad V_> \subseteq U_> \quad .)$$

A relation R is *covered* by $\mathcal{R}$ if $R = \cup\mathcal{R}$. The covering $\mathcal{R}$ is *non-redundant* if there is a total function $\mathcal{D}$ from indices of $\mathcal{R}$ to a set of completely disjoint subrectangles of $\cup\mathcal{R}$ that "defines" the elements of $\mathcal{R}$. To be precise, the covering $\mathcal{R}$ is *non-redundant* if there is a function $\mathcal{D}$ with the same source as $\mathcal{R}$ such that

$$\begin{array}{ll} & \langle \forall k \;::\; \mathrm{rectangle}.(\mathcal{D}.k) \wedge \mathcal{D}.k \subseteq \mathcal{R}.k \rangle \\ \wedge & \langle \forall j,k \;::\; \mathcal{D}.j \neq \mathcal{D}.k \;\equiv\; (\mathcal{D}.j)_< \cap (\mathcal{D}.k)_< = \perp\!\!\perp \;\wedge\; (\mathcal{D}.j)_> \cap (\mathcal{D}.k)_> = \perp\!\!\perp \rangle \\ \wedge & \langle \forall j,k \;::\; \mathcal{D}.j = \mathcal{D}.k \;\equiv\; \mathcal{R}.j = \mathcal{R}.k \rangle \quad . \end{array}$$

In such a case, we call the indexed bag $\mathcal{D}$ a *definiens* of $\mathcal{R}$.
□

Definition 210 Suppose $\mathcal{R}$ is a polar covering of relation R. The *polar ordering* of the elements of $\mathcal{R}$, denoted henceforth by the symbol $\sqsubseteq$, is defined by, for all indices j and k of $\mathcal{R}$,

$$\mathcal{R}.j \sqsubseteq \mathcal{R}.k \;\equiv\; (\mathcal{R}.j)_< \subseteq (\mathcal{R}.k)_< \quad .$$

Equivalently,

$$\mathcal{R}.j \sqsubseteq \mathcal{R}.k \;\equiv\; (\mathcal{R}.k)_> \subseteq (\mathcal{R}.j)_> \quad .$$

□

As suggested by the notation, the relation $\sqsubseteq$ is a provisional ordering on the elements of any indexed bag of relations; it is anti-symmetric whenever $\mathcal{R}$ is an indexed bag of polar rectangles by virtue of lemma 125 and definition 209 of "polar".

Definition 209 defines an indexed *bag* of rectangles rather than an indexed *set* of rectangles. (Recall that a set is an injective bag: see definition 129.) This is because it is easier to construct a bag rather than a set of polar rectangles that cover a given relation. Nevertheless, (indexed) sets are more desirable than (indexed) bags. The process we use to construct such sets is to first construct a bag and then show how to reduce the bag to a set. See theorem 215. Note that a definiens $\mathcal{D}$ of an indexed set $\mathcal{R}$ is also a set (because $\mathcal{R}.j = \mathcal{R}.k$ equivales $j = k$).

The adjective "polar" alludes to the property that the left and right domains of a covering are "polar" opposites: the larger the one, the smaller the other. The notion was introduced by Riguet [Rig51] in the context of a theorem on "relations de Ferrers". More precisely, Riguet introduced the notion of a *linear* polar covering. For further details of Riguet's theorem see section 11.

As we shall see, Riguet's theorem is straightforward. The following, equally straightforward theorem, is a generalisation of the "only-if" part of the theorem.

Theorem 211 Suppose R is a relation of type $A{\sim}B$. Define the function $\mathcal{R}$ by

$$\mathcal{R} \;=\; \langle b : b \subseteq R{>} : R \circ b \circ R\backslash R\rangle \;\; .$$

Then $\mathcal{R}$ is a polar covering of R.

Proof The elements of $\mathcal{R}$ are obviously rectangles because its index set is a set of points. (See lemma 124.) The property

$$R \;=\; \langle \cup b : b \subseteq R{>} : R \circ b \circ R\backslash R\rangle$$

is immediate from the saturation axiom (16), distributivity and cancellation.

The "polar" property is established as follows. For all b, b' such that $b \subseteq R{>}$ and $b' \subseteq R{>}$,

$$
\begin{array}{ll}
 & (R \circ b' \circ R\backslash R){>} \;\subseteq\; (R \circ b \circ R\backslash R){>} \\
= & \{ \quad \text{assumption: } b \subseteq R{>} \text{ and } b' \subseteq R{>}\text{, domains} \quad \} \\
 & (b' \circ R\backslash R){>} \;\subseteq\; (b \circ R\backslash R){>} \\
= & \{ \quad \text{lemma 60 with } R,a,a' := R\backslash R\,,\, b\,,\, b' \quad \} \\
 & b \circ \top\!\!\top \circ b' \;\subseteq\; (R\backslash R)/(R\backslash R) \\
= & \{ \quad (30) \quad \} \\
 & b \circ \top\!\!\top \circ b' \;\subseteq\; R\backslash R
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{lemma 60} \quad \} \\
& (R{\circ}b)_< \subseteq (R{\circ}b')_< \\
= & \{ \quad I \subseteq R\backslash R \text{, domains} \quad \} \\
& (R \circ b \circ R\backslash R)_< \subseteq (R \circ b' \circ R\backslash R)_< \quad .
\end{array}
$$

$\Box$

Example 212 The less-than relation on real numbers has a polar covering. Specifically, suppose x is a real number. Let $lt.x$ denote $\{y : y{\in}\mathbb{R} : y < x\}$ and $al.x$ denote $\{y : y{\in}\mathbb{R} : x \leq y\}$. Theorem 211 predicts that

$$\{x : x{\in}\mathbb{R} : lt.x \circ \top\!\!\top \circ al.x\}$$

is a polar covering of the less-than relation. (The only non-trivial part is to check that the at-most relation $\leq$ equals $<\backslash<$.)

This covering is, of course, not unique. More significantly, it is *not* non-redundant since

$$\langle \forall u,v \;:\; u < x \leq v \;:\; x \neq \tfrac{1}{2}(u+x) \;\wedge\; u < \tfrac{1}{2}(u+x) \leq v \rangle \quad .$$

For any real number x, it is possible to remove the rectangle defined by x without affecting the supremum.
$\Box$

Given the straightforwardness of theorem 211, it is inevitable that our focus is not on the polarity of coverings but on the existence of *non-redundant* coverings. The adjective “non-redundant” is meant to express the property that removal of any element from a covering $\mathcal{R}$ will have the effect of strictly reducing $\cup\mathcal{R}$. (Removal of an element may involve removing several elements of K since there is no requirement that $\mathcal{R}$ is injective.) Example 212 demonstrates that the less-than relation on real numbers has a polar covering but, as we shall see, the less-than relation on real numbers is an example of a relation for which there is no non-redundant covering.

The notation “$\mathcal{D}$” in definition 209 is chosen primarily to express the property that $\mathcal{D}.k$ uniquely “defines” (or “identifies”) $\mathcal{R}.k$. Conveniently, it also expresses the property that the relation covered by a definiens (the relation $\cup\mathcal{D}$) is always difunctional: see theorem 163.

A polar covering is not *obviously redundant* in the sense that, for all elements U and V of $\mathcal{R}$,

$$U \subseteq V \;\equiv\; U = V \quad .$$

(The easy proof is left to the reader.) That is, it is not possible to identify two elements U and V such that U is a proper subset of V and, thus, U can be removed from $\mathcal{R}$ without affecting $\cup\mathcal{R}$. Example 212 shows that the less-than relation on real numbers has a polar covering that has non-obvious redundancies. Example 213 is an example of a finite relation for which the polar covering constructed by theorem 211 has a non-obvious redundancy.

Example 213 Fig. 8 shows a relation R of type $\{A,B,C\}\sim\{\alpha,\beta,\gamma,\delta\}$. The four relations depicted in fig. 9 are rectangles of type $\{A,B,C\}\sim\{\alpha,\beta,\gamma,\delta\}$ (as indicated by the surrounding rectangular boxes); for greater clarity only edges connecting nodes in their left and right domains have been displayed.

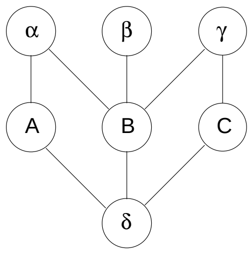


Figure 8: A Relation of Type $\{A,B,C\}\sim\{\alpha,\beta,\gamma,\delta\}$

These four rectangles are the elements of the polar covering constructed by theorem 211. The (reflexive-transitive reduction of the) ordering on the elements of the covering is depicted by arrowed brown lines. Take care to note how the depicted edges correspond to the ordering of the left domains of the rectangles:

$$\{B\}\subseteq\{A,B\} \wedge \{B\}\subseteq\{B,C\} \wedge \{A,B\}\subseteq\{A,B,C\} \wedge \{B,C\}\subseteq\{A,B,C\} \ ,$$

and to the "polar" ordering of their right domains:

$$\{\alpha,\beta,\gamma,\delta\}\supseteq\{\alpha,\delta\} \wedge \{\alpha,\beta,\gamma,\delta\}\supseteq\{\beta,\delta\} \wedge \{\alpha,\delta\}\supseteq\{\delta\} \wedge \{\beta,\delta\}\supseteq\{\delta\} \ .$$

The top rectangle is redundant (but not "obviously" so). By removing this rectangle, one obtains a non-redundant polar covering: this is the polar covering that is the dual of the covering detailed in theorem 211 (thus indexed by $\{A,B,C\}$ rather than $\{\alpha,\beta,\gamma,\delta\}$). The definiens of this covering is depicted by the bold green edges in fig. 9.

The red and blue squares surrounding instances of the elements of $\{A,B,C\}$ and $\{\alpha,\beta,\gamma,\delta\}$ should be ignored for the moment. We return to this example later; see example 284.

□

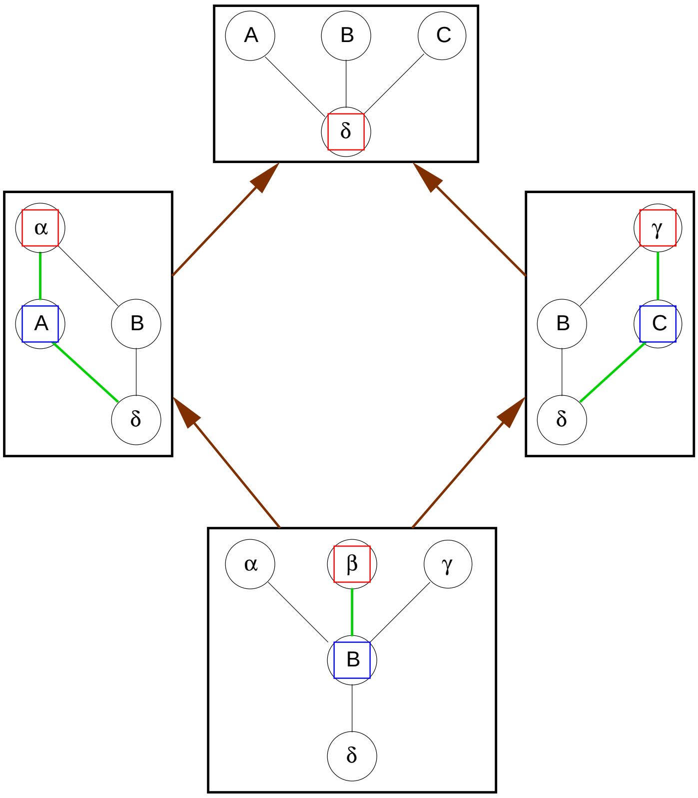


Figure 9: Polar Covering

8.1 Injective Polar Coverings

Separate from the issue of non-redundancy is the issue of duplications: our definition of a polar covering does not exclude the possibility of there being distinct indices j and k such that $\mathcal{R}.j = \mathcal{R}.k$. In general, this will be the case for the construction given in theorem 211. This can be remedied by taking as index set the equivalence classes of the per $R\succ$. With ρ being a functional relation such that $R\succ = \rho^{\cup} \circ \rho$ as in definition 191 (so, for all b such that $b \subseteq R>$, $\rho.b$ is the equivalence class of b according to the right per-domain $R\succ$), the function $\mathcal{R}$ defined by

$$\mathcal{R} \;=\; \left\langle c \,:\, c \subseteq \rho< \,:\, R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R \right\rangle$$

is a polar covering of R with the property that all elements are distinct. This is formalised in theorem 215. The crucial property is that, when applied to the core of a relation, the construction of theorem 211 yields an injective covering.

(Duplications are not evident in small examples because, as mentioned earlier, when constructing small examples, it is common to construct a relation that is isomorphic to its core. This is the case, for instance, for example 213.)

Lemma 214 The covering $\langle \cup c : c \subseteq |R|> : |R| \circ c \circ |R| \backslash |R| \rangle$ of the core $|R|$ of a relation R is injective.

Proof By the (pointwise) definition of injectivity, we have to prove that

$$\langle \forall c,c' \;:\; c \subseteq |R|> \wedge\, c' \subseteq |R|> \;:\; |R| \circ c \circ |R| \backslash |R| \;=\; |R| \circ c' \circ |R| \backslash |R| \;\equiv\; c = c' \rangle$$

where c and c' range over points in $|R|>$. We have

$$
\begin{array}{ll}
 & |R| \circ c \circ |R| \backslash |R| \;=\; |R| \circ c' \circ |R| \backslash |R| \\
= & \{ \quad \text{both terms are rectangles, lemma 125} \quad \} \\
 & (|R| \circ c)< \;=\; (|R| \circ c')< \quad \wedge \quad (c \circ |R| \backslash |R|)> \;=\; (c' \circ |R| \backslash |R|)> \\
= & \{ \quad \text{the covering is polar: theorem 211} \quad \} \\
 & (|R| \circ c)< \;=\; (|R| \circ c')< \\
= & \{ \quad c \text{ and } c' \text{ are points in } |R|>\text{, lemma 103} \quad \} \\
 & c \circ \top\!\top \circ c' \;\subseteq\; |R|\succ \\
= & \{ \quad \text{lemma 197} \quad \} \\
 & c \circ \top\!\top \circ c' \;\subseteq\; |R|> \\
= & \{ \quad c \text{ and } c' \text{ are points, (21)} \quad \} \\
 & c = c' \;\; .
\end{array}
$$

□

Theorem 215 Suppose R, λ and ρ are as in definition 191. Define the function $\mathcal{C}$ by

$$\mathcal{C} \;=\; \left\langle c : c \subseteq \rho< : \lambda \circ R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R \circ \rho^{\cup} \right\rangle \;,$$

where the dummy c ranges over points. Then $\mathcal{C}$ is a polar covering of $|R|$. It follows that the function $\mathcal{R}$ defined by

$$\mathcal{R} \;=\; \left\langle c : c \subseteq \rho< : R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R \right\rangle$$

is a polar covering of R. Moreover, both $\mathcal{C}$ and $\mathcal{R}$ are injective.

Proof First let us show that $\mathcal{C}$ is the same as the covering of $|R|$ defined by theorem 211.

$$\begin{array}{ll}
 & \mathcal{C} \\
= & \{ \quad \text{definition of } \mathcal{C} \quad \} \\
 & \langle c : c \subseteq \rho< : \lambda \circ R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R \circ \rho^{\cup} \rangle \\
= & \{ \quad (204) \quad \} \\
 & \langle c : c \subseteq \rho< : \lambda \circ R \circ \rho^{\cup} \circ c \circ \rho< \circ (\lambda \circ R \circ \rho^{\cup}) \backslash (\lambda \circ R \circ \rho^{\cup}) \circ \rho< \rangle \\
= & \{ \quad \text{lemma 195 and domains} \quad \} \\
 & \langle c : c \subseteq |R|> : \lambda \circ R \circ \rho^{\cup} \circ c \circ (\lambda \circ R \circ \rho^{\cup}) \backslash (\lambda \circ R \circ \rho^{\cup}) \rangle \\
= & \{ \quad \text{definition 191 of } |R| \quad \} \\
 & \langle c : c \subseteq |R|> : |R| \circ c \circ |R| \backslash |R| \rangle \;.
\end{array}$$

It follows, by theorem 211 that $\mathcal{C}$ is a polar covering of $|R|$.

Now we show that $\mathcal{R}$ is a polar covering of R. It is a covering of R since

$$\begin{array}{ll}
 & R \\
= & \{ \quad \text{lemma 193} \quad \} \\
 & \lambda^{\cup} \circ |R| \circ \rho \\
= & \{ \quad \mathcal{C} \text{ is a covering of } |R| \quad \} \\
 & \lambda^{\cup} \circ \langle \cup c : c \subseteq \rho< : \lambda \circ R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R \circ \rho^{\cup} \rangle \circ \rho \\
= & \{ \quad \text{distributivity} \quad \}
\end{array}$$

$$\langle \cup c : c \subseteq \rho_< : \lambda^\cup \circ \lambda \circ R \circ \rho^\cup \circ c \circ \rho \circ R\backslash R \circ \rho^\cup \circ \rho \rangle$$

$=$ { by definition 191, $\lambda^\cup \circ \lambda = R_\prec$ and $\rho^\cup \circ \rho = R_\succ$, $R_\prec \circ R \;=\; R \;=\; R \circ R_\succ$ }

$$\langle \cup c : c \subseteq \rho_< : R \circ \rho^\cup \circ c \circ \rho \circ R\backslash R \circ R_\succ \rangle$$

$=$ { (102) }

$$\langle \cup c : c \subseteq \rho_< : R \circ \rho^\cup \circ c \circ \rho \circ R\backslash R \circ R_> \rangle$$

$=$ { using properties of domains, lemma 195 and cancellation, $(\rho \circ R\backslash R)_> = R_>$ }

$$\langle \cup c : c \subseteq \rho_< : R \circ \rho^\cup \circ c \circ \rho \circ R\backslash R \rangle \;.$$

We conclude that the function $\mathcal{R}$ is a covering of R.

In order to prove that $\mathcal{R}$ is polar, we first note that

(216) $\quad (R \circ \rho^\cup \circ c)_< \;\subseteq\; (R \circ \rho^\cup \circ c')_< \;\;\equiv\;\; (\lambda \circ R \circ \rho^\cup \circ c)_< \;\subseteq\; (\lambda \circ R \circ \rho^\cup \circ c')_<$

and

(217) $\quad (c \circ \rho \circ R\backslash R)_> \;\supseteq\; (c' \circ \rho \circ R\backslash R)_> \;\;\equiv\;\; (c \circ \rho \circ R\backslash R \circ \rho^\cup)_> \;\supseteq\; (c' \circ \rho \circ R\backslash R \circ \rho^\cup)_>$

since

$$(R \circ \rho^\cup \circ c)_< \;\subseteq\; (R \circ \rho^\cup \circ c')_<$$

$\Rightarrow$ { domains and monotonicity }

$$(\lambda \circ R \circ \rho^\cup \circ c)_< \;\subseteq\; (\lambda \circ R \circ \rho^\cup \circ c')_<$$

$\Rightarrow$ { domains and monotonicity }

$$(\lambda^\cup \circ \lambda \circ R \circ \rho^\cup \circ c)_< \;\subseteq\; (\lambda^\cup \circ \lambda \circ R \circ \rho^\cup \circ c')_<$$

$=$ { $\lambda^\cup \circ \lambda = R_\prec$ and $R_\prec \circ R = R$ }

$$(R \circ \rho^\cup \circ c)_< \;\subseteq\; (R \circ \rho^\cup \circ c')_<$$

and

$$(c \circ \rho \circ R\backslash R)_> \;\supseteq\; (c' \circ \rho \circ R\backslash R)_>$$

$\Rightarrow$ { domains and monotonicity }

$$(c \circ \rho \circ R\backslash R \circ \rho^\cup)_> \;\supseteq\; (c' \circ \rho \circ R\backslash R \circ \rho^\cup)_>$$

$\Rightarrow$ { domains and monotonicity }

$$(c \circ \rho \circ R\backslash R \circ \rho^\cup \circ \rho)_> \;\supseteq\; (c' \circ \rho \circ R\backslash R \circ \rho^\cup \circ \rho)_>$$

$$
\begin{array}{ll}
= & \{ \quad \rho^{\cup} \circ \rho = R_{>} \text{ and } (102) \quad \} \\
& (c \circ \rho \circ R \backslash R \circ R_{>})_{>} \;\supseteq\; (c' \circ \rho \circ R \backslash R \circ R_{>})_{>} \\
= & \{ \quad (\rho \circ R \backslash R)_{>} \;=\; (R \circ R \backslash R)_{>} \;=\; R_{>} \quad \} \\
& (c \circ \rho \circ R \backslash R)_{>} \;\supseteq\; (c' \circ \rho \circ R \backslash R)_{>} \quad .
\end{array}
$$

We are now in a position to prove that $\mathcal{R}$ is polar:

$$
\begin{array}{ll}
& (R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R)_{<} \;\subseteq\; (R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R)_{<} \\
= & \{ \quad c \text{ and } c' \text{ are points, domains of rectangles} \quad \} \\
& (R \circ \rho^{\cup} \circ c)_{<} \;\subseteq\; (R \circ \rho^{\cup} \circ c')_{<} \\
= & \{ \quad (216) \quad \} \\
& (\lambda \circ R \circ \rho^{\cup} \circ c)_{<} \;\subseteq\; (\lambda \circ R \circ \rho^{\cup} \circ c')_{<} \\
= & \{ \quad \mathcal{C} \text{ is a polar covering of } |R| \quad \} \\
& (c \circ \rho \circ R \backslash R \circ \rho^{\cup})_{>} \;\supseteq\; (c' \circ \rho \circ R \backslash R \circ \rho^{\cup})_{>} \\
= & \{ \quad (217) \quad \} \\
& (c \circ \rho \circ R \backslash R)_{>} \;\supseteq\; (c' \circ \rho \circ R \backslash R)_{>} \\
= & \{ \quad c \text{ and } c' \text{ are points, domains of rectangles} \quad \} \\
& (R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R)_{>} \;\supseteq\; (R \circ \rho^{\cup} \circ c' \circ \rho \circ R \backslash R)_{>} \quad .
\end{array}
$$

Thus, by definition, the function $\mathcal{R}$ is polar.

Now we turn to the injectivity of $\mathcal{C}$ and $\mathcal{R}$. Lemma 214 establishes that $\mathcal{C}$ is injective. In order to show that $\mathcal{R}$ is injective, assume c and c' are points such that $c \subseteq \rho_{<}$ and $c' \subseteq \rho_{<}$. Then

$$
\begin{array}{ll}
& R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R \;=\; R \circ \rho^{\cup} \circ c' \circ \rho \circ R \backslash R \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
& \lambda \circ R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R \circ \rho^{\cup} \;=\; \lambda \circ R \circ \rho^{\cup} \circ c' \circ \rho \circ R \backslash R \circ \rho^{\cup} \\
= & \{ \quad \mathcal{C} \text{ is injective} \quad \} \\
& c = c' \quad .
\end{array}
$$

Since the converse follows from Leibniz's rule, we have thus proved that $\mathcal{R}$ is injective. We conclude that $\mathcal{R}$ is an injective, polar covering of R.

□

Our definition of a definiens does not include any maximality requirement. (In general, given a definiens $\mathcal{D}$ of a covering $\mathcal{R}$, a *minimal* definiens can be constructed by choosing exactly one point of each element of $\mathcal{D}$. On the other hand, maximality means that no additional points can be added without invalidating the definiens property.) It is possible that the definiens that we construct are indeed maximal but this is something we have not investigated.

If R is a finite relation, the construction of theorem 211 can be used to construct a non-redundant, injective, polar covering and its definiens. The covering is initialised to $\mathcal{R}$ as constructed by theorem 211 and the index set K of $\mathcal{R}$ is initialised to all points b in $R{>}$. The index set K' of $\mathcal{D}$ is initialised to the empty set. Then each point b in K is examined, one by one. If $R\circ b\circ R\backslash R$ is redundant (i.e. b can be removed from K without affecting $\cup\mathcal{R}$) then b is removed from K. If not, b is retained in K and added to K'. Also $\mathcal{D}.b$ is defined by

$$\mathcal{D}.b \;=\; R\circ b\circ R\backslash R \cap \neg\langle\cup b' : b'{\in}K \wedge b{\neq}b' : R\circ b'\circ R\backslash R\rangle \quad .$$

(So $\mathcal{D}.b$ is that part of the covering identified by b.) Assuming $R{>}$ is finite, this process will terminate with a non-redundant, injective, polar covering of R indexed by K.

8.2 Non-Redundant Polar Coverings

We have shown in theorem 215 how to construct an injective polar covering of a given relation R. Now we consider circumstances in which the covering is non-redundant. In the case that R is difunctional, it is straightforward to show that the covering constructed in theorem 211 is non-redundant and is its own definiens. (It is in this sense that theorem 211 generalises theorem 163.) This suggests that, in general, a covering of the diagonal of a relation R can be used as the definiens of a covering of R. This is indeed true so long as the diagonal is sufficiently large[5]. Specifically, we prove below that, for all relations R, if $(\Delta R){>} = R{>}$, the covering $\mathcal{R}$ defined by theorem 215 is non-redundant as witnessed by the function $\mathcal{D}$ defined by

$$\mathcal{D}.c \;=\; \Delta R\circ\rho^{\cup}\circ c\circ\rho \quad .$$

First, we show that it is a covering of ΔR.

Theorem 218 Suppose R is a relation and $R{\succ} = \rho^{\cup}\circ\rho$ where $\rho\circ\rho^{\cup} = \rho{<}$. Then the function $\mathcal{D}$ defined by

$$\mathcal{D} \;=\; \left\langle c : c\subseteq\rho{<} : \Delta R\circ\rho^{\cup}\circ c\circ\rho\right\rangle$$

is a covering of ΔR. That is,

$$\Delta R \;=\; \left\langle\cup c : c\subseteq\rho{<} : \Delta R\circ\rho^{\cup}\circ c\circ\rho\right\rangle \quad .$$

[5]But note example 224 below.

Moreover, if $(\Delta R)_> = R_>$, for all points c and c' such that $c \subseteq \rho_<$ and $c' \subseteq \rho_<$,

$$c \neq c' \quad \equiv \quad (\Delta R \circ \rho^\cup \circ c \circ \rho)_< \circ (\Delta R \circ \rho^\cup \circ c' \circ \rho)_< \;=\; \perp\!\!\perp$$

and

$$c \neq c' \quad \equiv \quad (\Delta R \circ \rho^\cup \circ c \circ \rho)_> \circ (\Delta R \circ \rho^\cup \circ c' \circ \rho)_> \;=\; \perp\!\!\perp \quad .$$

It follows that, if $(\Delta R)_> = R_>$, $\mathcal{D}$ is a completely disjoint, injective covering of ΔR.

Proof That each element of $\mathcal{D}$ is a rectangle is a consequence of lemma 124. Now we show that $\mathcal{D}$ covers ΔR:

$$
\begin{array}{ll}
& \langle \cup c : c \subseteq \rho_< : \Delta R \circ \rho^\cup \circ c \circ \rho \rangle \\
= & \{ \quad \text{distributivity} \quad \} \\
& \Delta R \circ \rho^\cup \circ \langle \cup c : c \subseteq \rho_< : c \rangle \circ \rho \\
= & \{ \quad \text{saturation axiom: (16)} \quad \} \\
& \Delta R \circ \rho^\cup \circ \rho_< \circ \rho \\
= & \{ \quad \text{domains, } R_\succ = \rho^\cup \circ \rho \quad \} \\
& \Delta R \circ R_\succ \\
= & \{ \quad \text{domains} \quad \} \\
& \Delta R \circ (\Delta R)_> \circ R_\succ \\
= & \{ \quad \text{lemma 189} \quad \} \\
& \Delta R \circ (\Delta R)_\succ \\
= & \{ \quad \text{(96) with } R := \Delta R \quad \} \\
& \Delta R \quad .
\end{array}
$$

We use lemma 132 to show that $\mathcal{D}$ is completely disjoint and injective. First, we show that the elements are non-empty.

$$
\begin{array}{ll}
& \Delta R \circ \rho^\cup \circ c \circ \rho \;=\; \perp\!\!\perp \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
& (\Delta R \circ \rho^\cup \circ c \circ \rho)_> \;=\; \perp\!\!\perp \\
= & \{ \quad \text{domains} \quad \} \\
& ((\Delta R)_> \circ \rho^\cup \circ c \circ \rho)_> \;=\; \perp\!\!\perp \\
= & \{ \quad \text{assumption: } (\Delta R)_> = R_> \quad \}
\end{array}
$$

$$(R{>} \circ \rho^{\cup} \circ c \circ \rho){>} \;=\; \bot\!\!\bot$$

$=\qquad \{ \qquad R{>} = (R{\succ}){>} = (\rho^{\cup} \circ \rho){>} = \rho{>} , \quad (\rho{>} \circ \rho^{\cup}){>} = \rho{<} \qquad \}$

$$(\rho{<} \circ c \circ \rho){>} = \bot\!\!\bot$$

$=\qquad \{ \qquad c \subseteq \rho{<} \qquad \}$

$$(c{\circ}\rho){>} = \bot\!\!\bot$$

$\Rightarrow\qquad \{$ domains: (45) and $\bot\!\!\bot$ is zero of composition $\}$

$$c{\circ}\rho = \bot\!\!\bot$$

$\Rightarrow\qquad \{$ $\bot\!\!\bot$ is zero of composition $\}$

$$c \circ \rho \circ \rho^{\cup} \circ c \;=\; \bot\!\!\bot$$

$=\qquad \{$ $\rho \circ \rho^{\cup} \;=\; \rho{<}$ and $c \subseteq \rho{<}$ $\}$

$$c = \bot\!\!\bot$$

$=\qquad \{$ c is a point $\}$

false .

That is,

(219) $\langle \forall c \;:\; c \subseteq \rho{<} \;:\; \Delta R \circ \rho^{\cup} \circ c \circ \rho \;\neq\; \bot\!\!\bot \rangle$.

For the second proof obligation (see lemma 132), assume that $c \neq c'$. Because the calculation is easier, we begin with the right domains. We have:

$$\Delta R \circ \rho^{\cup} \circ c \circ \rho \circ \rho^{\cup} \circ c' \circ \rho \circ \Delta R^{\cup}$$

$=\qquad \{$ $c \subseteq \rho{<}$ and $\rho \circ \rho^{\cup} = \rho{<}$ $\}$

$$\Delta R \circ \rho^{\cup} \circ c \circ c' \circ \rho \circ \Delta R^{\cup}$$

$=\qquad \{$ assumption: $c \neq c'$, (17) $\}$

$\bot\!\!\bot$.

That is, applying properties of converse,

(220) $\langle \forall c,c' \;:\; c \subseteq \rho{<} \wedge c \neq c' \;:\; (\Delta R \circ \rho^{\cup} \circ c \circ \rho) \circ (\Delta R \circ \rho^{\cup} \circ c' \circ \rho)^{\cup} = \bot\!\!\bot \rangle$.

The calculation for the left domains is similar but slightly more complex. We have:

$$\rho^{\cup} \circ c \circ \rho \circ (\Delta R)^{\cup} \circ \Delta R \circ \rho^{\cup} \circ c' \circ \rho$$

$=\qquad \{$ ΔR is difunctional, theorem 160 (with $R := \Delta R$) $\}$

$$\rho^{\cup} \circ c \circ \rho \circ (\Delta R){\succ} \circ \rho^{\cup} \circ c' \circ \rho$$

$$
\begin{array}{ll}
= & \{ \quad \text{assumption: } (\Delta R)_{>} = R_{>} \text{, lemma 190} \quad \} \\
& \rho^{\cup} \circ c \circ \rho \circ R_{\succ} \circ \rho^{\cup} \circ c' \circ \rho \\
= & \{ \quad R_{\succ} = \rho^{\cup} \circ \rho \,,\ \rho \circ \rho^{\cup} = \rho_{<} = \rho_{<} \circ \rho_{<} \text{ and } c \subseteq \rho_{<} \quad \} \\
& \rho^{\cup} \circ c \circ c' \circ \rho \\
= & \{ \quad \text{assumption: } c \neq c' \text{, (17)} \quad \} \\
& \perp\!\!\perp \ .
\end{array}
$$

That is, again applying properties of converse,

$$(221) \quad \langle \forall c,c' \ : \ c \subseteq \rho_{<} \wedge c \neq c' \ : \ (\Delta R \circ \rho^{\cup} \circ c \circ \rho)^{\cup} \circ (\Delta R \circ \rho^{\cup} \circ c' \circ \rho) = \perp\!\!\perp \rangle \ .$$

The combination of (219), (220) and (221) together with lemma 132 establishes that $\mathcal{D}$ is completely disjoint and injective.

□

It is now easy to see that $\mathcal{D}$ is a definiens of the injective polar covering of R defined in theorem 215:

Theorem 222 Suppose R is a relation such that $(\Delta R)_{>} = R_{>}$. Suppose also that $R_{\succ} = \rho^{\cup} \circ \rho$ where $\rho \circ \rho^{\cup} = \rho_{<}$. Then the indexed bag $\mathcal{R}$ of rectangles defined by

$$\mathcal{R} \ = \ \langle c : c \subseteq \rho_{<} : R \circ \rho^{\cup} \circ c \circ \rho \circ R \backslash R \rangle$$

is a non-redundant, injective polar covering of R. (In particular, $\mathcal{R}$ is an indexed set.) A definiens of the covering is the indexed set $\mathcal{D}$ defined by

$$\mathcal{D} \ = \ \langle c : c \subseteq \rho_{<} : \Delta R \circ \rho^{\cup} \circ c \circ \rho \rangle \ .$$

Moreover, by theorem 218, $\mathcal{D}$ is a covering of ΔR.

Proof Theorem 215 shows that $\mathcal{R}$ is an injective polar covering of R. It remains to show that it is non-redundant as witnessed by the function $\mathcal{D}$.

For all c such that $c \subseteq \rho_{<}$, the property $\mathcal{D}.c \subseteq \mathcal{R}.c$ is immediate from $\Delta R \subseteq R$, $I \subseteq R \backslash R$ and monotonicity of composition. That the elements of $\mathcal{D}$ form a completely disjoint set of rectangles was shown in theorem 218. It remains to show that $\mathcal{D}$ "defines" $\mathcal{R}$. We have, for all c and c' such that $c \subseteq \rho_{<}$ and $c' \subseteq \rho_{<}$,

$$
\begin{array}{ll}
& \mathcal{R}.c = \mathcal{R}.c' \\
= & \{ \quad \text{theorem 215} \quad \} \\
& c = c' \\
= & \{ \quad \text{theorem 218} \quad \} \\
& \mathcal{D}.c = \mathcal{D}.c' \ .
\end{array}
$$

□

Example 223

Fig. 10 pictures a small example of the theorems in this section. Fig. 10(a) depicts a relation R of type $\{\alpha,\beta,\gamma\}\sim\{A,B\}$; other parts of the figure depict the result of applying different functions to the relation R. (Heterogeneous relations are depicted as bipartite graphs whereas homogeneous relations are depicted as directed graphs.) Specifically, these are as follows.

$$(a)\ R\ ,\quad (b)\ \Delta R\ ,$$

$$(c)\ R\backslash R\ ,\quad (d)\ R/R\ ,$$

$$(e)\ R\circ A\circ R\backslash R\ ,\quad (f)\ R\circ B\circ R\backslash R\ ,$$

$$(g)\ \Delta R\circ A\circ R_{\succ}\ ,\quad (h)\ \Delta R\circ B\circ R_{\succ}\ ,$$

$$(i)\ R/R\circ\alpha\circ R\ ,\ (j)\ R/R\circ\beta\circ R\ ,\ (k)\ R/R\circ\gamma\circ R\ .$$

We have chosen to depict the relation as a graph (rather than a boolean matrix) because —for very small examples such as this— it is much easier for a human being to perform the necessary calculations by manipulating the graphs. For example, computing the composition of two relations is executed by chasing edges. Also —again for such very small examples— the definition of factors in terms of nested complements is much easier to use. This said, we leave the reader to check our calculations.

The example has been chosen deliberately to illustrate a number of aspects simultaneously. Note particularly that, for the relation depicted, $(\Delta R)_{>} = R_{>}$ but $(\Delta R)_{<} \neq R_{<}$. This means that theorem 222 is applicable but its dual is not.

Note that (as forewarned: see example 192) the relation R is isomorphic to its own core. So the functional ρ in theorem 222 is effectively the identity function and the construction given there is identical to the construction in theorem 211.

Considering the application of theorem 211, note that the combination of figs. 10(e) and 10(f) covers the relation R; also the relation depicted by 10(g) uniquely identifies the rectangle $R\circ A\circ R\backslash R$ shown in fig. 10(e) whilst 10(h) uniquely identifies the rectangle $R\circ A\circ R\backslash R$ shown in fig. 10(f). In contrast, figs. 10(i), (j) and (k) depict the relations $R/R\circ\alpha\circ R$, $R/R\circ\beta\circ R$ and $R/R\circ\gamma\circ R$ but none of these is identified by any subrectangle: the rectangles depicted by figs. 10(i) and (k) are disjoint but both have a non-empty intersection with the rectangle depicted by fig. 10(j).

□

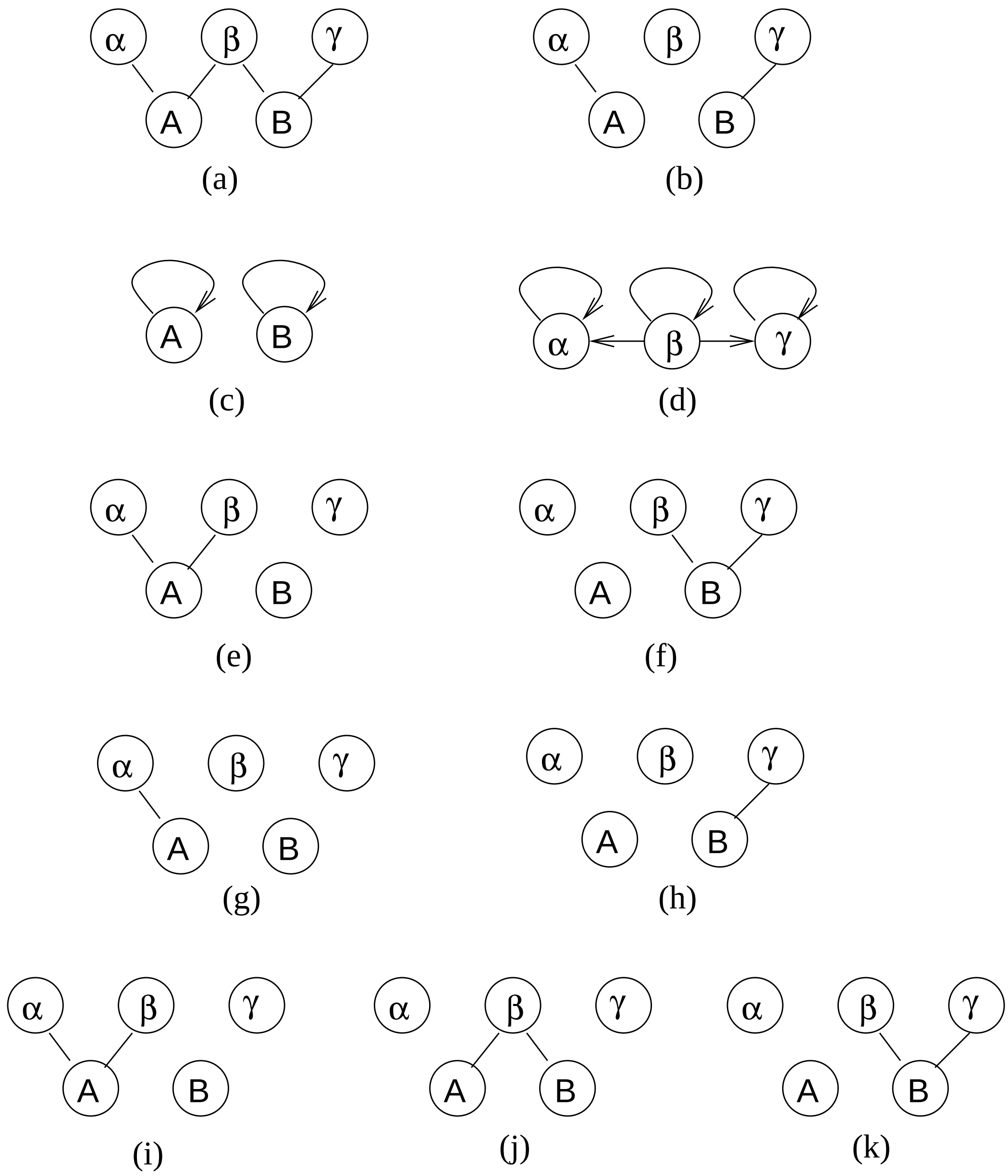


Figure 10: A Small Example

Example 223 is an example of a relation R such that $(\Delta R)_> = R_>$ but $(\Delta R)_< \neq R_<$. It is thus the case that, for this example,

$$R \;=\; \langle \cup b : b \subseteq (\Delta R)_> : R \circ b \circ R\backslash R \rangle \quad .$$

(Note the range restriction on the dummy b.) Curiously, in spite of the fact that $(\Delta R)_< \neq R_<$, it is also the case that

$$R \;=\; \langle \cup a : a \subseteq (\Delta R)_< : R/R \circ a \circ R \rangle \quad .$$

(Again, note the range restriction on the dummy a. To check the validity of the equation, it suffices to observe that the relation R is the union of the relations depicted by figs. 10(i) and (k).) This is also a non-redundant polar covering of R. One might thus conjecture that, in general, the diagonal ΔR is the key to finding a non-redundant polar covering of a given relation R. However, this is not always the case, as evidenced by the following example.

Example 224

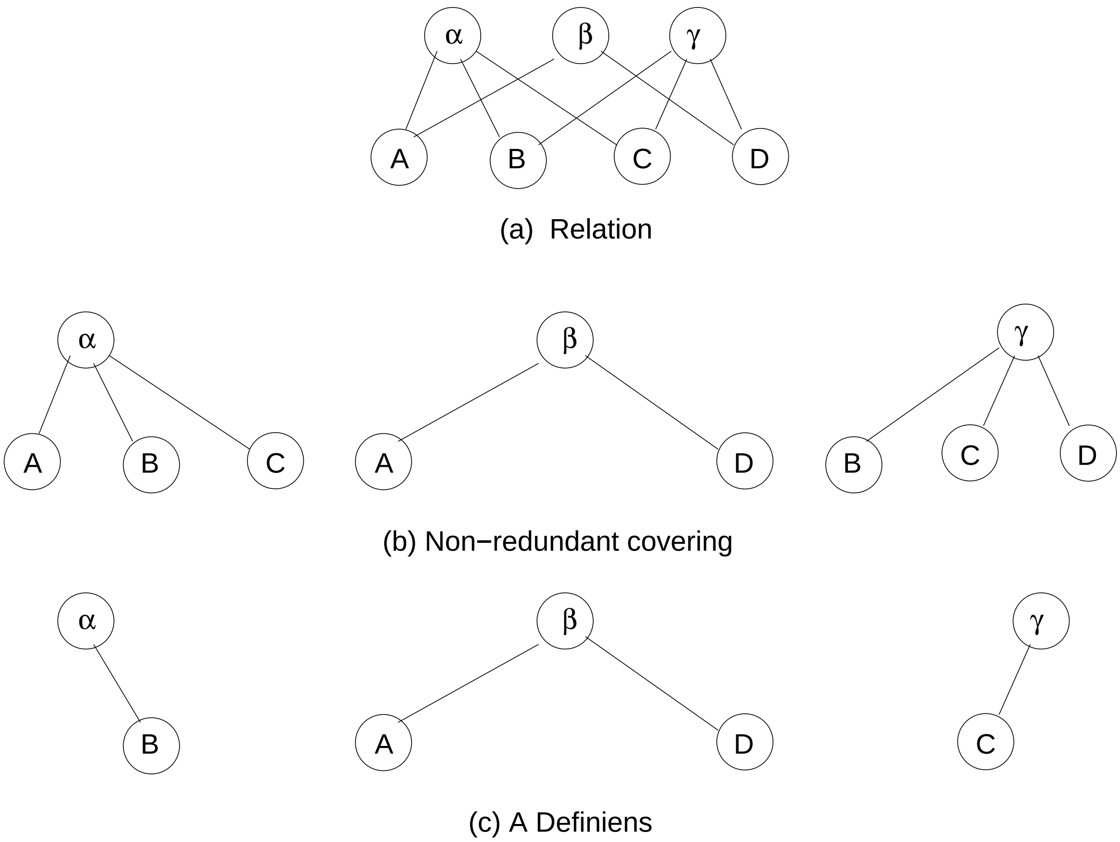


Figure 11: Empty Diagonal and Non-Redundant Covering

The top diagram of fig. 11 pictures a relation R of type $\{\alpha,\beta,\gamma\}\sim\{A,B,C,D\}$ such that ΔR is the empty relation. The example is a simplification of the example on p.161 of [KGJ00].

The three components of the polar covering predicted by the dual of theorem 211 are depicted in the second row. (The index set of the covering is $\{\alpha,\beta,\gamma\}$.) Note that the covering is non-redundant: the third row pictures a function that satisfies the definition of a definiens of the covering. (Again, the index set is $\{\alpha,\beta,\gamma\}$.)

Note that, although the definiens shown in fig. 11 is maximal, it is not unique: the edges from α to B and from γ to C may be replaced by edges from α to C and from γ to B. Other choices are also possible.

Note also that the relation R is not isomorphic to its core since $\{B,C\}$ is an equivalence class of $R\succ$. Conflating B and C to one node in figs. 11(a) and (b) does give a non-redundant covering of the core but this is not witnessed by the graph obtained by conflating B and C in fig. 11(c).

□

9 Block-Ordered Relations

In general, dividing a subset of a set A into blocks is formulated by specifying a functional relation f, say, with source[6] the set A; elements $a0$ and $a1$ are in the same block equivales $f.a0$ and $f.a1$ are both defined and $f.a0 = f.a1$. In mathematical terminology, a functional relation f defines the *partial equivalence relation* $f^{\cup} \circ f$ and the "blocks" are the equivalence classes of $f^{\cup} \circ f$. (Partiality means that some elements may not be in an equivalence class.)

Given functional relations f and g with sources A and B, respectively, and equal left domains, relation R of type $A{\sim}B$ is said to be *block-structured* by f and g if there is a relation S such that $R = f^{\cup} \circ S \circ g$. Informally, whether or not a and b are related by R depends entirely on the "block" $(f.a\,,\,g.b)$ to which they belong. Note that it is not required that f and g be total functions: it suffices that $f{>} = R{<}$ and $g{>} = R{>}$. The type of S is $C{\sim}C$ where C includes $\{a\colon\ a \circ f{>} = a\colon\ f.a\}$ (equally $\{b\colon\ b \circ f{>} = b\colon\ g.b\}$).

Definition 225 (Block-Ordered Relation) Suppose T is a relation of type $C{\sim}C$, f is a relation of type $C{\sim}A$ and g is a relation of type $C{\sim}B$. Suppose further that T is a provisional ordering, i.e. that

$$(226)\quad T \cap T^{\cup} \subseteq I \quad\wedge\quad T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad\wedge\quad T \circ T \subseteq T \ .$$

Suppose also that f and g are functional and onto the domain of T. That is, suppose

$$(227)\quad f \circ f^{\cup} \ =\ f{<} \ =\ T \cap T^{\cup} \ =\ g{<} \ =\ g \circ g^{\cup} \ .$$

Then we say that the relation $f^{\cup} \circ T \circ g$ is a *block-ordered relation*. A relation R of type $A{\sim}B$ is said to be *block-ordered* by f, g and T if $R = f^{\cup} \circ T \circ g$ and $f^{\cup} \circ T \circ g$ is a block-ordered relation.
□

Example 228 The archetypical example of a block-ordered relation is a preorder. Informally, if R is a preorder, its symmetric closure $R \cap R^{\cup}$ is an equivalence relation, and the relation R defines a partial ordering on the equivalence classes. Theorem 157 is a precise statement of the more general theorem that a provisional preorder is block-ordered. Briefly, if R is a provisional preorder, $R \cap R^{\cup}$ is a partial equivalence relation; so, by theorem 143, there is a functional relation f such that

$$R \cap R^{\cup} \ =\ f^{\cup} \circ f \ .$$

Since $R \ =\ (R \cap R^{\cup}) \circ R \circ (R \cap R^{\cup})$ (when R is a provisional preorder), it follows that

$$R \ =\ f^{\cup} \circ (f \circ R \circ f^{\cup}) \circ f \ .$$

[6] In the terminology we use, a relation of type $A{\sim}B$ has *target* A and *source* B.

The parenthesised relation is a provisional ordering of the equivalence classes of $R \cap R^{\cup}$. Thus a provisional preorder R is block-ordered by f, f and $f \circ R \circ f^{\cup}$.
□

Identifying a block-ordering of a relation —if it exists— is important for efficiency. Although a relation is defined to be a set of pairs, relations —even relations on finite sets— are rarely stored as such; instead some base set of pairs is stored and an algorithm used to generate, on demand, additional information about the relation. This is particularly so of ordering relations. For example, a test $m < n$ on integers m and n in a computer program is never implemented as a table lookup; instead an algorithm is used to infer from the basic relations $0 < 1$ together with the internal representation of m and n what the value of the test is. In the case of block-structured relations, functional relations f and g define partial equivalence relations $f^{\cup} \circ f$ and $g^{\cup} \circ g$ on their respective sources. (The relations $f^{\cup} \circ f$ and $g^{\cup} \circ g$ are partial because f and g are not required to be total.) Combining the functional relations with an ordering relation on their (common) target is an effective way of implementing a relation (assuming the ordering relation is also implemented effectively).

Example 229 Suppose G is the edge relation of a finite graph. The relation G^* is, of course, a preorder and so is block-ordered. The block-ordering of G^* given by theorem 157 —see example 228— is, however, not very useful. For practical purposes a block-ordering constructed from G (rather than G^*) is preferable. Here we outline how this is done.

Recall from example 185, that the diagonal $\Delta(G^*)$ is the relation $G^* \cap (G^{\cup})^*$ and that this is an equivalence relation on the nodes of G, whereby the equivalence classes are the *strongly connected components* of G. Let N denote the nodes of G and C denote the set of strongly connected components of G. By theorem 143, there is a function sc of type $C \leftarrow N$ such that

(230) $\quad G^* \cap (G^{\cup})^* \;=\; \mathsf{sc}^{\cup} \circ \mathsf{sc}$.

The relation $\mathcal{A}$ defined by

$$\mathsf{sc} \circ G \circ \mathsf{sc}^{\cup} \;\cap\; \neg I_C$$

is a homogeneous relation on the strongly connected components of G, i.e. a relation of type $C \sim C$. Informally, it is a graph obtained from the graph G by coalescing the nodes in a strongly connected component of G into a single node whilst retaining the edges of G that connect nodes in distinct strongly connected components[7]. A fundamental theorem is that

(231) $\quad G^* \;=\; \mathsf{sc}^{\cup} \circ \mathcal{A}^* \circ \mathsf{sc}$.

[7]Although we don't go into details, for any function f of appropriate type, the graph $f \circ G \circ f^{\cup}$ is "pathwise homomorphic" [McN67] to G .

Moreover, $\mathcal{A}$ is acyclic. That is,

(232) $\quad \mathrm{I_C} \cap \mathcal{A}^{+} \;=\; \perp\!\!\!\perp \quad .$

(See [BDGv22, BDGv21] for the details of the proof of (231) and (232). In fact the theorem is valid for all relations G; finiteness is not required.)

The relation $\mathcal{A}^{*}$ is, of course, transitive. It is also reflexive; combined with its acyclicity, it follows that

(233) $\quad \mathcal{A}^{*} \cap (\mathcal{A}^{*})^{\cup} \;=\; \mathrm{I_C} \quad .$

That is, $\mathcal{A}^{*}$ is a (total) provisional ordering on C. The conclusion is that G^{*} is block-ordered by sc, sc and $\mathcal{A}^{*}$.

Informally, a finite graph can always be decomposed into its strongly connected components together with an acyclic graph connecting the components.

Although the informal interpretation of this theorem is well-known, the formal proof is non-trivial. Although not formulated in the same way, it is essentially the "transitive reduction" of an arbitrary (not necessarily acyclic) graph formulated by Aho, Garey and Ullman [AGU72, Theorem 2].

The decomposition (231) is (implicitly) exploited when computing the inverse $\mathbf{A}^{-1}$ of a real matrix $\mathbf{A}$ in order to minimise storage requirements: using an elimination technique, a so-called "product form" is computed for each strongly connected component, whilst the process of "forward substitution" is applied to the acyclic-graph structure.
□

It is important to note the very strict requirement (227) on the functionals f and g. Note its similarity with the requirement on functionals f and g in the definition of the characterisation of a difunctional relation: definition 162. Were this requirement to be omitted (retaining only that f and g are functional relations *into* —not onto— the domain of T), there would be no guarantee of non-redundancy. As we shall see, our definition of block-ordering does guarantee the existence of a non-redundant polar covering (theorem 255) but not vice-versa (corollary 258). This suggests that the requirement may be too strong. See section 10 and the conclusions for further discussion.

Theorem 234 makes precise the statement that block orderings —where they exist— are unique "up to isomorphism".

Theorem 234 Suppose T is a provisional ordering. That is, suppose

$$\mathrm{T} \cap \mathrm{T}^{\cup} \subseteq \mathrm{I} \quad \wedge \quad \mathrm{T} = (\mathrm{T} \cap \mathrm{T}^{\cup}) \circ \mathrm{T} \circ (\mathrm{T} \cap \mathrm{T}^{\cup}) \quad \wedge \quad \mathrm{T} \circ \mathrm{T} \subseteq \mathrm{T} \quad .$$

Suppose also that f and g are functional and onto the domain of T. That is, suppose

$$\mathrm{f} \circ \mathrm{f}^{\cup} \;=\; \mathrm{f}{<} \;=\; \mathrm{T} \cap \mathrm{T}^{\cup} \;=\; \mathrm{g}{<} \;=\; \mathrm{g} \circ \mathrm{g}^{\cup} \quad .$$

Suppose further[8] that S, h and k satisfy the same properties as T, f and g (respectively) and that

$$(235)\quad f^{\cup}\circ T\circ g \;=\; h^{\cup}\circ S\circ k \;\; .$$

Then

$$(236)\quad f{>}=h{>} \;\wedge\; g{>}=k{>} \;\; ,$$

$$(237)\quad f^{\cup}\circ g = h^{\cup}\circ k \;\; ,$$

$$(238)\quad f^{\cup}\circ T^{\cup}\circ g \;=\; h^{\cup}\circ S^{\cup}\circ k \;\; \text{, and}$$

$$(239)\quad f\circ h^{\cup} \;=\; g\circ k^{\cup} \;\; .$$

Also, letting ϕ denote $f\circ h^{\cup}$ (equally, by (239), $g\circ k^{\cup}$),

$$(240)\quad \phi\circ\phi^{\cup} = T\cap T^{\cup} \;\wedge\; \phi^{\cup}\circ\phi = S\cap S^{\cup} \;\wedge\; \phi\circ T = S\circ\phi \;\; .$$

In words, ϕ is an order isomorphism of the domains of T and S.

Proof In combination with the assumption (235), properties (236), (238) and (237) are immediate from (249), (250) and (251), respectively.

Proof of (239) is a step on the way to proving (240). From symmetry considerations, it is an obvious first step.

$$\begin{array}{cl}
 & f\circ h^{\cup} \\
= & \{ \quad \text{assumption:} \;\; k\circ k^{\cup} = h{<} \quad \} \\
 & f\circ h^{\cup}\circ k\circ k^{\cup} \\
= & \{ \quad (237) \quad \} \\
 & f\circ f^{\cup}\circ g\circ k^{\cup} \\
= & \{ \quad \text{assumption:} \;\; f\circ f^{\cup} = g{<} \quad \} \\
 & g\circ k^{\cup} \;\; .
\end{array}$$

Now,

[8] The types of T and S may be different. The types of f and h, and of g and k will then also be different. As in lemma 248, the requirement is that the types are compatible with the type restrictions on the operators in all assumed properties. The symbol “ I ” in (240) is overloaded: if the type of T is $A{\sim}A$ and the type of S is $B{\sim}B$, $\phi\circ\phi^{\cup}$ has type $A{\sim}A$ and $\phi^{\cup}\circ\phi$ has type $B{\sim}B$.

$$
\begin{array}{ll}
 & \phi \circ \phi^{\cup} \\
= & \{ \quad \text{definition of } \phi \text{, converse} \quad \} \\
 & f \circ h^{\cup} \circ h \circ f^{\cup} \\
= & \{ \quad (239) \quad \} \\
 & g \circ k^{\cup} \circ h \circ f^{\cup} \\
= & \{ \quad (237) \text{ and converse} \quad \} \\
 & g \circ g^{\cup} \circ f \circ f^{\cup} \\
= & \{ \quad \text{assumption:} \quad f \circ f^{\cup} = T \cap T^{\cup} = g \circ g^{\cup} \quad \} \\
 & T \cap T^{\cup} \quad .
\end{array}
$$

Symmetrically, $\phi^{\cup} \circ \phi = T \cap T^{\cup}$. Finally,

$$
\begin{array}{ll}
 & T \circ \phi \\
= & \{ \quad \text{definition of } \phi \quad \} \\
 & T \circ f \circ h^{\cup} \\
= & \{ \quad \text{assumptions:} \quad f \circ f^{\cup} = T \cap T^{\cup} = g \circ g^{\cup} \\
 & \qquad\qquad T \ = \ (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \} \\
 & f \circ f^{\cup} \circ T \circ g \circ g^{\cup} \circ f \circ h^{\cup} \\
= & \{ \quad \text{assumption:} \quad f^{\cup} \circ T \circ g \ = \ h^{\cup} \circ S \circ k \quad \text{, (237) and converse} \quad \} \\
 & f \circ h^{\cup} \circ S \circ k \circ k^{\cup} \circ h \circ h^{\cup} \\
= & \{ \quad \text{assumption:} \quad h \circ h^{\cup} = S \cap S^{\cup} = k \circ k^{\cup} \quad \} \\
 & f \circ h^{\cup} \circ S \\
= & \{ \quad \text{definition of } \phi \quad \} \\
 & \phi \circ S \quad .
\end{array}
$$

□

9.1 Pair Algebras and Galois Connections

In order to find lots of examples of block-ordered relations one need look no further than the theory of Galois connections (which are, of course, ubiquitous). In this section, we briefly review the notion of a "pair algebra" —due to Hartmanis and Stearns [HS64, HS66]— and its relation to Galois connections.

Hartmanis and Stearns studied a particular practical problem: the so-called "state assignment problem". This is the problem of how to encode the states and inputs of a sequential machine in such a way that state transitions can be implemented economically using logic circuits. However, as they made clear in the preface of their book [HS66], their contribution was to "information science" in general:

> It should be stressed, however, that although many structure theory results describe possible physical realizations of machines, the theory itself is independent of the particular physical components of technology used in the realization.
>
> ...
>
> The mathematical foundations of this structure theory rest on an algebraization of the concept of "information" in a machine and supply the algebraic formalism necessary to study problems about the flow of this information.

Hartmanis and Stearns limited their analysis to finite, complete posets, and their analysis was less general than is possible. This work was extended in [Bac98] to non-finite posets and the current section is a short extract.

A Galois connection involves two posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$ and two functions, $F\in\mathcal{A}\leftarrow\mathcal{B}$ and $G\in\mathcal{B}\leftarrow\mathcal{A}$. These four components together form a *Galois connection* iff for all $b\in\mathcal{B}$ and $a\in\mathcal{A}$

(241) $F.b\sqsubseteq a \equiv b\preceq G.a$.

We refer to F as the *lower adjoint* and to G as the *upper adjoint*.

A Galois connection is thus a connection between two functions between posets. Typical accounts of the properties of Galois connections (for e.g. [GHK$^+$80]) focus on the properties of these *functions*. For example, given a function F, one may ask whether F is a lower adjoint in a Galois connection. The question posed by Hartmanis and Stearns was, however, rather different.

To motivate their question, note that the statement $F.b\sqsubseteq a$ defines a *relation* between $\mathcal{B}$ and $\mathcal{A}$. So too does $b\preceq G.a$. The existence of a Galois connection states that these two relations are equal. A natural question is therefore: under which conditions does an arbitrary (binary) relation between two posets define a Galois connection between the sets?

Exploring the question in more detail leads to two separate questions. The first is: suppose R is a relation between posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$. What is a necessary and sufficient condition that there exist a function F such that

$$(a,b)\in R \;\equiv\; F.b\sqsubseteq a \quad ?$$

The second is the dual of the first: given relation R, what is a necessary and sufficient condition that there exist a function G such that

$$(a,b)\in R \;\equiv\; b \preceq G.a \quad ?$$

The conjunction of these two conditions is a necessary and sufficient condition for a relation R to define a Galois connection. Such a relation is called a *pair algebra*.

Example 242 It is easy to demonstrate that the two questions are separate. To this end, fig. 12 depicts two posets and a relation between them. The posets are $\{\alpha,\beta\}$ and $\{A,B\}$; both are ordered lexicographically: the reflexive-transitive reduction of the lexicographic ordering is depicted by the directed edges. The relation of type $\{\alpha,\beta\}\sim\{A,B\}$ is depicted by the undirected edges.

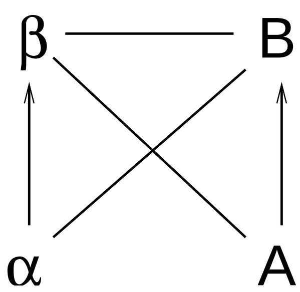


Figure 12: A Relation on Two Posets

Let the relation be denoted by R. Define the function F of type $\{\alpha,\beta\}\leftarrow\{A,B\}$ by $F.B=\alpha$ and $F.A=\beta$. Then it is easy to check that. for $a\in\{\alpha,\beta\}$ and $b\in\{A,B\}$,

$$(a,b)\in R \;\equiv\; F.b \sqsubseteq a \quad .$$

(There are just four cases to be considered.) On the other hand, there is no function G of type $\{A,B\}\leftarrow\{\alpha,\beta\}$ such that

$$(a,b)\in R \;\equiv\; b \preceq G.a \quad .$$

To check that this is indeed the case, it suffices to check that the assignment $G.A=\alpha$ is invalid (because $\alpha\sqsubseteq\alpha$ but $(\alpha,A)\notin R$) and the assignment $G.A=\beta$ is also invalid (because $\alpha\sqsubseteq\beta$ but $(\alpha,A)\notin R$).
□

Example 243 A less artificial, general way to demonstrate that the two questions are separate is to consider the membership relation. Specifically, suppose $\mathcal{S}$ is a set. Then the membership relation, denoted as usual by the —overloaded— symbol "$\in$", is a heterogeneous relation of type $\mathcal{S}\sim 2^{\mathcal{S}}$ (where $2^{\mathcal{S}}$ denotes the type of subsets of $\mathcal{S}$). Now, for all x of type $\mathcal{S}$ and X of type $2^{\mathcal{S}}$,

$$x\in X \;\equiv\; \{x\}\subseteq X \quad .$$

The right side of this equation has the form $F.b\sqsubseteq a$ where F is the function that maps an element into a singleton set and the ordering is the subset ordering. Also, its left side has the form $(a,b)\in R$, where the relation R is the membership relation and a and b are x and X, respectively. (This is where the overloading of notation can become confusing, for which our apologies!) It is, however, not possible to express $x\in X$ in the form $x\preceq G.X$ (except in the trivial cases where $\mathcal{S}$ has cardinality at most one). We leave the proof to the reader.
□

Example 244 An example of a Galois connection is the definition of the ceiling function on real numbers: for all real numbers x, $\lceil x\rceil$ is an integer such that, for all integers m,

$$x\leq m \;\equiv\; \lceil x\rceil\leq m \quad .$$

To properly fit the definition of a Galois connection, it is necessary to make explicit the implicit coercion from integers to real numbers in the left side of this equation. Specifically, we have, for all real numbers x and integers m,

$$x \leq_{\mathbb{R}} \mathsf{real}.m \quad\equiv\quad \lceil x\rceil \leq_{\mathbb{Z}} m$$

where real denotes the function that "coerces" an integer to a real, and $\leq_{\mathbb{R}}$ and $\leq_{\mathbb{Z}}$ denote the (homogeneous) at-most relations on, respectively, real numbers and integers. If, however, we consider the symbol "$\leq$" on the left side of the equation to denote the heterogeneous at-most relation of type $\mathbb{R}\sim\mathbb{Z}$, the fact that

$$x\leq m \quad\equiv\quad \lceil x\rceil \leq_{\mathbb{Z}} m$$

gives a representation of the (heterogeneous) "$\leq$" relation of type $\mathbb{R}\sim\mathbb{Z}$ as a block-ordered relation: referring to definition 225, the provisional ordering is $\leq_{\mathbb{Z}}$, f is the ceiling function and g is the identity function.

More interesting is if we take the contrapositive. We have, for all real numbers x and integers m,

$$m<x \;\equiv\; m\leq\lceil x\rceil-1 \quad .$$

On the right of this equation is the (homogeneous) at-most relation on integers. On the left is the (heterogeneous) less-than relation of type $\mathbb{Z}\sim\mathbb{R}$. The equation demonstrates

that this relation is block-ordered; the “blocks” of real numbers being all the numbers that have the same ceiling. (The functional f is the identity function, the functional g maps real number x to $\lceil x \rceil - 1$ and the provisional ordering is the ordering $\leq_{\mathbb{Z}}$.) The example is interesting because we show in theorem 319 that the (homogeneous) less-than relation on real numbers is *not* block-ordered.
□

Returning to the discussion immediately preceding example 242, the two separate questions are each of interest in their own right: a positive answer to either question may predict that a given relation has a block-ordering of a specific form: in the case of the first question, where the functional g in definition 225 is the identity function, and, in the case of the second question, where the functional f in definition 225 is the identity function. In both cases, a further step is to check the requirement on f and g: in the first case, one has to check that the function F is surjective and in the second case that the function G is surjective. (A Galois connection is said to be “perfect” if both F and G are surjective.) For example, the fact that

$$x \leq m \quad \equiv \quad x \leq_{\mathbb{R}} \mathsf{real}.m$$

does *not* define a block-ordering because the function real is not surjective.

The relevant theory predicting exactly when the first of the two questions has a positive answer is as follows. Suppose $(\mathcal{B}, \sqsubseteq)$ is a complete poset. Let $\sqcap$ denote the infimum operator for $\mathcal{B}$ and suppose p is a predicate on $\mathcal{B}$. Then we define *inf-preserving* by

$$(245) \quad p \text{ is inf-preserving} \quad \equiv \quad \langle \forall g :: p.(\sqcap g) \equiv \langle \forall x :: p.(g.x) \rangle \rangle \quad .$$

So, for a given a, the predicate $\langle b :: (a, b) \in R \rangle$ is inf-preserving equivales

$$\langle \forall g \ :: \ (a, \sqcap g) \in R \ \equiv \ \langle \forall x :: (a, g.x) \in R \rangle \rangle \quad .$$

Then we have:

Theorem 246 Suppose $\mathcal{A}$ is a set and $(\mathcal{B}, \sqsubseteq)$ is a complete poset. Suppose $R \subseteq \mathcal{A} \times \mathcal{B}$ is a relation between the two sets. Define F by

$$(247) \quad F.a \quad = \quad \langle \sqcap b \ : \ (a, b) \in R \ : \ b \rangle \quad .$$

Then the following two statements are equivalent.

- $\langle \forall a,b \ : \ a \in \mathcal{A} \wedge b \in \mathcal{B} \ : \ (a, b) \in R \ \equiv \ F.a \sqsubseteq b \rangle$.
- For all a, the predicate $\langle b :: (a, b) \in R \rangle$ is inf-preserving.

□

The answer to the second question is, of course, obtained by formulating the dual of theorem 246.

In general, for most relations occurring in practical information systems the answer to the pair-algebra questions will be negative: the required inf- and sup-preserving properties just do not hold. However, a common way to define a pair algebra is to extend a given relation to a relation between sets in such a way that the infimum and supremum preserving properties are automatically satisfied. Hartmanis and Stearns' [HS64, HS66] solution to the state assignment problem was to consider the lattice of partitions of a given set; in so-called "concept analysis", the technique is to extend a given relation to a relation between rectangles. For more detail of the latter, see section 10.

An important property of Galois connections is the (well-known) theorem we call the "unity of opposites": if F and G are the adjoint functions in a Galois connection of the posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$, then there is an isomorphism between the posets $(F.\mathcal{B},\sqsubseteq)$ and $(G.\mathcal{A},\preceq)$. ($F.\mathcal{B}$ denotes the "image" of the function F, and similarly for $G.\mathcal{A}$.) Knowledge of the unity-of-opposites theorem suggests theorem 234, which expresses an isomorphism between different representations of block-ordered relations.

9.2 Analogie Frappante

In this section, we relate block-orderings to diagonals. The main results are theorems 255 and 262. We have named theorem 262 the "analogie frappante" because it generalises Riguet's "analogie frappante" connecting "relation de Ferrers" to diagonals.

Lemma 248 Suppose T is a provisional ordering of type $C{\sim}C$. That is, suppose

$$T\cap T^{\cup} \subseteq I_C \quad\wedge\quad T = (T\cap T^{\cup})\circ T\circ(T\cap T^{\cup}) \quad\wedge\quad T\circ T\subseteq T \ .$$

Suppose also that f and g are functional and onto the domain of T. That is, suppose[9] that

$$f\circ f^{\cup} \ = \ f{<} \ = \ T\cap T^{\cup} \ = \ g{<} \ = \ g\circ g^{\cup} \ .$$

Let R denote $f^{\cup}\circ T\circ g$. Then

(249) $R{<} = f{>} \ \wedge \ R{>} = g{>}$,

(250) $f^{\cup}\circ T^{\cup}\circ g \ = \ R{<}\circ(R\backslash R/R)^{\cup}\circ R{>}$, and

[9]The ordering T must be homogeneous but f and g may be heterogeneous and of different type, so long as both have target C.

(251) $f^{\cup}\circ g \;=\; \Delta R$,

(252) $R_< = (\Delta R)_< \;\wedge\; R_> = (\Delta R)_>$,

(253) $R_{\prec} \;=\; \Delta R\circ\Delta R^{\cup} \;=\; f^{\cup}\circ f \;\wedge\; R_{\succ} \;=\; \Delta R^{\cup}\circ\Delta R \;=\; g^{\cup}\circ g$.

Proof Property (249) is a straightforward application of domain calculus:

$$
\begin{array}{ll}
 & R_> \\
= & \{ \quad \text{definition: } R = f^{\cup}\circ T\circ g \quad \} \\
 & (f^{\cup}\circ T\circ g)_> \\
= & \{ \quad \text{domains (specifically, } [\,(U\circ V)_> = (U_>\circ V)_>\,] \text{ and } [\,(U^{\cup})_> = U_<\,]) \quad \} \\
 & (f_<\circ T\circ g)_> \\
= & \{ \quad \text{assumption: } T = f_<\circ T\circ g_< \text{ (so } T = f_<\circ T\,) \quad \} \\
 & (T\circ g)_> \\
= & \{ \quad \text{domains (specifically, } [\,(U\circ V)_> = (U_>\circ V)_>\,]) \quad \} \\
 & (T_>\circ g)_> \\
= & \{ \quad \text{lemma 122 and assumption: } T\cap T^{\cup} = g_< \quad \} \\
 & g_> \quad .
\end{array}
$$

By a symmetric argument, $(f^{\cup}\circ T\circ g)_< = f_>$.

Now we consider (250). The raison d'être of (250) is that it expresses the left side as a function of $f^{\cup}\circ T\circ g$. In a pointwise calculation a natural step is to use indirect ordering. In a point-free calculation, this corresponds to using factors. That is, we exploit lemma 119:

$$
\begin{array}{ll}
 & f^{\cup}\circ T^{\cup}\circ g \\
= & \{ \quad \text{assumption: } T \text{ is a provisional ordering} \\
 & \quad\;\; \text{lemmas 116, 120 and 119} \quad \} \\
 & f^{\cup}\circ(T\cap T^{\cup})\circ T^{\cup}\backslash T^{\cup}/T^{\cup}\circ(T\cap T^{\cup})\circ g \\
= & \{ \quad \text{assumption: } f_< = T\cap T^{\cup} = g_< \quad \} \\
 & f^{\cup}\circ T^{\cup}\backslash T^{\cup}/T^{\cup}\circ g \\
= & \{ \quad \text{lemma 78 and assumption: } T = f_<\circ T\circ g_< \quad \} \\
 & f_>\circ(g^{\cup}\circ T^{\cup}\circ f)\backslash(g^{\cup}\circ T^{\cup}\circ f)/(g^{\cup}\circ T^{\cup}\circ f)\circ g_> \\
= & \{ \quad \text{(249) and definition of } R \quad \}
\end{array}
$$

$$R_< \circ R^\cup \backslash R^\cup / R^\cup \circ R_>$$
$$= \quad \{ \quad \text{factors} \quad \}$$
$$R_< \circ (R\backslash R/R)^\cup \circ R_> \quad .$$

Note the use of lemma 78. The discovery of this lemma is driven by the goal of the calculation.

The pointwise interpretation of $f^\cup \circ g$ is a relation expressing equality between values of f and g. This suggests that, in order to prove (251), we begin by exploiting the anti-symmetry of T:

$$f^\cup \circ g$$
$$= \quad \{ \quad f_< = T \cap T^\cup = g_< \text{ and domains} \quad \}$$
$$f^\cup \circ (T \cap T^\cup) \circ g$$
$$= \quad \{ \quad \text{distributivity (valid because } f \text{ and } g \text{ are functional)} \quad \}$$
$$f^\cup \circ T \circ g \;\; \cap \;\; f^\cup \circ T^\cup \circ g$$
$$= \quad \{ \quad \text{definition of } R \text{ and (250)} \quad \}$$
$$f^\cup \circ T \circ g \;\; \cap \;\; f_> \circ ((f^\cup \circ T \circ g) \backslash (f^\cup \circ T \circ g) / (f^\cup \circ T \circ g))^\cup \circ g_>$$
$$= \quad \{ \quad \text{(254) (see below)} \quad \}$$
$$f_> \circ f^\cup \circ T \circ g \circ g_> \;\; \cap \;\; ((f^\cup \circ T \circ g) \backslash (f^\cup \circ T \circ g) / (f^\cup \circ T \circ g))^\cup$$
$$= \quad \{ \quad \text{domains (specifically, } f_> \circ f^\cup = f^\cup \text{ and } g \circ g_> = g \text{)} \quad \}$$
$$f^\cup \circ T \circ g \;\; \cap \;\; ((f^\cup \circ T \circ g) \backslash (f^\cup \circ T \circ g) / (f^\cup \circ T \circ g))^\cup$$
$$= \quad \{ \quad \text{definition of } R \text{ and } \Delta R \quad \}$$
$$\Delta R \quad .$$

A crucial step in the above calculation is the use of the property

(254) $\quad U \cap p{\circ}V{\circ}q \quad = \quad p{\circ}(U{\cap}V){\circ}q \quad = \quad p{\circ}U{\circ}q \cap V$

for all relations U and V and coreflexive relations p and q. This is a frequently used property of domain restriction.

The remaining equations (252) and (253) are straightforward. First

$$(\Delta R)_<$$
$$= \quad \{ \quad (251) \quad \}$$
$$(f^\cup \circ g)_<$$

$$
\begin{array}{ll}
= & \{ \quad \text{domains and assumption: } f_< = g_< \quad \} \\
& f_> \\
= & \{ \quad \text{assumption: } f_< = T \cap T^{\cup} \quad \} \\
& ((T \cap T^{\cup}) \circ f)_> \\
= & \{ \quad \text{domains and converse} \quad \} \\
& (f^{\cup} \circ (T \cap T^{\cup}))_< \\
= & \{ \quad \text{lemma 122 and domains} \quad \} \\
& (f^{\cup} \circ T)_< \\
= & \{ \quad \text{domains and assumption: } g_< = T \cap T^{\cup} \\
& \quad\quad \text{and lemma 122} \quad \} \\
& (f^{\cup} \circ T \circ g)_< \ .
\end{array}
$$

That is $(\Delta R)_< = R_<$. The dual equation $(\Delta R)_> = R_>$ is immediate from the fact that $(\Delta R)^{\cup} = \Delta(R^{\cup})$ and properties of the domain operators. For the per domains, we have:

$$
\begin{array}{ll}
& R_{\prec} \\
= & \{ \quad R_< = (\Delta R)_< \text{ and } R_> = (\Delta R)_> \text{ (above); lemma 190} \quad \} \\
& (\Delta R)_{\prec} \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 160 (with } R := \Delta R \text{)} \quad \} \\
& \Delta R \circ \Delta R^{\cup} \\
= & \{ \quad \text{lemma 248 and definition of } \Delta R \quad \} \\
& f^{\cup} \circ g \circ (f^{\cup} \circ g)^{\cup} \\
= & \{ \quad \text{converse and } f_< = g_< = g \circ g^{\cup} \quad \} \\
& f^{\cup} \circ f \ .
\end{array}
$$

Again, the dual equation is immediate.
□

Theorem 255 Suppose $R = f^{\cup} \circ T \circ g$ where f, g and T have the properties stated in definition 225. Then the function $\mathcal{R}$ defined by

(256) $\mathcal{R} = \langle c : c \subseteq T \cap T^{\cup} : f^{\cup} \circ T \circ c \circ T \circ g \rangle$

is a non-redundant, injective, polar covering of R, and the function $\mathcal{D}$ defined by

(257) $\mathcal{D} = \langle c : c \subseteq T \cap T^{\cup} : f^{\cup} \circ c \circ g \rangle$

is a definiens of $\mathcal{R}$ such that $\cup\mathcal{D} = \Delta R$. That is, a block-ordered relation has a non-redundant, injective, polar covering such that the definiens of the covering is a covering of the diagonal of R.

Proof The theorem is a consequence of lemma 248, theorem 222 and theorem 218. Specifically, lemma 248 (in particular (253) and (252)) states that the conditions required to apply theorem 222 are met with ρ instantiated to g. Thus,

$$\mathcal{R} \;=\; \langle c : c \subseteq g_< : R \circ g^\cup \circ c \circ g \circ R\backslash R \rangle$$

is a non-redundant, injective polar covering of R. The definition of $\mathcal{R}$ is simplified as follows. First,

$$\begin{array}{ll} & g \circ R\backslash R \\ = & \{ \quad R = f^\cup \circ T \circ g \quad \} \\ & g \circ (f^\cup \circ T \circ g)\backslash(f^\cup \circ T \circ g) \\ = & \{ \quad \text{lemma 79 with } R,S,f,g := T\,,\,T{\circ}g\,,\,f\,,\,g \quad \} \\ & g \circ g^\cup \circ T\backslash(T{\circ}g) \\ = & \{ \quad g \circ g^\cup = g_< \quad \} \\ & g_< \circ T\backslash(T{\circ}g) \quad . \end{array}$$

So, for all c such that $c \subseteq g_<$,

$$\begin{array}{ll} & R \circ g^\cup \circ c \circ g \circ R\backslash R \\ = & \{ \quad \mathcal{R} \text{ covers } R\text{, so } (R \circ g^\cup \circ c \circ g \circ R\backslash R)_> \subseteq R_> \;;\; R_> = g_> \\ & \quad \text{(in preparation for lemma 77)} \quad \} \\ & R \circ g^\cup \circ c \circ g \circ R\backslash R \circ g_> \\ = & \{ \quad R = f^\cup \circ T \circ g \text{ and } g \circ R\backslash R = g_< \circ T\backslash(T{\circ}g) \text{ (see above)} \quad \} \\ & f^\cup \circ T \circ g \circ g^\cup \circ c \circ g_< \circ T\backslash(T{\circ}g) \circ g_> \\ = & \{ \quad g \circ g^\cup = g_<\text{, assumption: } c \subseteq g_<\text{, lemma 77 with } R,f := T,g \quad \} \\ & f^\cup \circ T \circ c \circ T\backslash T \circ g \\ = & \{ \quad T \text{ is a provisional ordering, } T \cap T^\cup = g_<\text{,} \\ & \quad \text{lemma 118} \quad \} \\ & f^\cup \circ T \circ c \circ T \circ g \quad . \end{array}$$

Since $g< = T \cap T^{\cup}$ by assumption, we have established (256).

Theorem 222 defines the definiens of the covering as the indexed set $\mathcal{D}$ where

$$\mathcal{D} \;=\; \langle c : c \subseteq g< : \Delta R \circ g^{\cup} \circ c \circ g \circ R\succ \rangle \quad .$$

But, for all c such that $c \subseteq g<$,

$$\begin{array}{ll} & \Delta R \circ g^{\cup} \circ c \circ g \circ R\succ \\ = & \{ \quad (253) \text{ and } (251) \quad \} \\ & f^{\cup} \circ g \circ g^{\cup} \circ c \circ g \circ g^{\cup} \circ g \\ = & \{ \quad g \circ g^{\cup} = g<, \text{ assumption: } c \subseteq g< \quad \} \\ & f^{\cup} \circ c \circ g \quad . \end{array}$$

Using the assumption that $g< = T \cap T^{\cup}$ once again, we have established (257). That $\cup\mathcal{D} = f^{\cup} \circ g = \Delta R$ follows from $f^{\cup} \circ g = \Delta R$ and the saturation axiom.

□

Lemma 248 has as immediate corollary that the converse of theorem 255 is invalid.

Corollary 258 There are relations that have a non-redundant polar covering but are not block-ordered.

Proof Examples 223 and 224 are both examples of finite relations that have non-redundant polar coverings. Example 223 has the property that $(\Delta R)< \neq R<$; however, $(\Delta R)> = R>$. Example 224 has an empty diagonal; that is, $(\Delta R)< \neq R<$ (and $(\Delta R)> \neq R>$). So by (the converse of) lemma 248 (specifically, (252)), neither relation is block-ordered.

□

We now prove the converse of lemma 248.

Lemma 259 A relation R is block-ordered if $R< = (\Delta R)<$ and $R> = (\Delta R)>$.

Proof Suppose $R< = (\Delta R)<$ and $R> = (\Delta R)>$. Our task is to construct relations f, g and T such that

$$R = f^{\cup} \circ T \circ g \quad ,$$

$$T \cap T^{\cup} \subseteq I \quad \wedge \quad T \;=\; (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \wedge \quad T \circ T \subseteq T \quad \text{and}$$

$$f \circ f^{\cup} \;=\; f< \;=\; T \cap T^{\cup} \;=\; g< \;=\; g \circ g^{\cup} \quad .$$

Since ΔR is difunctional, theorem 161 is the obvious place to start. Applying the theorem, we can construct f and g such that $\Delta R = f^{\cup} \circ g$ and

$$\Delta R = f^{\cup} \circ g \;\wedge\; f \circ f^{\cup} \;=\; f< \;=\; g \circ g^{\cup} \;=\; g< \quad .$$

(The proof of theorem 161 gives several ways of doing this.) Using standard properties of the domain operators together with the assumption that $R< = (\Delta R)<$ and $R> = (\Delta R)>$, it follows that

$$R< = f> \;\wedge\; R> = g> \quad .$$

It remains to construct the provisional ordering T. The appropriate construction is suggested by lemma 248, in particular (250). Specifically, we define T by the equation

(260) $\quad T \;=\; g \circ R\backslash R/R \circ f^{\cup} \quad .$

The proof that $R = f^{\cup} \circ T \circ g$ is by mutual inclusion. First note that

(261) $\quad f^{\cup} \circ T \circ g \;=\; \Delta R \circ R\backslash R/R \circ \Delta R$

since

$$\begin{array}{ll} & f^{\cup} \circ T \circ g \\ = & \quad \{ \quad (260) \quad \} \\ & f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \circ g \\ = & \quad \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\ & \Delta R \circ R\backslash R/R \circ \Delta R \quad . \end{array}$$

So

$$\begin{array}{ll} & f^{\cup} \circ T \circ g \\ \subseteq & \quad \{ \quad (261) \text{ and } \Delta R \subseteq R \quad \} \\ & R \circ R\backslash R/R \circ R \\ \subseteq & \quad \{ \quad \text{cancellation} \quad \} \\ & R \quad . \end{array}$$

Also,

$$\begin{array}{ll} & R \subseteq f^{\cup} \circ T \circ g \\ = & \quad \{ \quad (261) \quad \} \\ & R \;\subseteq\; \Delta R \circ R\backslash R/R \circ \Delta R \\ = & \quad \{ \quad \text{per domains: (98)} \quad \} \\ & R\prec \circ R \circ R\succ \;\subseteq\; \Delta R \circ R\backslash R/R \circ \Delta R \\ = & \quad \{ \quad \text{assumption: } R< = (\Delta R)< \text{ and } R> = (\Delta R)>\text{, lemma 190} \quad \} \end{array}$$

$$
\begin{array}{ll}
 & (\Delta R)\!\prec \circ R \circ (\Delta R)\!\succ \;\subseteq\; \Delta R \circ R\backslash R/R \circ \Delta R \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 160 (with } R := \Delta R) \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R \circ \Delta R^{\cup} \circ \Delta R \;\subseteq\; \Delta R \circ R\backslash R/R \circ \Delta R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & \Delta R^{\cup} \circ R \circ \Delta R^{\cup} \;\subseteq\; R\backslash R/R \\
\Leftarrow & \{ \quad \Delta R^{\cup} \subseteq R\backslash R/R \text{, monotonicity} \quad \} \\
 & R\backslash R/R \circ R \circ R\backslash R/R \;\subseteq\; R\backslash R/R \\
= & \{ \quad \text{factors} \quad \} \\
 & R \circ R\backslash R/R \circ R \circ R\backslash R/R \circ R \;\subseteq\; R \\
= & \{ \quad \text{cancellation} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

Combining the two inclusions we conclude that indeed $R = f^{\cup} \circ T \circ g$.

We now establish the requirements on T. First,

$$
\begin{array}{ll}
 & T \cap T^{\cup} \\
= & \{ \quad \text{definition and converse} \quad \} \\
 & g \circ R\backslash R/R \circ f^{\cup} \;\cap\; f \circ (R\backslash R/R)^{\cup} \circ g^{\cup} \\
\subseteq & \{ \quad \text{modular law} \quad \} \\
 & f \circ (f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \circ g \;\cap\; (R\backslash R/R)^{\cup}) \circ g^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & f \circ (\Delta R \circ R\backslash R/R \circ \Delta R \;\cap\; (R\backslash R/R)^{\cup}) \circ g^{\cup} \\
\subseteq & \{ \quad \Delta R \subseteq R \text{, monotonicity and cancellation} \quad \} \\
 & f \circ (R \;\cap\; (R\backslash R/R)^{\cup}) \circ g^{\cup} \\
= & \{ \quad \Delta R = R \;\cap\; (R\backslash R/R)^{\cup} \quad \} \\
 & f \circ \Delta R \circ g^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & f \circ f^{\cup} \circ g \circ g^{\cup} \\
= & \{ \quad f \circ f^{\cup} \;=\; f\!< \;=\; g \circ g^{\cup} \;=\; g\!< \quad \} \\
 & f\!< \quad .
\end{array}
$$

Thus $T \cap T^{\cup} \subseteq f\!<$. So $T \cap T^{\cup} \subseteq I$. Now

$$
\begin{array}{ll}
 & f< \subseteq T \cap T^{\cup} \\
= & \{ \quad \text{infima and } f< \text{ is coreflexive} \quad \} \\
 & f< \subseteq T \\
\Leftarrow & \{ \quad \text{domains} \quad \} \\
 & f \circ f^{\cup} \subseteq T \\
\Leftarrow & \{ \quad \text{definition of } T \text{ and monotonicity} \quad \} \\
 & f \subseteq g \circ R \backslash R / R \\
\Leftarrow & \{ \quad f< = g \circ g^{\cup}, \text{ domains and monotonicity} \quad \} \\
 & g^{\cup} \circ f \subseteq R \backslash R / R \\
= & \{ \quad f^{\cup} \circ g = \Delta R \quad \} \\
 & \Delta R^{\cup} \subseteq R \backslash R / R \\
= & \{ \quad \Delta R = R \cap (R \backslash R / R)^{\cup}, \text{ converse} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

So, by anti-symmetry we have established that $T \cap T^{\cup} = f<$. Since also $f< = g<$, we conclude from the definition of T and properties of domains that

$$T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad .$$

The final task is to show that T is transitive:

$$
\begin{array}{ll}
 & T \circ T \\
= & \{ \quad \text{definition} \quad \} \\
 & g \circ R \backslash R / R \circ f^{\cup} \circ g \circ R \backslash R / R \circ f^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & g \circ R \backslash R / R \circ \Delta R \circ R \backslash R / R \circ f^{\cup} \\
\subseteq & \{ \quad \Delta R \subseteq R \quad \} \\
 & g \circ R \backslash R / R \circ R \circ R \backslash R / R \circ f^{\cup} \\
\subseteq & \{ \quad \text{factors} \quad \} \\
 & g \circ R \backslash R / R \circ f^{\cup} \\
= & \{ \quad \text{definition} \quad \} \\
 & T \quad .
\end{array}
$$

□

It is interesting to reflect on the proof of lemma 259. The hardest part is to find appropriate definitions of f, g and T such that $R = f^{\cup} \circ T \circ g$. The key to constructing f and g is Riguet's "analogie frappante" [Rig51] whereby he introduced the "différence" —the diagonal ΔR— of the relation R. Expressing the diagonal in terms of factors as we have done makes many parts of the calculations very straightforward. One much less straightforward step is the use of lemma 190 in the proof that $R \subseteq f^{\cup} \circ T \circ g$. Note how calculational needs drive the search for the lemma: in order to simplify the inclusion it is necessary to expose the term $R\backslash R/R$ on the right side, and that is precisely what the lemma enables.

We conclude with the theorem that we call the "analogie frappante". It is not the theorem that Riguet presented but we have chosen to give it this name in order to recognise Riguet's contribution.

Theorem 262 (Analogie Frappante) A relation R is block-ordered if and only if $R_< = (\Delta R)_<$ and $R_> = (\Delta R)_>$.

Proof Lemma 248 establishes "only-if" and lemma 259 establishes "if".
□

Example 263 Recall that example 223 is of a relation R such that $R_< = (\Delta R)_<$ but $R_> \neq (\Delta R)_>$. Because of the simplicity of the example, it is possible to check, by exhausting all possible assignments to f and g, that the relation is not block-ordered. For suppose, on the contrary, that $R = f^{\cup} \circ T \circ g$, where f, T and g satisfy the conditions for a block-ordering. Then it must be the case that $g.A \neq g.B$ (since $(R \circ A)_< \neq (R \circ B)_<$). But also it must be the case that $f.\alpha$, $f.\beta$ and $f.\gamma$ are distinct (because, eg., $(\alpha \circ R)_> \neq (\beta \circ R)_>$). This has the consequence that $f_< \neq g_<$. But, by defining $f.\alpha = x$, $f.\beta = y$, $f.\gamma = z$, $g.A = x$, $g.B = z$ and $y \sqsubseteq x$ and $y \sqsubseteq z$, it is the case that $R = f^{\cup} \circ \sqsubseteq \circ g$. We say that the relation has an "imperfect" block-ordering. See section 10.
□

Example 264 A generic way to construct examples of relations that are not block-ordered is to exploit example 187. In order to avoid unnecessary repetition, we refer the reader to that example for the definition of the relation in given a finite set $\mathcal{X}$ and a set $\mathcal{S}$ of subsets of $\mathcal{X}$.

(Example 263 is a slightly disguised instance of the generic construction: the nodes A and B can be identified with, respectively, $\{\alpha,\beta\}$ and $\{\beta,\gamma\}$.)

Recall that the diagonal Δin of type $\mathcal{X} \sim \mathcal{S}$ is injective. It follows that the size of $(\Delta in)_<$ is at most the size of $\mathcal{S}$. If, however, the set $\mathcal{S}$ has $\mathcal{X}$ as one of its elements, the size of $\mathsf{in}_<$ equals the size of $\mathcal{X}$. Theorem 262 thus predicts that, if $\mathcal{X}$ is an element of $\mathcal{S}$, a necessary condition for in to be block-ordered is that the sizes of $\mathcal{X}$ and $\mathcal{S}$ must be equal; conversely, if $\mathcal{X}$ is an element of $\mathcal{S}$, in is not block-ordered if the sizes of $\mathcal{X}$ and $\mathcal{S}$ are different.

Fig. 6 (example 187) shows that, even if the sizes of $\mathcal{X}$ and $\mathcal{S}$ are equal, the relation in may not be block-ordered: as remarked then, for the choice of $\mathcal{S}$ shown in fig. 6, $\mathsf{in}_<$ and $(\Delta\mathsf{in})_<$ are different since 0 and 3 are elements of the former but not the latter.

It is straightforward to construct instances of $\mathcal{X}$ and $\mathcal{S}$ such that the relation in is block-ordered. It suffices to ensure that three conditions are satisfied: $\mathcal{X}$ is an element of $\mathcal{S}$, the sizes of $\mathcal{X}$ and $\mathcal{S}$ are equal, and, for each x in $\mathcal{X}$, the set of sets represented by $(x{\circ}\mathsf{in})_>$ is totally ordered. Fig. 13 is one such. Referring to definition 225, the functional f is $\Delta\mathsf{in}^{\cup}$ (depicted by rectangles) and the functional g is $\mathrm{I}_{\mathcal{S}}$; the ordering relation is the subset relation $\mathsf{in}\backslash\mathsf{in}$ (depicted by the directed graph).

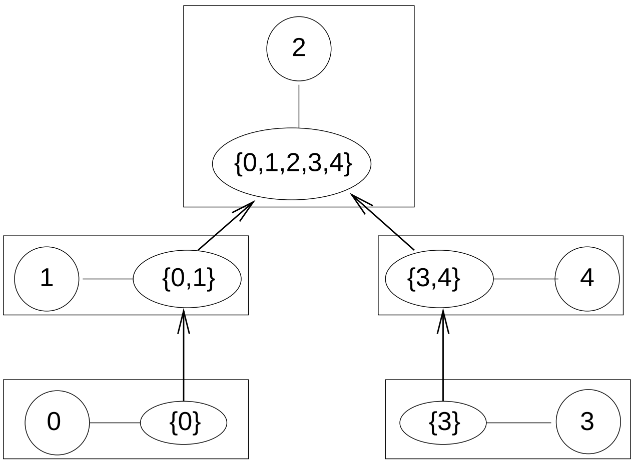


Figure 13: A Block-Ordered Membership Relation

□

The following theorem is a corollary of theorem 207. In combination with theorem 262 it states that a relation is block-ordered iff its core is block-ordered. Testing whether or not a given relation is block-ordered can thus be decomposed into computing the core of the relation and then testing whether or not that is block-ordered.

Theorem 265 Suppose R is an arbitrary relation. Then

$$R_< = (\Delta R)_< \quad \equiv \quad |R|_< = (\Delta|R|)_< \ .$$

Dually,

$$R_> = (\Delta R)_> \quad \equiv \quad |R|_> = (\Delta|R|)_> \ .$$

Proof Suppose R, λ and ρ are as in definition 191. Then

$$
\begin{array}{ll}
 & |R|_< = (\Delta|R|)_< \\
= & \{ \quad \text{definition 191 and theorem 207} \quad \} \\
 & (\lambda \circ R \circ \rho^\cup)_< = (\lambda \circ \Delta R \circ \rho^\cup)_< \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & (\lambda^\cup \circ (\lambda \circ R \circ \rho^\cup)_<)_< = (\lambda^\cup \circ (\lambda \circ \Delta R \circ \rho^\cup)_<)_< \\
= & \{ \quad \text{domains} \quad \} \\
 & (\lambda^\cup \circ \lambda \circ R \circ \rho^\cup)_< = (\lambda^\cup \circ \lambda \circ \Delta R \circ \rho^\cup)_< \\
= & \{ \quad \lambda^\cup \circ \lambda \circ R = R_\prec \circ R = R, \\
 & \quad\quad (\rho^\cup)_< = (\rho^\cup \circ \rho)_< = (R_\succ)_< = R_>, \text{ and domains} \quad \} \\
 & R_< = (\lambda^\cup \circ \lambda \circ \Delta R \circ \rho^\cup)_< \\
= & \{ \quad (\rho^\cup)_< = (\rho^\cup \circ \rho)_< \text{ and domains} \quad \} \\
 & R_< = (\lambda^\cup \circ \lambda \circ \Delta R \circ \rho^\cup \circ \rho)_< \\
= & \{ \quad \text{theorem 207} \quad \} \\
 & R_< = (\lambda^\cup \circ \Delta|R| \circ \rho)_< \\
= & \{ \quad \text{theorem 207} \quad \} \\
 & R_< = (\Delta R)_< \quad .
\end{array}
$$

Similarly,

$$
\begin{array}{ll}
 & R_< = (\Delta R)_< \\
= & \{ \quad \text{definition 191, theorem 207 and Leibniz} \quad \} \\
 & (\lambda^\cup \circ |R| \circ \rho)_< = (\lambda^\cup \circ \Delta|R| \circ \rho)_< \\
\Rightarrow & \{ \quad \text{Leibniz and domains} \quad \} \\
 & (\lambda \circ \lambda^\cup \circ |R| \circ \rho)_< = (\lambda \circ \lambda^\cup \circ \Delta|R| \circ \rho)_< \\
= & \{ \quad \rho_< = (\rho \circ \rho^\cup)_< \text{ and domains} \quad \} \\
 & (\lambda \circ \lambda^\cup \circ |R| \circ \rho \circ \rho^\cup)_< = (\lambda \circ \lambda^\cup \circ \Delta|R| \circ \rho \circ \rho^\cup)_< \\
= & \{ \quad \text{theorem 207 (applied twice)} \quad \} \\
 & |R|_< = (\Delta|R|)_< \quad .
\end{array}
$$

The property

$$R_< = (\Delta R)_< \quad \equiv \quad |R|_< = (\Delta|R|)_<$$

follows by mutual implication. The dual follows by instantiating R to $R^{\cup}$ and applying the properties of converse.

□

By combining the definition of block-ordering with theorem 207, it is immediately clear that R is block-ordered if $|R|$ is a provisional ordering. In general, a core of a block-ordered relation will not be a provisional ordering. This is because the types of the targets of the components λ and ρ in the definition of a core are not required to be the same; on the other hand, orderings are required to be homogeneous relations. However by carefully restricting the choice of core, it is possible to construct a core that is indeed a provisional ordering.

Theorem 266 Suppose R is an arbitrary relation. Then if R is block-ordered it has a core that is a provisional ordering.

Proof Suppose R is block-ordered. That is, suppose that f, g and T are relations such that T is a provisional ordering,

$$R = f^{\cup} \circ T \circ g$$

and

$$f \circ f^{\cup} \;=\; f_{<} \;=\; T \cap T^{\cup} \;=\; g_{<} \;=\; g \circ g^{\cup} \;\;.$$

Then, by lemma 248, $R_{\prec} = f^{\cup} \circ f$ and , $R_{\succ} = g^{\cup} \circ g$. Thus f and g satisfy the conditions for defining $|R|$. (See definition 191.) Consequently,

$$\begin{array}{ll}
 & |R| \\
= & \{ \quad \text{definition 191} \quad \} \\
 & f \circ R \circ g^{\cup} \\
= & \{ \quad R = f^{\cup} \circ T \circ g \quad \} \\
 & f \circ f^{\cup} \circ T \circ g \circ g^{\cup} \\
= & \{ \quad f \circ f^{\cup} \;=\; f_{<} \;=\; T \cap T^{\cup} \;=\; g_{<} \;=\; g \circ g^{\cup} \quad \} \\
 & (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \\
= & \{ \quad T \text{ is a provisional ordering, lemma 122 and domains} \quad \} \\
 & T \;\;.
\end{array}$$

We conclude that $|R|$ is the provisional ordering T.

□

Combining theorem 266 with theorem 194, we conclude that any core of a block-ordered relation is isomorphic to a provisional ordering. Loosely speaking, block-ordered relations are provisional orderings up to isomorphism and reduction to the core.

Example 267 From the Galois connection, for all reals x and integers m,

$$\lceil x \rceil \leq m \;\equiv\; x \leq m$$

defining the ceiling function, we deduce that the heterogeneous relation ${}_{\mathbb{R}}{\leq}_{\mathbb{Z}}$ has core the provisional ordering $\leq_{\mathbb{Z}}$. This is because the ceiling function is surjective. Its core in *not* the ordering $\leq_{\mathbb{R}}$ because the coercion real from integers to reals is not surjective. (See also example 244.)

On the other hand, if a Galois connection

$$F.b \sqsubseteq a \equiv b \preceq G.a$$

of posets $(\mathcal{A}, \sqsubseteq)$ and $(\mathcal{B}, \preceq)$ is "perfect" (i.e. both F and G are surjective), both the orderings $\sqsubseteq$ and $\preceq$ are cores of the defined heterogeneous relation. That the orderings are isomorphic is an instance of the unity-of-opposites theorem [Bac02].
□

10 Imperfect Block-Orderings

Following definition 225 we remarked that the condition on the functional relations f and g in a block-ordering is very strict. Later we remarked that a Galois connection satisfies the condition if it is so-called "perfect". (See the discussion following example 244 and also example 267.) In this section we study what might be called "(possibly) imperfect" block-orderings. The results presented here are used later to show that finite "staircase relations" are indeed block-ordered.

Some of the results presented in this section are inspired by what has been called "concept analysis" (the English translation of the German "Begriffenanalyse"). "Concept analysis" was briefly mentioned in section 9.1 as an example of how Hartmanis and Stearns' theory of pair algebras leads to the identification of Galois connections. As we shall see, the fundamental notion in "concept analysis" is closely related to Riguet's polar coverings.

Aside The research presented here was undertaken under the restrictions of the coronavirus pandemic an unfortunate consequence of which has been that access to library facilities has been impossible. This means that I have not been able to investigate the original (or, indeed, subsequent) literature in order to determine to what extent the relationship between Riguet's work and "concept analysis" is already known. The sole source of my knowedge of "concept analysis" is the text by Davey and Priestley [DP90, chapter 11]. **End of Aside**

10.1 Grips

Suppose R is a relation of type $A{\sim}B$ and suppose U is a rectangle such that $U \subseteq R$. Then, because $U = U \circ \top\!\!\top \circ U$ (by definition of a rectangle), we have

$$(268) \quad (U \subseteq R/(\top\!\!\top \circ U)) \;=\; (U \subseteq R) \;=\; (U \subseteq (U \circ \top\!\!\top)\backslash R) \;\; .$$

The equality between the outer two terms immediately suggests the identification of a Galois connection, which possibility we now explore.

It is easy to check that, for all relations R and S,

$$R/(\top\!\!\top \circ S) \;=\; R/(\top\!\!\top \circ S) \circ \top\!\!\top \;\; .$$

(For completeness, the proof is given in section 10.2.) That is, $R/(\top\!\!\top \circ S)$ is a left condition[10] for all relations R and S. Also, for all relations R and S,

$$(S \circ \top\!\!\top)\backslash R \;=\; \top\!\!\top \circ (S \circ \top\!\!\top)\backslash R \;\; .$$

[10]Recall that a left condition is a relation R such that $R = R \circ \top\!\!\top$. Dually, a right condition is a relation R such that $R = \top\!\!\top \circ R$.

That is, $(S\circ\top\!\!\top)\backslash R$ is a right condition for all relations R and S.

Returning to (268), we recognise the equality between the outer two terms as an instance of the equality, for all X and Y such that $X = R{<}\circ X\circ\top\!\!\top$ and $Y = \top\!\!\top\circ Y\circ R{>}$,

$$(269)\quad X\subseteq R/Y \quad\equiv\quad X\backslash R\supseteq Y \ .$$

The relation $R{<}\circ X\circ\top\!\!\top$ is a left condition representing a subset of the left domain of R, and the relation $\top\!\!\top\circ Y\circ R{>}$ is a right condition representing a subset of the right domain of R. Conversely, if U is such that $U\subseteq R$, $U\circ\top\!\!\top = R{<}\circ(U\circ\top\!\!\top)\circ\top\!\!\top$ and $\top\!\!\top\circ U = \top\!\!\top\circ(\top\!\!\top\circ U)\circ R{>}$. Thus the equality between the outer two terms of (268) is the Galois connection (269) between the (left condition representation of the) subsets of the left domain of R and the (right condition representation of the) subsets of the right domain of R, where in one case the ordering relation is the subset relation and in the other case the ordering relation is the superset relation.

One of the most important characteristics of a Galois connection is the theorem that we have dubbed the unity-of-opposites theorem [Bac02] and which we have already mentioned several times. Specifically, if

$$F.b\sqsubseteq a\equiv b\preceq G.a$$

is a Galois connection of posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$, elements a and b are *opposites* if

$$F.b = a \ \wedge\ b = G.a \ .$$

The unity-of-opposites theorem states that opposites form isomorphic sub-posets of $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$ and, moreover, completeness properties of $\mathcal{A}$ and/or $\mathcal{B}$ are inherited by these sub-posets.

Guided by (268), it is convenient to package two "opposites" into one rectangle. Such rectangles we call "grips":

Definition 270 (Grip) A rectangle U is said to be a *grip* of relation R if

$$U\circ\top\!\!\top = R/(\top\!\!\top\circ U) \quad\wedge\quad \top\!\!\top\circ U = (U\circ\top\!\!\top)\backslash R \ .$$

□

The word "grip" is an abbreviation of the Dutch word "begrip" which has the same meaning as the German word "Begriff". One meaning of the word "grip" in both Dutch and English is "handle"; the same is true of the German word "Griff". In American-English, the word "grip" also means "bag" or "holder". Thus our notion of a "grip" is a "handle" or "holder" for two opposites in the Galois connection defined by (269).

We have chosen to introduce new terminology partly in order to emphasise a subtle but important difference between our use of rectangles as holders of opposites and the way such holders are defined in "Begriffenanalyse". In the field of "Begriffenanalyse", the opposites of a relation of type $A{\sim}B$ are elements of 2^A and 2^B (the sets of subsets of A and B) and a "Begriff" is a pair (U,V) where U and V are opposites of each other. Typically, although not always, $\emptyset_A$ (the empty subset of A) and B are opposites, as are A and $\emptyset_B$ (the empty subset of B). In such cases, $(\emptyset_A, B)$ and $(A, \emptyset_B)$ are by definition "Begriffen". Our definition of a grip excludes this possibility because a grip of a relation R is always a non-empty rectangle. (A disadvantage of our definition is that greater care needs to be exercised when applying the unity-of-opposites theorem. Fortunately this is not relevant here.)

Note how the subset ordering on the left side of (269) is flipped to become the superset ordering on the right side. The "opposites" are thus "polar" opposites in the sense that if U and V are grips of relation R then

$$U{\circ}\top\!\top \subseteq V{\circ}\top\!\top \;\;\equiv\;\; \top\!\top{\circ}U \supseteq \top\!\top{\circ}V \;\; .$$

Example 271 Fig. 14 shows the grips of a relation of type $\{V,E,P,J,U\}{\sim}\{x,n,s,y,f,l,m\}$. The example is a simplification[11] of one presented by Davey and Priestley [DP90, table 11.1 and figure 11.1].

The grips are depicted by (larger black) rectangles, the left domain of each rectangle being formed by the set of upper-case letters listed vertically and the right domain of each rectangle being formed by the set of lower-case letters listed horizontally. The graph structure anticipates results presented in section 10.2, namely that the grips of a relation form a polar covering. The significance of the blue and red squares is explained in example 285. For the moment, it suffices to note that there is no least and no greatest grip whereas the relation does have a least and greatest "Begriff", the least "Begriff" having the empty set as its left component and the greatest "Begriff" having the empty set as its right component.
□

10.2 Polar Covering and Properties

In this section we show that the set of grips of a relation R is a polar covering of R. (See definition 209.) Simultaneously we show that the grips of a relation define a "(possibly) imperfect" block-ordering of the relation.

[11] The upper-case letters V, E, etc. stand for planets: Venus, Earth, etc. The lower-case letters stand for attributes of the planets: for example, y stands for "has a moon" whilst x stands for "does not have a moon". The simplification that has been made is to reduce the relation presented by Davey and Priestley to its core. The letter V, for example, represents the equivalence class {Venus,Mercury} in Davey and Priestley's presentation.

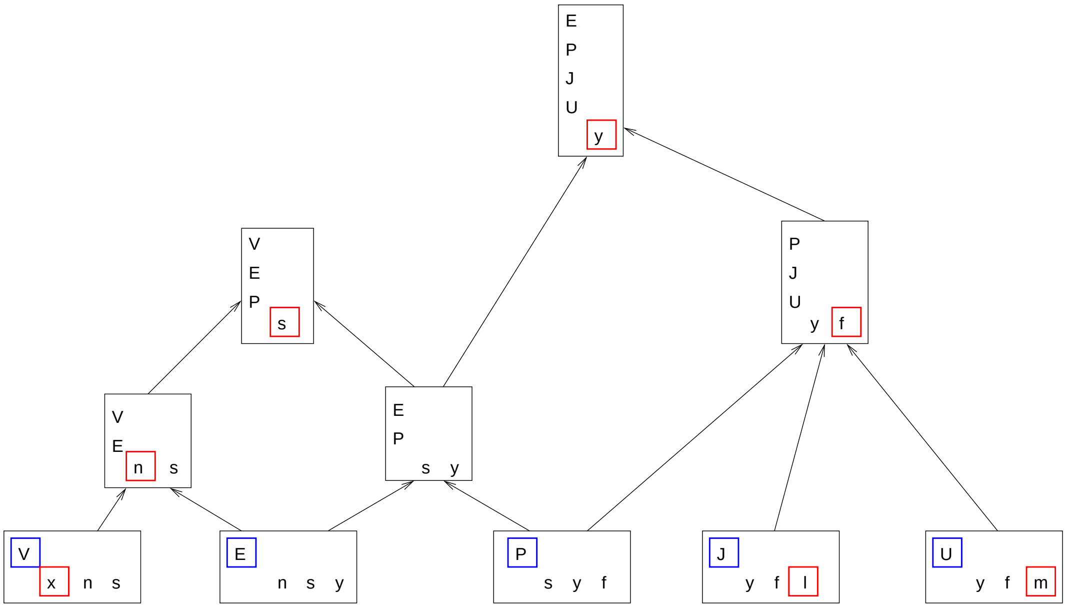


Figure 14: Grips

An important insight is that the polar coverings indexed by points in the left and right domain of a given relation that formed the basis of theorem 211 define a subset of the grips of the relation. It is this subset that defines the "(possibly) imperfect" block-ordering; it also enables one to construct the diagonal of the relation.

We begin with a couple of lemmas that are needed later.

Lemma 272 For all R and S of the same type,

$$R/(\top\!\top\circ S) \;=\; R/(\top\!\top\circ S)\circ\top\!\top \;\;.$$

Proof

$$
\begin{array}{ll}
 & R/(\top\!\top\circ S) \;=\; R/(\top\!\top\circ S)\circ\top\!\top \\
= & \{ \quad \text{anti-symmetry of the subset relation} \\
 & \quad\;\; \text{assumption: } R \text{ and } S \text{ have the same type, so } I\subseteq\top\!\top \quad \} \\
 & R/(\top\!\top\circ S) \supseteq R/(\top\!\top\circ S)\circ\top\!\top \\
= & \{ \quad \text{factors} \quad \} \\
 & R \supseteq R/(\top\!\top\circ S)\circ\top\!\top\circ\top\!\top\circ S \\
= & \{ \quad \text{by cone rule, } \top\!\top\circ\top\!\top=\top\!\top \quad \} \\
 & R \supseteq R/(\top\!\top\circ S)\circ\top\!\top\circ S \\
= & \{ \quad \text{cancellation} \quad \}
\end{array}
$$

$\quad$ true .

□

Lemma 273 For all relations R and all points b (of appropriate type),

$$R/(\top\!\!\top\circ b) \;=\; R\circ b\circ\top\!\!\top \quad .$$

Proof The proof is by indirect equality. Suppose U is a left condition (i.e. $U = U\circ\top\!\!\top$). Then

$$
\begin{array}{ll}
 & U\circ\top\!\!\top\circ b \subseteq R \\
\Rightarrow & \{ \quad b \text{ is coreflexive, so } b = b\circ b\text{ ; monotonicity} \quad \} \\
 & U\circ\top\!\!\top\circ b\circ\top\!\!\top \subseteq R\circ b\circ\top\!\!\top \\
= & \{ \quad b \neq \perp\!\!\perp \text{, cone rule} \quad \} \\
 & U\circ\top\!\!\top \subseteq R\circ b\circ\top\!\!\top \\
\Rightarrow & \{ \quad I \subseteq \top\!\!\top \quad \} \\
 & U \subseteq R\circ b\circ\top\!\!\top \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & U\circ\top\!\!\top\circ b \subseteq R\circ b\circ\top\!\!\top\circ b \\
= & \{ \quad b \text{ is a point, so } b\circ\top\!\!\top\circ b = b \quad \} \\
 & U\circ\top\!\!\top\circ b \subseteq R\circ b \\
\Rightarrow & \{ \quad b \text{ is coreflexive, i.e. } b \subseteq I \quad \} \\
 & U\circ\top\!\!\top\circ b \subseteq R \quad .
\end{array}
$$

We have thus shown (by mutual implication) that, for all left conditions U,

$$U\circ\top\!\!\top\circ b \subseteq R \;\equiv\; U \subseteq R\circ b\circ\top\!\!\top \quad .$$

But $U\circ\top\!\!\top\circ b \subseteq R \equiv U \subseteq R/(\top\!\!\top\circ b)$. That is, for all left conditions U,

$$U \subseteq R/(\top\!\!\top\circ b) \;\equiv\; U \subseteq R\circ b\circ\top\!\!\top \quad .$$

The lemma follows by applying lemma 272 and the rule of indirect equality.

□

We now turn to the proof that the grips of a relation form a polar covering of the relation.

Lemma 274 For all relations R and all rectangles U of the same type as R, if U is a grip of R then $U \subseteq R$.

Proof Suppose U is a grip of R. Then

$$
\begin{array}{ll}
 & U \\
= & \{ \quad U \text{ is a rectangle, definition 123} \quad \} \\
 & U\circ\top\!\!\top\circ U \\
= & \{ \quad \text{definition 270} \quad \} \\
 & R/(\top\!\!\top\circ U)\circ U \\
= & \{ \quad \text{lemma 272} \quad \} \\
 & R/(\top\!\!\top\circ U)\circ\top\!\!\top\circ U \\
\subseteq & \{ \quad \text{cancellation} \quad \} \\
 & R \ .
\end{array}
$$

□

Lemma 275 Suppose U and V are grips of R. Then

$$U< \subseteq V< \ \equiv \ U> \supseteq V> \ .$$

Proof

$$
\begin{array}{ll}
 & U< \subseteq V< \\
= & \{ \quad \text{condition-coreflexive isomorphism} \quad \} \\
 & U\circ\top\!\!\top \subseteq V\circ\top\!\!\top \\
= & \{ \quad U \text{ and } V \text{ are grips of } R\text{, definition 270 and Leibniz} \quad \} \\
 & R/(\top\!\!\top\circ U) \subseteq R/(\top\!\!\top\circ V) \\
\Leftarrow & \{ \quad \text{factors} \quad \} \\
 & \top\!\!\top\circ U \supseteq \top\!\!\top\circ V \\
= & \{ \quad \text{condition-coreflexive isomorphism} \quad \} \\
 & U> \supseteq V> \ .
\end{array}
$$

That is, $U< \subseteq V< \Leftarrow U> \supseteq V>$ for all grips U and V of R.

Dually, $U> \subseteq V> \Leftarrow U< \supseteq V<$. Since the latter property holds for all grips U and V of R, we can interchange U and V to get $V> \subseteq U> \Leftarrow V< \supseteq U<$. That is, $U> \supseteq V> \Leftarrow U< \subseteq V<$ for all grips U and V of R.

Combining the two implications, we conclude that, for all grips U and V of R,

$$U< \subseteq V< \ \equiv \ U> \supseteq V> \ .$$

□

Lemma 275 is the first step in showing that the grips satisfy definition 209 of a polar covering. Specifically, the lemma allows us to introduce an ordering on grips as per the definition. For future reference, here is the definition.

Definition 276 Suppose U and V are grips of a relation R. Then we define the relation $\sqsubseteq$ by

$$U \sqsubseteq V \;\equiv\; U{<} \subseteq V{<} \quad .$$

Equivalently (in view of lemma 275)

$$U \sqsubseteq V \;\equiv\; U{>} \supseteq V{>} \quad .$$

□

Lemma 277 The relation $\sqsubseteq$ of definition 276 is a provisional ordering of grips.

Proof That $\sqsubseteq$ is reflexive and transitivity is a straightforward conseqence of the reflexivity and transitivity of the subset relation. That it is anti-symmetric is a consequence of the fact that grips are rectangles, lemma 275 and lemma 125.

□

Theorem 211 showed how to construct a polar covering of a given relation R, indexed by points b in $R{>}$. Dually, one can construct a polar covering of R indexed by points a in $R{<}$. The elements of these two coverings are particularly special grips of R. Specifically —see lemma 279— comparing the grip with index a with the grip with index b enables the determination of whether or not a and b are related by R.

First, we show that both coverings define grips.

Lemma 278 For all relations R and all points b such that $b \subseteq R{>}$, the rectangle $R \circ b \circ R\backslash R$ is a grip of R. Dually, for all relations R and all points a such that $a \subseteq R{<}$, the rectangle $R/R \circ a \circ R$ is a grip of R.

Proof Assume that b is a point such that $b \subseteq R{>}$. Then

$$
\begin{array}{ll}
 & R/(\top\!\top \circ R \circ b \circ R\backslash R) \\
= & \{ \quad [\ \top\!\top \circ R = \top\!\top \circ R{>}\ ]\text{; assumption: } b \subseteq R{>}\text{, so } R{>} \circ b = b \quad \} \\
 & R/(\top\!\top \circ b \circ R\backslash R) \\
= & \{ \quad \text{factors, specifically } [\ R/(S \circ T) = (R/T)/S\ ] \quad \} \\
 & (R/(R\backslash R))/(\top\!\top \circ b)
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{factors, specifically (29)} \quad \} \\
& R/(\top\!\!\top \circ b) \\
= & \{ \quad \text{assumption: } b \text{ is a point; lemma 273} \quad \} \\
& R \circ b \circ \top\!\!\top \\
= & \{ \quad (R\backslash R)_< = I \quad \} \\
& R \circ b \circ R\backslash R \circ \top\!\!\top \ .
\end{array}
$$

Also

$$
\begin{array}{ll}
& (R \circ b \circ R\backslash R \circ \top\!\!\top)\backslash R \\
= & \{ \quad R\backslash R \supseteq I \text{, so } (R\backslash R)_> = I \quad \} \\
& (R \circ b \circ \top\!\!\top)\backslash R \\
= & \{ \quad \text{factors, specifically } [\ (S \circ T)\backslash R = T\backslash(S\backslash R)\ ] \text{ with } R,S,T := R, R, b \circ \top\!\!\top \quad \} \\
& (b \circ \top\!\!\top)\backslash(R\backslash R) \\
= & \{ \quad \text{dual of lemma 273 with } R := R\backslash R \quad \} \\
& \top\!\!\top \circ b \circ R\backslash R \\
= & \{ \quad [\ \top\!\!\top \circ R = \top\!\!\top \circ R_>\ ]\text{; assumption: } b \subseteq R_> \text{, so } R_> \circ b = b \quad \} \\
& \top\!\!\top \circ R \circ b \circ R\backslash R \ .
\end{array}
$$

Combining the two calculations, we have shown that $R \circ b \circ R\backslash R$ satisfies the condition on U in definition 270.
□

Now we show how to use the two polar coverings to determine whether or not points are related. Recalling definition 276 of the ordering $\sqsubseteq$ on grips, we have:

Lemma 279 For all relations R and all points a and b such that $a \subseteq R_<$,

(280) $a \circ \top\!\!\top \circ b \subseteq R \quad \equiv \quad R/R \circ a \circ R \ \sqsubseteq\ R \circ b \circ R\backslash R$.

That is,

(281) $a \circ \top\!\!\top \circ b \subseteq R \quad \equiv \quad (R/R \circ a)_< \subseteq (R \circ b)_<$.

Dually, for all relations R and all points a and b such that $b \subseteq R_>$,

(282) $a \circ \top\!\!\top \circ b \subseteq R \quad \equiv \quad (b \circ R\backslash R)_> \subseteq (a \circ R)_>$.

Proof We begin by proving (281) by mutual implication. Note that, by lemma 58, the left side of (281) is equivalent to $a \subseteq (R \circ b)_<$. This fact is exploited below.

$$
\begin{array}{ll}
 & (R/R\circ a)_< \;\subseteq\; (R\circ b)_< \\
\Rightarrow & \{ \quad I\subseteq R/R\,,\ \text{monotonicity and transitivity} \quad \} \\
 & a \;\subseteq\; (R\circ b)_< \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & (R/R\circ a)_< \;\subseteq\; (R/R\circ (R\circ b)_<)_< \\
= & \{ \quad \text{domains} \quad \} \\
 & (R/R\circ a)_< \;\subseteq\; (R/R\circ R\circ b)_< \\
= & \{ \quad \text{cancellation: (28)} \quad \} \\
 & (R/R\circ a)_< \;\subseteq\; (R\circ b)_< \quad .
\end{array}
$$

Applying lemma 58, we have proved (281). Property (280) now follows easily:

$$
\begin{array}{ll}
 & R/R\circ a\circ R \;\sqsubseteq\; R\circ b\circ R\backslash R \\
= & \{ \quad \text{definition 276 of } \sqsubseteq \quad \} \\
 & (R/R\circ a\circ R)_< \;\subseteq\; (R\circ b\circ R\backslash R)_< \\
= & \{ \quad \text{domains and assumption: } \ a\subseteq R_< \ ; \ (R\backslash R)_< = I \quad \} \\
 & (R/R\circ a)_< \;\subseteq\; (R\circ b)_< \\
= & \{ \quad \text{(281) and lemma 58} \quad \} \\
 & a\circ\top\!\!\top\circ b \subseteq R \quad .
\end{array}
$$

□

Theorem 283 For all relations R, the set of grips of R is a polar covering of R. That is,

$$R \;=\; \langle \cup U \;:\; \mathsf{grip}.U.R \;:\; U\rangle$$

where the grips of R are ordered by the relation $\sqsubseteq$ introduced in definition 276. Moreover,

$$R \;=\; f^{\cup}\circ\sqsubseteq\circ\, g$$

where the functional f mapping points a in $R_<$ to grips of R is defined by

$$f.a \;\;=\;\; R/R\circ a\circ R \quad ,$$

the functional g mapping points b in $R_>$ to grips of R is defined by

$$g.b \;\;=\;\; R\circ b\circ R\backslash R \quad .$$

Proof We have

$$\langle\cup\mathcal{U} \;:\; \mathsf{grip}.\mathcal{U}.\mathsf{R} \;:\; \mathcal{U}\rangle$$

$$\supseteq \qquad \{ \qquad \text{lemma 278 and monotonicity} \qquad \}$$

$$\langle\cup b \;:\; b \subseteq \mathsf{R}{>} \;:\; \mathsf{R}\circ b\circ \mathsf{R}\backslash\mathsf{R}\rangle$$

$$= \qquad \{ \qquad \text{theorem 211} \qquad \}$$

$$\mathsf{R}$$

$$\supseteq \qquad \{ \qquad \text{lemma 274} \qquad \}$$

$$\langle\cup\mathcal{U} \;:\; \mathsf{grip}.\mathcal{U}.\mathsf{R} \;:\; \mathcal{U}\rangle \quad .$$

Thus, by anti-symmetry of the subset relation, , $\mathsf{R} = \langle\cup\mathcal{U} : \mathsf{grip}.\mathcal{U}.\mathsf{R} : \mathcal{U}\rangle$.

That $\mathsf{R} \;=\; \mathsf{f}^{\cup}\circ\sqsubseteq\circ\,\mathsf{g}$ is immediate from lemma 279 and the definition of function application (as discussed in section 3.5).

□

Note that theorem 283 does *not* prove that every relation is block-ordered: the functionals f and g are not surjective onto the domain of the provisional ordering as required by definition 225. The equation

$$\mathsf{R} \;=\; \mathsf{f}^{\cup}\circ\sqsubseteq\circ\,\mathsf{g}$$

in theorem 283 expresses a *(possibly) imperfect* block-ordering of R .

Example 284 As discussed in example 213, fig. 8 (page 134) shows a relation R of type $\{A,B,C\}{\sim}\{\alpha,\beta,\gamma,\delta\}$ and fig. 9 (page 135) shows the (reflexive-transitive reduction of the) provisional ordering defined by theorem 211 .

Recall that the four relations depicted in fig. 9 are rectangles of the same type as R . These four rectangles are the values of the functional relation g . Specifically, the topmost rectangle depicts the relation $\mathsf{g}.\delta$, the middle-left rectangle depicts $\mathsf{g}.\alpha$, the middle-right rectangle depicts $\mathsf{g}.\gamma$ and the bottom rectangle depicts $\mathsf{g}.\beta$. This is indicated by the small red squares.

The bottom three rectangles are also the values of the functional relation f . Specifically, the bottom-most rectangle depicts the relation $\mathsf{f}.B$, the middle-left rectangle depicts $\mathsf{f}.A$ and the middle-right rectangle depicts $\mathsf{f}.C$. This is indicated by the small blue squares.

The provisional ordering $\sqsubseteq$ on the rectangles is depicted by the brown arrowed edges. We leave the reader to check that $\mathsf{R} = \mathsf{f}^{\cup}\circ\sqsubseteq\circ\,\mathsf{g}$.

These four rectangles are the only grips of the relation. (This is not generally the case.) The ordering shown is thus also the ordering of grips introduced in definition 276.

Note that we have *not* constructed a block-ordering of the relation R because $f{<} \neq g{<}$. (That is, f is not surjective.) The diagonal ΔR is the relation depicted by the three edges that connect red and blue squares. (Theorem 292 establishes that this is a general property of the diagonal.) Thus it is the case that $R{<} \neq (\Delta R){<}$ but $R{>} = (\Delta R){>}$.
□

Several lemmas and theorems we present (including theorem 283 and lemma 279) require points a and b to be elements of the left and right domain, respectively, of the relation R. This is very important to note since many authors assume, often without mention, that relations are "total", i.e. that their source and targets equal their right and left domains.

Assuming this requirement is met, for finite relations whether or not points are related can be determined by a graph searching algorithm. The nodes of the graph are the grips of the relation and the edges of the graph are defined by the reflexive-transitive reduction of the polar ordering of grips. (Borrowing terminology from ordered-set theory, the graph might sometimes be called the "Hasse diagram" of the polar ordering of grips.) Example 285 provides further explanation.

Example 285 As explained in example 271, the black rectangles in fig. 14 depict the grips of a relation; the edges connecting these rectangles depict the polar ordering on the grips in a way that should be self-explanatory. The collection of rectangles marked by small blue squares depicts the polar covering of the relation indexed by elements of its left domain, whilst the collection of rectangles marked by small red squares depicts the polar covering of the relation indexed by elements of its right domain; the squares identify the point defining the enclosing rectangle. (Cf. lemma 278.) For example, the bottom-left grip corresponds to V and to x.

Taken together, theorem 283 and lemma 279 state formally how the blue and red squares enable one to calculate whether or not the corresponding points are related. The blue squares depict a function f whose source is the left domain of the relation and whose target is the set of grips; similarly, the red squares depict a function g whose source is the right domain of the relation and whose target is also the set of grips. The ordering $\sqsubseteq$ on grips is the reflexive-transitive closure G^* of the graph G and the relation R is $f^{\cup} \circ G^* \circ g$. That is, for points a and b, $a \circ \top\!\!\top \circ b \subseteq R$ iff there is a path in the graph G from the grip enclosing the blue square labelled a to the grip enclosing the red square labelled b. For example, V and y are *not* related by R because there is not a path from the bottom-left grip to the topmost grip whereas E and y *are* related by R because there is such a path.

As in example 284, the blue and red squares also enable the identification of the diagonal of the relation. Specifically, consider the rectangles that have both a blue and a red square; then the pairs of points identified by the squares form the diagonal of the

relation. That is, the diagonal is the set of pairs $\{(V,x),(J,l),(U,m)\}$. (See theorem 292.)
□

The final task in this section is to formulate and prove the assertion mentioned in examples 284 and 285 that the diagonal of a relation is determined by coincident blue and red squares. The property we prove —see theorem 292— is, in fact, much stronger, although difficult to put in words.

Lemma 286 Suppose R is an arbitrary relation. Suppose a and b are points such that $a\circ\top\!\!\top\circ b \subseteq R$. Then the following properties of a, b and R are all equivalent.

(287) $(a\circ R)_> = (b\circ R\backslash R)_>$,

(288) $(a\circ R)_> \subseteq (b\circ R\backslash R)_>$,

(289) $R/R\circ a\circ R = R\circ b\circ R\backslash R$,

(290) $(R\circ b)_< \subseteq (R/R\circ a)_<$.

(291) $(R/R\circ a)_< = (R\circ b)_<$,

Proof The equivalence of (287), (288) and (289) is proved as follows.

$$
\begin{array}{cl}
 & (a\circ R)_> = (b\circ R\backslash R)_> \\
= & \{ \quad \text{anti-symmetry of the subset relation} \quad \} \\
 & (a\circ R)_> \subseteq (b\circ R\backslash R)_> \;\wedge\; (b\circ R\backslash R)_> \subseteq (a\circ R)_> \\
= & \{ \quad \text{assumption: } a\circ\top\!\!\top\circ b \subseteq R \text{ (so } a\subseteq R_< \text{ and } b\subseteq R_>\text{) ; (282)} \quad \} \\
 & (a\circ R)_> \subseteq (b\circ R\backslash R)_> \\
= & \{ \quad \text{definition 276 of } \sqsupseteq \text{ and domains} \quad \} \\
 & R/R\circ a\circ R \sqsupseteq R\circ b\circ R\backslash R \\
= & \{ \quad \text{assumption: } a\circ\top\!\!\top\circ b \subseteq R \text{ (so } a\subseteq R_< \text{ and } b\subseteq R_>\text{);} \\
 & \quad\quad \text{(280) and anti-symmetry of } \sqsubseteq \quad \} \\
 & R/R\circ a\circ R = R\circ b\circ R\backslash R \;.
\end{array}
$$

The equivalence of (290) and (291) with (289) is the converse dual.
□

Theorem 292 Suppose R is an arbitrary relation. Suppose a and b are points such that $a{\circ}\top\!\top{\circ}b \subseteq R$. Then the following three properties of a, b and R are all equivalent.

(293) $\langle\forall a' : a'{\circ}\top\!\top{\circ}b \subseteq R : R/R \circ a' \circ R \sqsubseteq R/R \circ a \circ R\rangle$,

(294) $R/R \circ a \circ R = R \circ b \circ R\backslash R$,

(295) $\langle\forall b' : a{\circ}\top\!\top{\circ}b' \subseteq R : R \circ b \circ R\backslash R \sqsubseteq R \circ b' \circ R\backslash R\rangle$.

It follows that all three properties are also equivalent to the property

(296) $a{\circ}\top\!\top{\circ}b \subseteq \Delta R$.

Proof We prove the equivalence of (294) and (295) by mutual implication. The equivalence of (293) and (294) is the converse-dual.

For the "if" part we exploit the fact that the two sides of the equation to be proved are grips of R and the grips of R form a polar covering of R. Specifically, assuming a and b are points such that $a{\circ}\top\!\top{\circ}b \subseteq R$,

$$
\begin{array}{ll}
 & R/R \circ a \circ R = R \circ b \circ R\backslash R \\
= & \{ \quad \text{assumption: } a{\circ}\top\!\top{\circ}b \subseteq R \text{ (so } b \subseteq R{>}\text{) ; lemma 286} \quad \} \\
 & (a{\circ}R){>} \subseteq (b \circ R\backslash R){>} \\
= & \{ \quad \text{saturation axiom: (16)} \quad \} \\
 & \langle\forall b' : b' \subseteq (a{\circ}R){>} : b' \subseteq (b \circ R\backslash R){>}\rangle \\
\Leftarrow & \{ \quad [\, b' \subseteq (b' \circ R\backslash R){>} \,] \text{, transitivity of the subset relation} \quad \} \\
 & \langle\forall b' : b' \subseteq (a{\circ}R){>} : (b' \circ R\backslash R){>} \subseteq (b \circ R\backslash R){>}\rangle \\
= & \{ \quad \text{lemma 58} \quad \} \\
 & \langle\forall b' : a{\circ}\top\!\top{\circ}b' \subseteq R : (b' \circ R\backslash R){>} \subseteq (b \circ R\backslash R){>}\rangle \ .
\end{array}
$$

Also

$$
\begin{array}{ll}
 & R/R \circ a \circ R = R \circ b \circ R\backslash R \\
= & \{ \quad \text{assumption: } a{\circ}\top\!\top{\circ}b \subseteq R, \text{ (so } b \subseteq R{>}\text{); lemma 286} \quad \} \\
 & (a{\circ}R){>} = (b \circ R\backslash R){>} \\
= & \{ \quad \text{saturation axiom: (16)} \quad \} \\
 & \langle\forall b' :: b' \subseteq (a{\circ}R){>} \equiv b' \subseteq (b \circ R\backslash R){>}\rangle \\
\Rightarrow & \{ \quad \text{weakening equivalence to implication and lemma 58} \quad \}
\end{array}
$$

$$\langle\forall b' : a\circ\top\top\circ b' \subseteq R : b' \subseteq (b\circ R\backslash R)_{>}\rangle$$

$$\Rightarrow \qquad \{ \quad \text{monotonicity} \quad \}$$

$$\langle\forall b' : a\circ\top\top\circ b' \subseteq R : (b'\circ R\backslash R)_{>} \subseteq ((b\circ R\backslash R)_{>}\circ R\backslash R)_{>}\rangle$$

$$\Rightarrow \qquad \{ \quad \text{domains and } R\backslash R\circ R\backslash R \subseteq R\backslash R \quad \}$$

$$\langle\forall b' : a\circ\top\top\circ b' \subseteq R : (b'\circ R\backslash R)_{>} \subseteq (b\circ R\backslash R)_{>}\rangle \ .$$

Putting the two calculations together we have shown that (294) is equivalent to

(297) $\langle\forall b' : a\circ\top\top\circ b' \subseteq R : (b'\circ R\backslash R)_{>} \subseteq (b\circ R\backslash R)_{>}\rangle$.

That (297) is equivalent to (295) is an immediate consequence of the definition of $\sqsubseteq$ and properties of the domain operator. (Take care with applying definition 276.)

The dual of (297) is

(298) $\langle\forall a' : a'\circ\top\top\circ b \subseteq R : (R/R\circ a')_{<} \subseteq (R/R\circ a)_{<}\rangle$.

That (298) is equivalent to (295) is also an immediate consequence of the definition of $\sqsubseteq$ and properties of the domain operator. (Again, take care with applying definition 276.)

The proof of (296) is now straightforward:

$$b\circ\top\top\circ a\circ R \subseteq R\backslash R$$

$$\Rightarrow \qquad \{ \quad b \text{ is a point so } b = b\circ b \text{ ; monotonicity} \quad \}$$

$$b\circ\top\top\circ a\circ R \subseteq b\circ R\backslash R$$

$$\Rightarrow \qquad \{ \quad \text{monotonicity and domains} \quad \}$$

$$(a\circ R)_{>} \subseteq (b\circ R\backslash R)_{>}$$

$$\Rightarrow \qquad \{ \quad \text{monotonicity and domains} \quad \}$$

$$b\circ\top\top\circ a\circ R \subseteq b\circ\top\top\circ b\circ R\backslash R$$

$$\Rightarrow \qquad \{ \quad b \text{ is a point so } b = b\circ\top\top\circ b \quad \}$$

$$b\circ\top\top\circ a\circ R \subseteq R\backslash R \ .$$

So

$$a\circ\top\top\circ b \subseteq \Delta R$$

$$= \qquad \{ \quad \Delta R = R\cap(R\backslash R/R)^{\cup} \quad \}$$

$$a\circ\top\top\circ b \subseteq R \ \wedge \ a\circ\top\top\circ b \subseteq (R\backslash R/R)^{\cup}$$

$$= \qquad \{ \quad \text{assumption: } a\circ\top\top\circ b \subseteq R \text{ ; converse and factors} \quad \}$$

$$b\circ\top\top\circ a\circ R \subseteq R\backslash R$$

$$
\begin{array}{ll}
= & \{ \quad \text{monotonicity and domains} \quad \} \\
 & (a \circ R)_{>} \subseteq (b \circ R \backslash R)_{>} \\
= & \{ \quad \text{lemma 286} \quad \} \\
 & R/R \circ a \circ R \;=\; R \circ b \circ R \backslash R \ .
\end{array}
$$

□

Theorem 292 is difficult to express precisely in words. Informally (and very imprecisely), it characterises the diagonal ΔR of a relation R as the collection of rectangles each of which is simultaneously the infimum of the grips indexed by points a in the left domain of R and the supremum of the grips indexed by points b in the right domain of R. Careful study of examples 284 and 271, as outlined below, will hopefully make this clear. (Example 271 is not such a good example because the duality between left and right domains is not evident.)

Example 299 We refer to example 284 (page 180). As remarked, the diagonal ΔR is the collection of rectangles having both a blue and a red square.

Note carefully how the rectangles making up the diagonal ΔR are each the infima of a subset of the ordered set of grips indexed by points in the left domain of R. For example the rectangle defined by the pair (A, α) is the infimum of itself and the topmost rectangle; these are the grips indexed by A. The same is true with "left" replaced by "right" and "infimum" replaced by "supremum": the rectangle defined by the pair (A, α) is the supremum of itself and the bottom-most rectangle, these being the rectangles indexed by α.

□

Example 300 We return again to example 271, in particular fig. 14 on page 174.

As in example 284, the blue and red squares enable the identification of the diagonal of the relation. Specifically, consider the rectangles that have both a blue and a red square; then the pairs of points identified by the squares form the diagonal of the relation. That is, the diagonal is the set of pairs $\{(V,x), (J,l), (U,m)\}$.

Note carefully how the rectangles making up the diagonal ΔR are each the infima of the subset of the ordered set of grips indexed by points in the left domain of R. For example the rectangle defined by the pair (V,x) is the infimum of the three rectangles with the point V in their left domains. The same is true with "left" replaced by "right" and "infimum" replaced by "supremum" but the sets of grips degenerates to a singleton set.

□

Example 301 Fig. 15 shows the grips of the converse[12] $in^{\cup}$ of the membership relation in defined in example 187. The blue and red squares have the same function as in examples 299 and 300. As in those examples, the diagonal of the relation is identified by the rectangles that have both a blue and a red square.

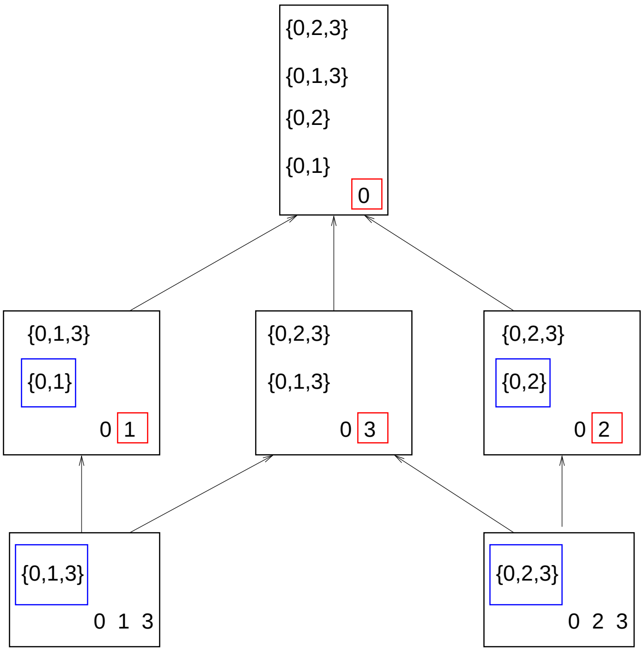


Figure 15: Grips of a Membership Relation

□

10.3 Grips of Provisional Orderings

If grips are to be used to represent membership of a relation, a practical question is just how many grips might a relation have (as a function of the sizes of its left and right domains). Some insight into this question can be obtained by considering an interesting special case: when the relation is a provisional ordering.

Suppose T is a provisional ordering and x is a point such that $x \subseteq T \cap T^{\cup}$. Then $T \circ x \circ T$ is a grip of T. Indeed, by lemma 118,

$$T \circ x \circ T \backslash T \quad = \quad T \circ x \circ T \quad = \quad T/T \circ x \circ T \quad .$$

This raises the question whether every grip of T is of this form.

The answer is no and a very instructive counterexample is given by the provisional at-most ordering on rational numbers, which we denote by $\leq_Q$. For a given rational

[12]The converse $in^{\cup}$ has been used simply because the figure would have been too wide if in had been used.

number q, the grip $\leq_Q \circ q \circ \leq_Q$ is a rectangle that relates rational number p to rational number r whenever $p \leq q \leq r$ (using the conventional overloaded notation). An easily proved property is that, for all rational numbers q and q',

$$\leq_Q \circ q \circ \leq_Q \;=\; \leq_Q \circ q' \circ \leq_Q \quad\equiv\quad q = q'$$

so each grip of $\leq_Q$ equals $\leq_Q \circ q \circ \leq_Q$ for *at most one* rational number q.

To see that not every grip of $\leq_Q$ is of the form $\leq_Q \circ q \circ \leq_Q$, consider all the rational numbers p and r such that $p^2 \leq 2 \leq r^2$ (again using the conventional overloaded notation). We leave it to the reader to check that the corresponding rectangle is a grip of $\leq_Q$. However, it cannot be expressed in the form $\leq_Q \circ q \circ \leq_Q$ for any rational number q since, as is well-known, $\sqrt{2}$ is an irrational number.

The so-called "Dedekind-MacNeille completion" of the rationals Q defines $\mathbb{R}$, the set of real numbers, to be the grips of $\leq_Q$; in so doing, the rational number q is identified with the grip $\leq_Q \circ q \circ \leq_Q$ and the irrational numbers (such as $\sqrt{2}$) are identified with the grips that are different from $\leq_Q \circ q \circ \leq_Q$ for all rational numbers q.

We see from this example that the cardinality of the grips of a relation may be greater than the cardinality of the relation. This suggests that the number of grips of a finite relation may, in the worst case, be an exponential function of the size of the relation. If so, representing a finite relation by the transitive-reflexive reduction of the polar ordering of its grips and testing membership of the relation via a graph-searching algorithm may not be practical. However, this is not something I have investigated.

11 Staircase Relations

As mentioned immediately after its definition, the notion of a polar covering was introduced by Riguet in connection with what he called "relations de Ferrers". Riguet [Rig51] states the following theorem:

> Pour que R soit une relation de Ferrers, il faut et il suffit que R soit réunion de rectangles dont les projections de même nom sont totalement ordonnées par inclusion et tels que si la première projection de l'un des rectangles est contenue dans la première projection d'un autre rectangle, la seconde projection du second est contenue dans la seconde projection du premier.

(For those unable to read French, the theorem states a necessary and sufficient condition for a relation to be "de Ferrers". The formal statement and proof of the theorem is given below: see theorem 334. The theorem clearly begs the question what is the definition of a "relation de Ferrers". We postpone answering this question until later. The reason for doing so is that Riguet gives both a formal definition and a mental picture —the picture shown in fig. 1 of what we call a "staircase relation"— but it is far from obvious how Riguet's definition and the mental picture are related.)

Riguet does not give a proof of the theorem. He also states that there is a striking analogy ("analogie frappante") between the definitions and properties of "relations de Ferrers" and difunctional relations but leaves the analogy unclear. In this section, we formalise the mental picture of a "staircase relation" (fig. 1) in several different but equivalent ways, one of which is Riguet's orginal definition. We then prove Riguet's theorem. This is quite straightforward. However, clarifying the "analogie frappante" is more difficult. To this end, we formulate the notion of a "polar covering" of a staircase relation and a "non-redundant" polar covering. We show how Riguet's theorem predicts that the less-than relation on real numbers has a polar covering but not a non-redundant polar covering. The non-redundancy property is the vital link between difunctional relations and (a proper subclass of) staircase relations. It is also the link between (a proper subclass of) staircase relations and block-ordered relations.

11.1 Formal Definition

Let us now turn to the formalisation of the mental picture of a "staircase" relation.

Suppose the relation R of type $A{\sim}B$ can be depicted as a "staircase". Then, for any element b of B, the set of elements a of A such that a and b are related by R is depicted by the region where a vertical line drawn at the point that depicts b intersects with the shaded area in the staircase depiction of R. See fig. 16. (Conversely, the set of elements b of B that are related to a given element a of A is depicted by drawing a horizontal line at the point depicted by a.)

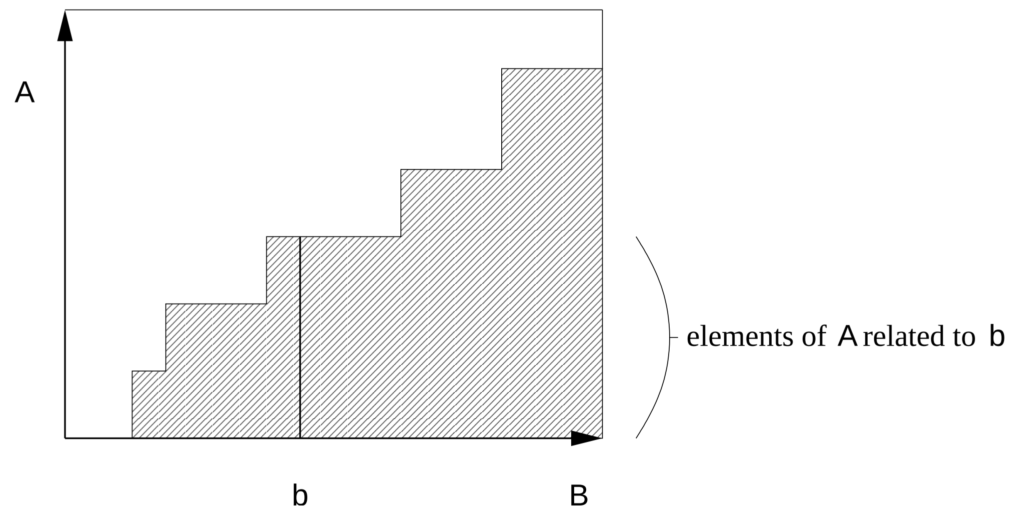


Figure 16: Preordering Defined By a Staircase Relation

The characteristic property of a "staircase" is that such lines increase in length —of course, not strictly— as one proceeds from the left to the right of the picture. But "length" and "left" and "right" are features of pictures and not properties of relations. A better characterisation that is not specific to drawing pictures is suggested by focusing on the subset of A comprising elements related by R to a given element b of B. In relation algebra, this is denoted by $(R\circ b)_<$ and the characteristic property of a "staircase" is that, for any two elements $b0$ and $b1$ of B, either $(R\circ b0)_<$ is a subset of $(R\circ b1)_<$ or, vice-versa, $(R\circ b1)_<$ is a subset of $(R\circ b0)_<$. In terms of the mental picture, $b0$ is to the left or to the right of $b1$.

At this point, certain concepts central to relation algebra spring to mind. First, the subset relation is an ordering relation. This immediately leads to the observation that the relation S defined by

$$b0[\![S]\!]b1 \quad \equiv \quad (R\circ b0)_< \subseteq (R\circ b1)_<$$

is a preorder. Then the "vice-versa" statement also looks familiar: it is the statement that $S\cup S^{\cup}$ is total (i.e. equal to the universal relation).

Those familiar with factors will immediately spot a much better characterisation. For any binary relation R, the relations $R\backslash R$ and R/R are preorders. That is, both are transitive and reflexive. (If R has type $A\sim B$ then $R\backslash R$ has type $B\sim B$ and R/R has type $A\sim A$.) If R is itself a preorder, then $R = R\backslash R = R/R = R\backslash R/R$. (Transitivity of R is equivalent to $R\subseteq R\backslash R$ and reflexivity of R implies $R\backslash R\subseteq R$; similarly for R/R.) This fact underlies the use of the rule called indirect ordering.

The pointwise formulation of the relation $R\backslash R$ is

$$b0[\![R\backslash R]\!]b1 \quad \equiv \quad \langle \forall a : a[\![R]\!]b0 : a[\![R]\!]b1 \rangle \quad .$$

That is $R\backslash R$ is the relation S defined above. This is the eureka moment in this preliminary investigation: that relation R is a "staircase" relation means formally that the preorder $R\backslash R$ is linear[13]. (Later we show that this is equivalent to R/R being linear.) For brevity, we denote this property by SC. That is:

Definition 302 The predicate SC on (binary) relations is defined by, for all R,

$$SC.R \quad \equiv \quad R\backslash R \cup (R\backslash R)^{\cup} \ = \ \top\!\!\top \quad .$$

□

The boolean $SC.R$ should be read as "R is a staircase relation". This section is thus about the properties of $R\backslash R$, for arbitrary relation R, when $R\backslash R$ is linear. The properties we investigate are driven by the need to provide further justification for the "correctness" of the formal definition with respect to the informal mental picture of such a relation.

Inevitably, we sometimes need to exploit pointwise definitions of "staircase" relations. Such a definition is formulated in lemma 303. Informally, the lemma states that there is a linear ordering on the depths of the "stairs" of a "staircase" relation. (Later we see that this is equivalent to there being a linear ordering on the heights of the "stairs".)

Lemma 303 The property $SC.R$ is equivalent to:

$$\langle \forall b,b' : b \subseteq R_{>} \wedge b' \subseteq R_{>} : (R \circ b)_{<} \subseteq (R \circ b')_{<} \ \vee \ (R \circ b')_{<} \subseteq (R \circ b)_{<} \rangle \quad .$$

(Dummies b and b' range over points of the appropriate type.)

Proof

$$\begin{array}{ll}
 & SC.R \\
= & \{ \quad \text{definition 302} \quad \} \\
 & R\backslash R \cup (R\backslash R)^{\cup} \ = \ \top\!\!\top \\
= & \{ \quad \text{saturation axiom: (16)} \quad \} \\
 & \langle \forall b,b' :: b \circ \top\!\!\top \circ b' \ \subseteq \ R\backslash R \cup (R\backslash R)^{\cup} \rangle \\
= & \{ \quad b \circ \top\!\!\top \circ b' \text{ is an (irreducible) atom, and converse} \quad \}
\end{array}$$

[13] An ordering S —of any sort— is said to be *linear* if $S \cup S^{\cup} \ = \ \top\!\!\top$. Sometimes the word "total" is used instead of linear. For example, Riguet [Rig51] uses the term "totalement ordonnées".

$$\langle\forall b,b' :: b\circ\top\!\!\top\circ b' \subseteq R\backslash R \ \vee\ b'\circ\top\!\!\top\circ b \subseteq R\backslash R\rangle$$

$= \quad \{ \quad$ lemma 60 $\quad \}$

$$\langle\forall b,b' :: (R\circ b)_< \subseteq (R\circ b')_< \ \vee\ (R\circ b')_< \subseteq (R\circ b)_<\rangle$$

$= \quad \{ \quad$ b and b' are points;

hence, $(b\subseteq R_> \wedge b'\subseteq R_>) \ \vee\ (R\circ b)_< = \bot\!\!\bot \ \vee\ (R\circ b')_< = \bot\!\!\bot$

case analysis (further details omitted) $\quad \}$

$$\langle\forall b,b' : b\subseteq R_> \wedge b'\subseteq R_> : (R\circ b)_< \subseteq (R\circ b')_< \ \vee\ (R\circ b')_< \subseteq (R\circ b)_<\rangle \ .$$

□

The final step in the proof of lemma 303 restricts the range of the dummies b and b'. This is an indication that our definition of SC demands refinement: the relation $R\backslash R$ typically includes irrelevant information. We return to this topic in section 11.6.

11.2 Equivalent Formulations

Lemma 34 enables a simple proof that linearity of $R\backslash R$ is equivalent to linearity of R/R. Specifically:

Lemma 304 The following are all equivalent formulations of $SC.R$:

(305) $\quad R\backslash R \cup (R\backslash R)^\cup \ = \ \top\!\!\top \ ,$

(306) $\quad R/R \cup (R/R)^\cup \ = \ \top\!\!\top \ ,$

(307) $\quad R \cup (R\backslash R/R)^\cup \ = \ \top\!\!\top \ ,$

(308) $\quad R\circ\neg R^\cup\circ R \ \subseteq\ R \ .$

Proof We prove first that (306) and (308) are equivalent:

$$R\circ\neg R^\cup\circ R \ \subseteq\ R$$

$= \quad \{ \quad$ factors $\quad \}$

$$R\circ\neg R^\cup \ \subseteq\ R/R$$

$= \quad \{ \quad$ complements $\quad \}$

$$\top\!\!\top \ \subseteq\ R/R \cup \neg(R\circ\neg R^\cup)$$

$= \quad \{ \quad$ (38) with $R,S := R^\cup, R^\cup$ (and $R = (R^\cup)^\cup$) $\quad \}$

$$\top\!\!\top \ \subseteq\ R/R \ \cup\ R^\cup\backslash R^\cup$$

$$
\begin{array}{ll}
= & \{ \quad (35) \text{ with } R,S := R,R \quad \} \\
& \top\!\top \ \subseteq\ R/R \cup (R/R)^{\cup} \\
= & \{ \quad [S \subseteq \top\!\top] \text{ with } S := R/R \cup (R/R)^{\cup} \text{ and anti-symmetry} \quad \} \\
& \top\!\top \ =\ R/R \cup (R/R)^{\cup} \ .
\end{array}
$$

A symmetric argument establishes the equivalence of (305) and (308):

$$
\begin{array}{ll}
& R \circ \neg R^{\cup} \circ R \ \subseteq\ R \\
= & \{ \quad \text{factors} \quad \} \\
& \neg R^{\cup} \circ R \ \subseteq\ R\backslash R \\
= & \{ \quad \text{complements} \quad \} \\
& \top\!\top \ \subseteq\ R\backslash R \cup \neg(\neg R^{\cup} \circ R) \\
= & \{ \quad (38) \text{ with } S,T := R^{\cup}, R^{\cup} \quad \} \\
& \top\!\top \ \subseteq\ R\backslash R \ \cup\ R^{\cup} / R^{\cup} \\
= & \{ \quad (36) \text{ with } R,S := R,R \ (\text{and } R = (R^{\cup})^{\cup}) \quad \} \\
& \top\!\top \ \subseteq\ R\backslash R \cup (R\backslash R)^{\cup} \\
= & \{ \quad [S \subseteq \top\!\top] \text{ with } S := R\backslash R \cup (R\backslash R)^{\cup} \text{ and anti-symmetry} \quad \} \\
& \top\!\top \ =\ R\backslash R \cup (R\backslash R)^{\cup} \ .
\end{array}
$$

Finally,

$$
\begin{array}{ll}
& R \circ \neg R^{\cup} \circ R \ \subseteq\ R \\
= & \{ \quad \text{factors} \quad \} \\
& \neg R^{\cup} \ \subseteq\ R\backslash R/R \\
= & \{ \quad \text{converse and complements} \quad \} \\
& \top\!\top \ \subseteq\ R \cup (R\backslash R/R)^{\cup} \\
= & \{ \quad [S \subseteq \top\!\top] \text{ with } S := R \cup (R\backslash R/R)^{\cup} \text{ and anti-symmetry} \quad \} \\
& \top\!\top \ =\ R \cup (R\backslash R/R)^{\cup} \ .
\end{array}
$$

□

Note that, in lemma 304, the symbol “ $\top\!\top$ ” denoting the universal relation is overloaded: if R has type $A{\sim}B$, its occurrence in (305) has type $B{\sim}B$, its occurrence in (306) has type $A{\sim}A$ and its occurrence in (307) has type $A{\sim}B$. This means that any attempt to prove, for example, that

$$R \cup (R\backslash R/R)^{\cup} = R/R \cup (R/R)^{\cup}$$

is doomed to fail. One might conjecture that it is possible to establish the equivalence of (305) and (306) without introducing complements by showing that both are equivalent to (307). However, the use of (308) is inevitable because of the algebraic properties of set union: when a set union is on the greater side of a set inclusion, there is no other choice but to introduce set negation.

11.3 General Constructions

Two general methods for identifying examples of staircase relations are given in lemmas 309 and 310.

Lemma 309 A linear preorder is a staircase relation. That is, for all (homogeneous) R,

$$\mathsf{SC.R} \quad \Leftarrow \quad R{\circ}R \subseteq R \;\wedge\; I \subseteq R \;\wedge\; R \cup R^{\cup} = \top\!\!\top \quad .$$

Proof We have

$$R = R\backslash R/R \quad \Leftarrow \quad R{\circ}R \subseteq R \;\wedge\; I \subseteq R$$

since

$$\begin{array}{ll} & R \subseteq R\backslash R/R \\ = & \{ \quad \text{factors} \quad \} \\ & R{\circ}R{\circ}R \subseteq R \\ \Leftarrow & \{ \quad \text{monontonicity and transitivity} \quad \} \\ & R{\circ}R \subseteq R \end{array}$$

and

$$\begin{array}{ll} & R\backslash R/R \subseteq R \\ = & \{ \quad [\, R = I\backslash R/I \,] \quad \} \\ & R\backslash R/R \subseteq I\backslash R/I \\ \Leftarrow & \{ \quad \text{(anti)monotonicity} \quad \} \\ & I \subseteq R \quad . \end{array}$$

Also,

$$R^{\cup} \circ R^{\cup} \subseteq R^{\cup} \;\wedge\; I \subseteq R^{\cup} \quad \equiv \quad R{\circ}R \subseteq R \;\wedge\; I \subseteq R \quad .$$

(The converse of a preorder is a preorder.) So

$$
\begin{array}{ll}
 & \mathsf{SC.R} \\
= & \{ \quad \text{lemma 304, in particular (307)} \quad \} \\
 & \mathsf{R} \cup (\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup} \;=\; \top\!\top \\
= & \{ \quad \text{assumption: } \mathsf{R} \text{ is a preorder} \\
 & \quad\;\; \text{(hence, } \mathsf{R}^{\cup} \text{ is a preorder and } \mathsf{R}^{\cup} = \mathsf{R}^{\cup}\backslash\mathsf{R}^{\cup}/\mathsf{R}^{\cup}\text{)} \\
 & \quad\;\; \text{lemma 34, in particular (37)} \quad \} \\
 & \mathsf{R} \cup \mathsf{R}^{\cup} \;=\; \top\!\top \\
= & \{ \quad \text{assumption: } \mathsf{R} \text{ is linear (i.e. } \mathsf{R}\cup\mathsf{R}^{\cup} = \top\!\top\text{)} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

□

An example of a staircase relation predicted by lemma 309 is the at-most relation — on natural numbers, integers, rational numbers or reals.

The second way of constructing a staircase relation is to reduce a linear preorder by eliminating its reflexive part (making it so-called “strict”). For example, the less-than relation (on natural numbers, integers, rational numbers or reals) is a staircase relation. (Lemma 311 is an alternative way of establishing that the less-than relation is a staircase relation. See example 315.) Formally, we have:

Lemma 310 For all (homogeneous) R,

$$\mathsf{SC.R} \;\Leftarrow\; \mathsf{R}\circ\mathsf{R}\subseteq\mathsf{R} \;\wedge\; \mathsf{R}\cup\mathrm{I}\cup\mathsf{R}^{\cup} = \top\!\top \quad .$$

Proof

$$
\begin{array}{ll}
 & \mathsf{SC.R} \\
= & \{ \quad (307) \quad \} \\
 & \mathsf{R} \cup (\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup} \;=\; \top\!\top \\
= & \{ \quad [\mathsf{X}\subseteq\top\!\top] \text{ and antisymmetry} \quad \} \\
 & \top\!\top \subseteq \mathsf{R} \cup (\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup} \\
\Leftarrow & \{ \quad \text{assumption: } \mathsf{R}\cup\mathrm{I}\cup\mathsf{R}^{\cup} = \top\!\top\text{, so } \top\!\top \subseteq \mathsf{R}\cup\mathrm{I}\cup\mathsf{R}^{\cup} \\
 & \quad\;\; \text{monotonicity and transitivity} \quad \} \\
 & \mathrm{I}\cup\mathsf{R}^{\cup} \subseteq (\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup} \\
= & \{ \quad \text{converse, factors and distributivity} \quad \} \\
 & \mathsf{R}\circ\mathrm{I}\circ\mathsf{R}\cup\mathsf{R}\circ\mathsf{R}\circ\mathsf{R} \;\subseteq\; \mathsf{R}
\end{array}
$$

$= \quad \{ \quad$ supremum and monotonicity $\quad \}$

$\quad R \circ R \subseteq R$

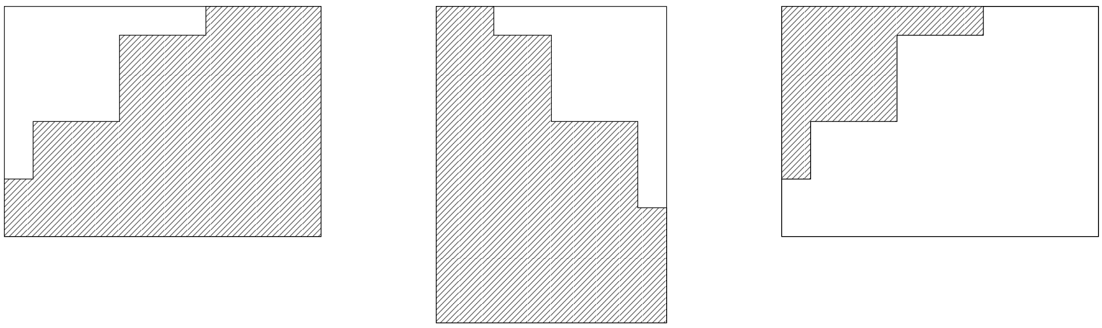
$= \quad \{ \quad$ assumption $\quad \}$

$\quad \mathsf{true}$.

□

11.4 Invariant Properties

In this section, we prove that the class of linear preorders characterised by the predicate SC is invariant under a variety of operators. Lemma 311 is supported by the mental picture shown in fig. 17.



(a) Staircase (b) Converse (c) Complement

Figure 17: Staircase Invariants

Lemma 311 For all R,

$$\mathsf{SC}.R \;=\; \mathsf{SC}.\neg R \;=\; \mathsf{SC}.R^{\cup} \;.$$

(As always, equality is used conjunctionally.)

Proof

$\quad \mathsf{SC}.R$

$= \quad \{ \quad$ definition 302 $\quad \}$

$\quad R\backslash R \cup (R\backslash R)^{\cup} \;=\; \top\!\!\top$

$= \quad \{ \quad$ corollary 39 $\quad \}$

$$
\begin{array}{ll}
 & \neg R \backslash \neg R \cup (\neg R \backslash \neg R)^{\cup} \ = \ \top\!\!\top \\
= & \{ \quad \text{definition 302} \quad \} \\
 & \mathsf{SC}.\neg R \ .
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & \mathsf{SC}.R \\
= & \{ \quad \text{definition 302} \quad \} \\
 & R\backslash R \cup (R\backslash R)^{\cup} \ = \ \top\!\!\top \\
= & \{ \quad \text{lemma 304 (in particular (308))} \quad \} \\
 & R \circ \neg R^{\cup} \circ R \ \subseteq \ R \\
= & \{ \quad \text{properties of converse} \quad \} \\
 & R^{\cup} \circ \neg R \circ R^{\cup} \ \subseteq \ R^{\cup} \\
= & \{ \quad \text{lemma 304 (in particular (308)) with } R := R^{\cup} \quad \} \\
 & \mathsf{SC}.R^{\cup} \ .
\end{array}
$$

□

Lemma 312 The functions $\langle R :: R\backslash R \rangle$ and $\langle R :: R/R \rangle$ are closure operators. That is

$$(R\backslash R)\backslash(R\backslash R) = R\backslash R \ \wedge \ (R/R)/(R/R) = R/R \ .$$

Proof This is a straightforward application of standard properties of factors:

$$
\begin{array}{ll}
 & (R\backslash R)\backslash(R\backslash R) \\
= & \{ \quad [\, R\backslash(S\backslash T) = (S \circ R)\backslash T \,] \text{ with } R,S,T := R,R,R \quad \} \\
 & (R \circ R\backslash R)\backslash R \\
= & \{ \quad (28){:}\ [\ R \circ R\backslash R = R\ ] \quad \} \\
 & R\backslash R \ .
\end{array}
$$

The second equation is proved in the same way.
□

Lemma 313 For all R,

$$\mathsf{SC}.R \ = \ \mathsf{SC}.(R\backslash R) \ = \ \mathsf{SC}.(R/R) \ .$$

Proof Straightforward application of definition 302 and lemma 312.
□

Lemma 314 For all S, R and T (of appropriate type),

$$\mathsf{SC}.(S\circ R\circ T) \;\Leftarrow\; \mathsf{SC}.R \;\; .$$

Proof

$$
\begin{array}{ll}
 & \mathsf{SC}.(S\circ R\circ T) \\
= & \{ \quad \text{lemma 304, in particular (308) with } R := S\circ R\circ T \quad \} \\
 & S\circ R\circ T\circ\neg(S\circ R\circ T)^{\cup}\circ S\circ R\circ T \;\subseteq\; S\circ R\circ T \\
\Leftarrow & \{ \quad \text{monotonicity of composition} \quad \} \\
 & R\circ T\circ\neg(S\circ R\circ T)^{\cup}\circ S\circ R \;\subseteq\; R \\
= & \{ \quad \text{middle-exchange rule (and double negation)} \quad \} \\
 & (R\circ T)^{\cup}\circ\neg R\circ(S\circ R)^{\cup} \;\subseteq\; (S\circ R\circ T)^{\cup} \\
= & \{ \quad \text{converse} \quad \} \\
 & T^{\cup}\circ R^{\cup}\circ\neg R\circ R^{\cup}\circ S^{\cup} \;\subseteq\; T^{\cup}\circ R^{\cup}\circ S^{\cup} \\
\Leftarrow & \{ \quad \text{monotonicity of composition} \quad \} \\
 & R^{\cup}\circ\neg R\circ R^{\cup} \;\subseteq\; R^{\cup} \\
= & \{ \quad R = (R^{\cup})^{\cup} \text{ and lemma 304 with } R := R^{\cup} \quad \} \\
 & \mathsf{SC}.R^{\cup} \\
= & \{ \quad \text{lemma 311} \quad \} \\
 & \mathsf{SC}.R \;\; .
\end{array}
$$

□

Example 315 The above properties allow us to identify a number of examples of staircase relations that prove to be significant later.

The at-most relation (commonly denoted by the symbol "$\leq$") is a linear ordering relation — on the integers, on the rationals and on the real numbers. By lemma 309 all three relations are staircase relations. By applying lemma 311 it is thus the case that the greater-than relation (commonly denoted by "$>$"), the less-than relation (commonly denoted by the symbol "$<$") and the at-least relation (commonly denoted by the symbol "$\geq$") are all staircase relations — again, on the integers, on the rationals and on the real numbers. This is because the greater-than relation is the complement of the at-most relation, the less-than relation is the converse of the greater-than relation, and, in turn, the at-least relation is the complement of the less-than relation.

Note that the less-than relation is not a preorder. (It is transitive but not reflexive.) Thus it is an example of a relation R such that $R \neq R\backslash R$ (and $R \neq R/R$) but is nevertheless a staircase relation according to definition 302.

The reader is invited to picture the less-than relation on the integers as a "staircase". Picturing the less-than relation on the rational numbers (or on the real numbers) as a "staircase" is, however, more difficult — in fact impossible in a formal sense to be made precise later. This raises doubts as to whether definition 302 is the appropriate abstraction from the mental picture of a "staircase".
□

We conclude this section with a property due to Riguet [Rig51]. (See the discussion following the lemma.)

Lemma 316 For all R, the relation $R\backslash R/R$ is a staircase relation if R is a staircase relation.

Proof For brevity, let S denote $R\backslash R/R$. Then

$$
\begin{array}{ll}
 & \mathsf{SC}.S \\
= & \{ \quad \text{lemma 304} \quad \} \\
 & S \circ \neg S^{\cup} \circ S \;\subseteq\; S \\
= & \{ \quad \text{lemma 32 and definition of } S \quad \} \\
 & R\backslash R/R \circ R \circ \neg R^{\cup} \circ R \circ R\backslash R/R \;\subseteq\; R\backslash R/R \\
= & \{ \quad \text{definition of factors} \quad \} \\
 & R \circ R\backslash R/R \circ R \circ \neg R^{\cup} \circ R \circ R\backslash R/R \circ R \;\subseteq\; R \\
\Leftarrow & \{ \quad \text{cancellation} \quad \} \\
 & R \circ \neg R^{\cup} \circ R \\
= & \{ \quad \text{lemma 304} \quad \} \\
 & \mathsf{SC}.R \quad .
\end{array}
$$

□

The combination of lemmas 182 and 316 is the second of two theorems stated by Riguet [Rig51]. More precisely, he states that $R \circ \neg R^{\cup} \circ R$ is a "relation de Ferrers" if R is a "relation de Ferrers" (cf. lemma 316) and their "différence" $R \cap \neg(R \circ \neg R^{\cup} \circ R)$ (i.e. ΔR) is a difunctional relation (cf. lemma 182). This explains his use of the term "différence" for what we call the "diagonal" of a relation.

11.5 Linear Orderings

In this section and the next we return to the mental picture of "staircases" as illustrated by fig. 1. An alternative perspective on a staircase relation of type $A{\sim}B$ is that it divides the elements of A into "blocks"; similarly the elements of B are also divided into "blocks". Fig. 18 is an example where A and B are each divided into five blocks. The effect is to divide the "staircase" into fifteen ($1{+}2{+}3{+}4{+}5$) blocks. A pair (a,b) is related by the staircase relation if the number assigned to a is at most the number assigned to b. Note that the at-most relation on numbers is a linear ordering.

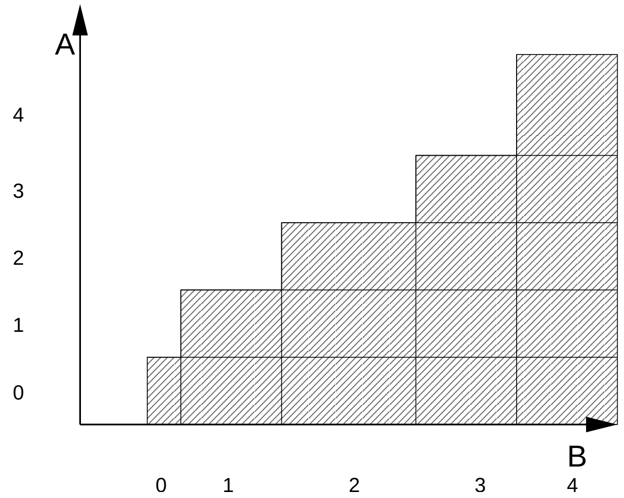


Figure 18: Block Structure of a Staircase Relation

In section 11.6, we show that every *linearly* block-ordered relation is a staircase relation. However, as we show in this section, a staircase relation does not necessarily have a block-ordering. See theorem 319. Thus, contrary to claims made in the literature —see section 12— it is not the case that these two concepts are equivalent.

Lemma 317 Suppose R has type $A{\sim}B$ and f and g are relations with targets A and B, respectively, such that $f\circ f^{\cup} = R{<}$ and $g\circ g^{\cup} = R{>}$. Then

$$SC.(f^{\cup}\circ R\circ g) \;\equiv\; SC.R \;\; .$$

Proof The equivalence is proved by mutual implication.

$$
\begin{array}{ll}
 & SC.R \\
= & \{ \quad \text{assumption: } f\circ f^{\cup} = R{<} \text{ and } g\circ g^{\cup} = R{>} \;\; \text{; domains} \quad \} \\
 & SC.(f\circ f^{\cup}\circ R\circ g\circ g^{\cup}) \\
\Leftarrow & \{ \quad \text{lemma 314 with } S,T := f\,,g^{\cup} \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & SC.(f^{\cup}\circ R\circ g) \\
\Leftarrow & \{ \quad \text{lemma 314 with } S,T := f^{\cup}, g \quad \} \\
 & SC.R \ .
\end{array}
$$

□

Corollary 318 Suppose T of type $C{\sim}C$ is a linear ordering and suppose f and g are functional and surjective relations of types $C{\sim}A$ and $C{\sim}B$, respectively. Then $f^{\cup}\circ T\circ g$ is a staircase relation.

Proof An ordering is also a preorder (and a linear ordering is a linear preorder). So the corollary follows immediately from the combination of lemmas 309 and 317.
□

Theorem 319 Not every staircase relation is block-ordered. Specifically, the less-than relation on the real numbers (or the rational numbers) is a staircase relation but is not block-ordered.

Proof We remarked in example 315 that the less-than relation on the real numbers is a staircase relation. To show that it is not block-ordered, we exploit lemma 248.

Suppose that the less-than relation on the real numbers is block-ordered by the functions f and g and the provisional ordering T. That is, suppose

$$R \ = \ f^{\cup}\circ T\circ g \ \ ,$$

where f and g are functionals of type $A{\leftarrow}\mathbb{R}$, for some A, and

$$f\circ f^{\cup} = T\cap T^{\cup} = g\circ g^{\cup} \quad \wedge \quad I_{\mathbb{R}} \subseteq f^{\cup}\circ f \cap g^{\cup}\circ g \ \ ,$$

T is a provisional ordering of type $A{\sim}A$ and for all x and y of type $\mathbb{R}$, and

$$x{\circ}\top\!\top{\circ}y \subseteq R \ \ \equiv \ \ x{<}y \ \ .$$

We begin by showing that $f^{\cup}\circ g$ is the empty relation. Inevitably, we need to exploit the pointwise definition of the diagonal, as formulated in lemma 40.

$$
\begin{array}{ll}
 & x{\circ}\top\!\top{\circ}y \ \subseteq \ f^{\cup}\circ g \\
= & \{ \quad \text{lemma 248, in particular (251)} \quad \} \\
 & x{\circ}\top\!\top{\circ}y \ \subseteq \ R\cap R^{\cup}\backslash R^{\cup}/R^{\cup} \\
= & \{ \quad \text{definition of intersection and lemma 40} \quad \} \\
 & x{\circ}\top\!\top{\circ}y \subseteq R \ \wedge \ \langle \forall u,v : u{\circ}\top\!\top{\circ}y \subseteq R \wedge x{\circ}\top\!\top{\circ}v \subseteq R : u{\circ}\top\!\top{\circ}v \subseteq R \rangle
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{definition of } R \quad \} \\
& x<y \ \wedge \ \langle \forall u,v : u<y \wedge x<v : u<v \rangle \\
\Rightarrow & \{ \quad u,v := y-(y-x)\times\frac{2}{3} \, , \, x+(y-x)\times\frac{1}{3} \\
& \quad (\text{Note that } x<y \ \Rightarrow \ y-(y-x)\times\frac{2}{3} < y \ \wedge \ x < x+(y-x)\times\frac{1}{3}) \quad \} \\
& y-(y-x)\times\frac{2}{3} < x+(y-x)\times\frac{1}{3} \\
= & \{ \quad \text{arithmetic} \quad \} \\
& y-x < y-x \\
= & \{ \quad \text{the less-than relation is irreflexive} \quad \} \\
& \mathsf{false} \ .
\end{array}
$$

That is, by the saturation axiom (16), $f^{\cup}\circ g = \perp\!\!\perp_{\mathbb{R}}$. This contradicts theorem 262 since the left (and right) domain of the empty relation is the empty relation and the left and right domains of the less-than relation are both non-empty.
□

A brief, informal summary of the proof of theorem 319 is that the less-than relation on real numbers is indeed a staircase relation but has no "diagonal" (more formally its "diagonal" is the empty relation) and no such staircase relation can be block-ordered. The informal contrapositive is that a necessary step in the process of block-ordering a staircase relation is to begin by identifying its diagonal; this is a difunctional relation and can be represented by $f^{\cup}\circ g$ where f and g are functional. If the right domain of g equals the right domain, and the right domain of f equals the left domain of the given relation, the process is completed by identifying the ordering relation T.

For example, the less-than relation on the integers is block-ordered. Indeed, for all integers m and n

$$m<n \ \equiv \ m+1\leq n \ .$$

The relation f is thus the successor function, the relation T is the at-most relation and the relation g is the identity function (on the integers). The "diagonal" is the set of pairs $(m\,,\,m+1)$.

The less-than relation on the natural numbers is also block-ordered but more care needs to be taken in the definition of the block-ordering. The relation f is the successor function; its source is the natural numbers and its target is the strictly positive natural numbers. The provisional ordering T is a subset of the at-most relation on natural numbers (specifically, the at-most relation restricted to the strictly positive natural numbers) and g is the *partial* identity relation on the natural numbers with left (and right) domain the strictly positive natural numbers. (Thus no number is related by g to the number 0.)

That the less-than relation on the real numbers is not block-ordered is a consequence of the fact that if $x<y$ the interval between x and y can always be subdivided at will. (That is, it is always possible to find a real number z such that $x<z$ and $z<y$.) The same is also true of the rationals and the proof of theorem 319 is equally valid in this case. Abstracting from the details of the less-than relation, we get the following theorem.

Theorem 320 Suppose R is a homogeneous relation such that

$$R \neq \perp\!\!\perp \quad \wedge \quad I \cap R = \perp\!\!\perp \quad \wedge \quad R = R \circ R \quad \wedge \quad R \cup I \cup R^{\cup} = \top\!\!\top \quad .$$

Then R is a staircase relation and $\Delta R = \perp\!\!\perp$.

It follows that any such relation is not block-ordered.

Proof Lemma 310 proves that R is a staircase relation.

Comparing the above conditions on R with those in lemma 310, the additions are the non-emptiness property $R \neq \perp\!\!\perp$, the "strictness" property $I \cap R = \perp\!\!\perp$ and the "subdivision" property $R \subseteq R \circ R$. (The less-than relation on real numbers has the subdivision property whereas the less-than relation on the integers does not.) Applying lemma 321 (below), the subdivision and strictness properties imply that $\Delta R = \perp\!\!\perp$. That R is not block-ordered follows from theorem 262 and the assumption that $R \neq \perp\!\!\perp$.

□

The lemma used to prove theorem 320 is the following:

Lemma 321

$$R \subseteq R \circ R \Rightarrow (\Delta R = \perp\!\!\perp \equiv I \cap R \subseteq \perp\!\!\perp) \quad .$$

Proof

$$
\begin{array}{ll}
 & R \subseteq R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & I \cap R \subseteq I \cap R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \{ \quad \text{modular law} \quad \} \\
 & I \cap R \subseteq R \circ (R^{\cup} \circ R^{\cup} \cap \neg R^{\cup}) \circ R \\
= & \{ \quad \text{assumption: } R \subseteq R \circ R \quad \} \\
 & I \cap R \subseteq R \circ (R^{\cup} \cap \neg R^{\cup}) \circ R \\
= & \{ \quad \text{complements} \quad \} \\
 & I \cap R \subseteq \perp\!\!\perp \\
= & \{ \quad I = I^{\cup} \text{, converse and shunting} \quad \}
\end{array}
$$

$$\begin{array}{ll}
& I \subseteq \neg R^{\cup} \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
& R \circ R \ \subseteq \ R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \{ \quad \text{assumption: } R \subseteq R \circ R \text{ and transitivity} \quad \} \\
& R \ \subseteq \ R \circ \neg R^{\cup} \circ R \ .
\end{array}$$

That is,

(322) $R \subseteq R \circ R \ \Rightarrow \ (R \ \subseteq \ R \circ \neg R^{\cup} \circ R \ \equiv \ I \cap R \subseteq \perp\!\!\perp)$.

So

$$\begin{array}{ll}
& \Delta R = \perp\!\!\perp \\
= & \{ \quad [\perp\!\!\perp \subseteq X] \text{ and antisymmetry, definition of } \Delta R \quad \} \\
& R \cap (R \backslash R / R)^{\cup} \subseteq \perp\!\!\perp \\
= & \{ \quad \text{shunting} \quad \} \\
& R \subseteq \neg (R \backslash R / R)^{\cup} \\
= & \{ \quad (32) \quad \} \\
& R \ \subseteq \ R \circ \neg R^{\cup} \circ R \\
= & \{ \quad \text{assumption: } R \subseteq R \circ R \text{, } (322) \quad \} \\
& I \cap R \subseteq \perp\!\!\perp \ .
\end{array}$$

□

The assumption that $R \neq \perp\!\!\perp$ in theorem 320 is necessary. The relation $\neg I_{\mathbb{1}}$ (see (33)) is the empty relation; it is also a block-ordered staircase relation on a finite type that satisfies all the assumptions of theorem 320 except for the assumption that it is non-empty.

Note that, if R is a homogeneous relation such that

$$R \neq \perp\!\!\perp \ \wedge \ R = R \circ R \ \wedge \ I \cap R = \perp\!\!\perp \ ,$$

the left and right domains of R cannot be finite. (The easy proof involves constructing an infinite sequence of points $\langle i : i \in \mathbb{N} : a_i \rangle$ such that,

$$\langle \forall i :: a_i \circ \top\!\!\top \circ a_{i+1} \subseteq R \rangle \ \wedge \ \langle \forall i,j :: a_i = a_j \ \equiv \ i = j \rangle \ .$$

This raises the question whether all finite staircase relations are linearly block-ordered.

11.6 Linear Block Ordering

Recall that, immediately following lemma 303, we remarked that the definition of SC demands refinement. This is more evident from the limitations of corollary 318: the corollary assumes a linear ordering —and not a provisional linear ordering— and, more importantly, that f and g are surjective. In practice, one might be tempted to fudge the application of the corollary by restricting a given ordering to a subset of the elements on which it is defined (for example, restricting the at-most ordering on integers to the at-most ordering on even integers). Rather than resort to such measures, we prefer to make the process precise within our axiom system. Indeed, it is necessary for us to do so in order to establish a sufficient condition for a staircase relation to be linearly block-ordered. See theorem 333 below.

In the following lemmas $R{>\!\!\bullet}$ denotes the complement of $R{>}$ in the lattice of coreflexives. That is, for arbitrary relation R, we have

$$(323)\quad R{>} \cup R{>\!\!\bullet} = I \quad\wedge\quad R{>} \cap R{>\!\!\bullet} = \bot\!\!\bot$$

(where I and $\bot\!\!\bot$ denote the identity and empty relations of appropriate type). Similarly $R{\bullet\!\!<}$ denotes the complement of $R{<}$ That is

$$(324)\quad R{<} \cup R{\bullet\!\!<} = I \quad\wedge\quad R{<} \cap R{\bullet\!\!<} = \bot\!\!\bot \quad .$$

Domain calculus enables the proof of the following:

$$(325)\quad R \circ R{>\!\!\bullet} = \bot\!\!\bot \quad\wedge\quad R{\bullet\!\!<} \circ R = \bot\!\!\bot \quad .$$

Given a relation R, the points in $R{\bullet\!\!<}$ (or, dually $R{>\!\!\bullet}$) are arguably irrelevant since they are precisely the points that are not related to any other point by R. Similar statements can be made about factors. In general, for arbitrary relations R and S, the factor $R\backslash S$ is arguably too big because its left domain includes $R{>\!\!\bullet}$. Similarly, the factor R/S is too big because its right domain includes $S{\bullet\!\!<}$, as is shown in the following lemma.

Lemma 326 For all R and S,

$$R{>\!\!\bullet} \circ R\backslash S \;=\; R{>\!\!\bullet} \circ \top\!\!\top \quad\wedge\quad R/S \circ S{\bullet\!\!<} \;=\; \top\!\!\top \circ S{\bullet\!\!<} \quad .$$

Proof We prove the first equation:

$$\begin{array}{ll}
 & R{>\!\!\bullet} \circ \top\!\!\top \\
= & \{ \quad \text{complements} \quad \} \\
 & R{>\!\!\bullet} \circ (R\backslash S \cup \neg(R\backslash S)) \\
= & \{ \quad \text{distributivity} \quad \}
\end{array}$$

$$
\begin{array}{ll}
 & R{>}{\bullet} \circ R\backslash S \;\cup\; R{>}{\bullet} \circ \neg(R\backslash S) \\
= & \{\quad (38) \quad\} \\
 & R{>}{\bullet} \circ R\backslash S \;\cup\; R{>}{\bullet} \circ R^{\cup} \circ \neg S \\
= & \{\quad (325) \text{ and converse} \quad\} \\
 & R{>}{\bullet} \circ R\backslash S \;\cup\; \perp\!\!\perp \circ \neg S \\
= & \{\quad \perp\!\!\perp \text{ is zero of composition and unit of union} \quad\} \\
 & R{>}{\bullet} \circ R\backslash S \quad .
\end{array}
$$

□

The argument that factors typically include irrelevant information extends to the preorders $R\backslash R$ and R/R. In particular, note the terms involving $R{>}{\bullet}$ in the following lemma.

Lemma 327 For all R,

$$R\backslash R \cup (R\backslash R)^{\cup} \;=\; R{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R{>} \;\cup\; R{>}{\bullet} \circ \top\!\!\top \;\cup\; \top\!\!\top \circ R{>}{\bullet} \quad .$$

Proof

$$
\begin{array}{ll}
 & R\backslash R \cup (R\backslash R)^{\cup} \\
= & \{\quad (323) \quad\} \\
 & (R{>} \cup R{>}{\bullet}) \circ R\backslash R \;\cup\; (R\backslash R)^{\cup} \circ (R{>} \cup R{>}{\bullet}) \\
= & \{\quad \text{distributivity} \quad\} \\
 & \quad R{>} \circ R\backslash R \;\cup\; R{>}{\bullet} \circ R\backslash R \\
\cup & \quad (R\backslash R)^{\cup} \circ R{>} \;\cup\; (R\backslash R)^{\cup} \circ R{>}{\bullet} \\
= & \{\quad \text{lemma 326 and rearranging} \quad\} \\
 & R{>} \circ R\backslash R \;\cup\; \top\!\!\top \circ R{>}{\bullet} \\
\cup & (R\backslash R)^{\cup} \circ R{>} \;\cup\; R{>}{\bullet} \circ \top\!\!\top \\
= & \{\quad (323) \text{ and distributivity (as in first two steps)} \quad\} \\
 & R{>} \circ R\backslash R \circ R{>} \;\cup\; R{>} \circ R\backslash R \circ R{>}{\bullet} \;\cup\; \top\!\!\top \circ R{>}{\bullet} \\
\cup & R{>} \circ (R\backslash R)^{\cup} \circ R{>} \;\cup\; R{>}{\bullet} \circ (R\backslash R)^{\cup} \circ R{>} \;\cup\; R{>}{\bullet} \circ \top\!\!\top \\
= & \{\quad R{>} \circ R\backslash R \subseteq \top\!\!\top \text{ and } (R\backslash R)^{\cup} \circ R{>} \subseteq \top\!\!\top \\
 & \qquad \text{and definition of subset relation} \quad\} \\
 & R{>} \circ R\backslash R \circ R{>} \;\cup\; \top\!\!\top \circ R{>}{\bullet}
\end{array}
$$

$$\cup \quad R_{>} \circ (R\backslash R)^{\cup} \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top$$

$$= \quad \{ \quad \text{rearranging and distributivity} \quad \}$$

$$R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} \;\cup\; \top\!\!\top \circ R_{>\bullet} \;\cup\; R_{>\bullet} \circ \top\!\!\top \quad .$$

□

(Lemma 327 is essentially the case analysis that was omitted in the proof of lemma 303.) Avoiding the useless information introduced by the factor operators was the motivation for our introducing the notion of "provisional" (pre)orders. The following lemma enables the conventional notion of a linear ordering to be extended to provisional orderings.

Lemma 328 For all R,

$$R\backslash R \cup (R\backslash R)^{\cup} = \top\!\!\top \;\equiv\; R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} = R_{>} \circ \top\!\!\top \circ R_{>} \quad .$$

Proof By mutual implication. First,

$$R\backslash R \cup (R\backslash R)^{\cup} = \top\!\!\top$$

$$\Rightarrow \quad \{ \quad \text{Leibniz} \quad \}$$

$$R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} \;=\; R_{>} \circ \top\!\!\top \circ R_{>} \quad .$$

Second,

$$R\backslash R \cup (R\backslash R)^{\cup}$$

$$= \quad \{ \quad \text{lemma 327} \quad \}$$

$$R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top \;\cup\; \top\!\!\top \circ R_{>\bullet}$$

$$= \quad \{ \quad \text{assume: } R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} = R_{>} \circ \top\!\!\top \circ R_{>} \quad \}$$

$$R_{>} \circ \top\!\!\top \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top \;\cup\; \top\!\!\top \circ R_{>\bullet}$$

$$= \quad \{ \quad \text{(323), distributivity and rearranging (as in proof of lemma 327)} \quad \}$$

$$R_{>} \circ \top\!\!\top \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top$$

$$\cup \quad R_{>} \circ \top\!\!\top \circ R_{>} \;\cup\; R_{>} \circ \top\!\!\top \circ R_{>\bullet} \;\cup\; \top\!\!\top \circ R_{>\bullet}$$

$$= \quad \{ \quad \text{(323), distributivity and rearranging} \quad \}$$

$$\top\!\!\top \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top$$

$$\cup \quad R_{>} \circ \top\!\!\top \;\cup\; \top\!\!\top \circ R_{>\bullet}$$

$$= \quad \{ \quad \text{(323), distributivity and rearranging} \quad \}$$

$$\top\!\!\top \quad .$$

(Note the assumption in the second step.) That is,

$$R{>}\circ(R\backslash R\cup(R\backslash R)^{\cup})\circ R{>} = R{>}\circ\top\!\!\top\circ R{>} \;\Rightarrow\; R\backslash R\cup(R\backslash R)^{\cup} = \top\!\!\top \quad .$$

□

As always, for practical purposes it is preferable to express properties in terms of the core of a relation rather than the relation itself. Lemma 328 is easily rewritten accordingly:

Theorem 329 Suppose $|R|$ is a core of relation R as determined by the functionals λ and ρ. (See definition 191.) Then R is a staircase relation iff

$$|R|{<}\circ(|R|\backslash|R| \;\cup\; (|R|\backslash|R|)^{\cup})\circ|R|{<} \;=\; |R|{>}\circ\top\!\!\top\circ|R|{>} \quad .$$

Proof

$$\begin{array}{ll}
 & R{>}\circ(R\backslash R\cup(R\backslash R)^{\cup})\circ R{>} \;=\; R{>}\circ\top\!\!\top\circ R{>} \\
= & \{\quad \text{lemma 195} \quad\} \\
 & \rho{>}\circ(R\backslash R\cup(R\backslash R)^{\cup})\circ\rho{>} \;=\; \rho{>}\circ\top\!\!\top\circ\rho{>} \\
= & \{\quad \text{by lemma 193, } R \;=\; \lambda^{\cup}\circ|R|\circ\rho\,\text{, (81) with } f,g:=\lambda,\rho \quad\} \\
 & \rho^{\cup}\circ(|R|\backslash|R| \;\cup\; (|R|\backslash|R|)^{\cup})\circ\rho \;=\; \rho{>}\circ\top\!\!\top\circ\rho{>} \\
= & \{\quad (\Rightarrow)\text{ monotonicity and domains; } (\Leftarrow)\text{ ditto and } \rho^{\cup}\circ\rho = \rho{<} \quad\} \\
 & \rho\circ\rho^{\cup}\circ(|R|\backslash|R| \;\cup\; (|R|\backslash|R|)^{\cup})\circ\rho\circ\rho^{\cup} \;=\; \rho{<}\circ\top\!\!\top\circ\rho{<} \\
= & \{\quad \text{by lemma 195, } \rho\circ\rho^{\cup} = \rho{<} = |R|{>} \quad\} \\
 & |R|{>}\circ(|R|\backslash|R| \;\cup\; (|R|\backslash|R|)^{\cup})\circ|R|{>} \;=\; |R|{>}\circ\top\!\!\top\circ|R|{>} \quad .
\end{array}$$

The theorem follows by combining lemma 328 with definition 302.
□

Lemma 330 A linear provisional ordering is a staircase relation.

Proof Suppose T is a linear provisional ordering. Then

$$\begin{array}{ll}
 & SC.T \\
= & \{\quad \text{definition 302} \quad\} \\
 & T\backslash T\cup(T\backslash T)^{\cup} = \top\!\!\top \\
= & \{\quad \text{lemma 328} \quad\} \\
 & T{>}\circ(T\backslash T\cup(T\backslash T)^{\cup})\circ T{>} = T{>}\circ\top\!\!\top\circ T{>}
\end{array}$$

$$
\begin{array}{cl}
= & \{ \quad \text{lemma 122} \quad \} \\
& (T \cap T^{\cup}) \circ (T \backslash T \cup (T \backslash T)^{\cup}) \circ (T \cap T^{\cup}) \;=\; (T \cap T^{\cup}) \circ \top\!\!\top \circ (T \cap T^{\cup}) \\
= & \{ \quad \text{assumption: } T \text{ is a provisional ordering} \\
& \quad\;\; \text{lemma 119 and definition 121} \quad \} \\
& (T \cap T^{\cup}) \circ (T \cup T^{\cup}) \circ (T \cap T^{\cup}) \;=\; (T \cap T^{\cup}) \circ \top\!\!\top \circ (T \cap T^{\cup}) \\
= & \{ \quad \text{assumption: } T \text{ is linear, definition 121} \quad \} \\
& \mathsf{true} \;\; .
\end{array}
$$

□

Lemma 331 Suppose R is a linearly block-ordered relation. Then R is a staircase relation.

Proof This is an immediate consequence of lemmas 317 and 330. Specifically, by definition 225, R is a block-ordered relation if $R = f^{\cup} \circ T \circ g$ where f and g satisfy (227) and T is a provisional ordering (i.e. satisfies (226)). It is a linearly block-ordered relation if, in addition, T is a linear provisional ordering. Applying lemma 317 (with $R{:=}T$), R is a staircase relation if T is a staircase relation. But this is indeed the case by lemma 330.
□

Lemma 332 Suppose R is a staircase relation. Then

$$
R \text{ is linearly block-ordered} \;\Leftarrow\; (\Delta R)_{<} \;=\; R_{<} \;\wedge\; (\Delta R)_{>} \;=\; R_{>} \;\; .
$$

Proof By lemma 259, R is block-ordered. Specifically, lemma 259 shows how to construct functionals f and g and a provisional ordering T satisfying the properties (227) and (226) such that $R = f^{\cup} \circ T \circ g$. The task is thus to prove that T is linear if R is a staircase relation.

We have:

$$
\begin{array}{cl}
& R_{<} \circ (R \backslash R / R)^{\cup} \circ R_{>} \\
= & \{ \quad R = f^{\cup} \circ T \circ g \;\text{ and (227)} \quad \} \\
& f_{>} \circ ((f^{\cup} \circ T \circ g) \backslash (f^{\cup} \circ T \circ g) / (f^{\cup} \circ T \circ g))^{\cup} \circ g_{>} \\
= & \{ \quad \text{converse and factors: (37)} \quad \} \\
& f_{>} \circ (g^{\cup} \circ T^{\cup} \circ f) \backslash (g^{\cup} \circ T^{\cup} \circ f) / (g^{\cup} \circ T^{\cup} \circ f) \circ g_{>} \\
= & \{ \quad \text{lemma 78 with } U,V,W := T^{\cup}, T^{\cup}, T^{\cup} \quad \}
\end{array}
$$

$f^\cup \circ (T\backslash T/T)^\cup \circ g$

$=$ { domains and (227) }

$f^\cup \circ (T\cap T^\cup) \circ (T\backslash T/T)^\cup \circ (T\cap T^\cup) \circ g$

$=$ { T is a provisional ordering, lemmas 116 and 118 }

$f^\cup \circ T^\cup \circ g$.

So

$SC.R$

$=$ { (307) }

$R \cup (R\backslash R/R)^\cup \;=\; \top\!\!\top$

$\Rightarrow$ { $[S \subseteq \top\!\!\top]$, domains and monotonicity }

$R \;\cup\; R_< \circ (R\backslash R/R)^\cup \circ R_> \;=\; R_< \circ \top\!\!\top \circ R_>$

$=$ { $R = f^\cup \circ T \circ g$ and above calculation }

$f^\cup \circ T \circ g \;\cup\; f^\cup \circ T^\cup \circ g \;=\; f_> \circ \top\!\!\top \circ g_>$

$=$ { distributivity }

$f^\cup \circ (T \cup T^\cup) \circ g \;=\; f_> \circ \top\!\!\top \circ g_>$

$\Rightarrow$ { Leibniz }

$f \circ f^\cup \circ (T \cup T^\cup) \circ g \circ g^\cup \;\supseteq\; f \circ f_> \circ \top\!\!\top \circ g_> \circ g^\cup$

$=$ { definition 225 of block-ordered
in particular (227); domains }

$(T\cap T^\cup) \circ (T \cup T^\cup) \circ (T\cap T^\cup) \;=\; (T\cap T^\cup) \circ \top\!\!\top \circ (T\cap T^\cup)$

$=$ { lemma 122 and definition 121 }

T is linear .

□

Theorem 333 Suppose R is a staircase relation. Then

$$R \text{ is linearly block-ordered} \;\equiv\; (\Delta R)_< = R_< \;\wedge\; (\Delta R)_> = R_> \;.$$

Proof By mutual implication. “Only-if” is an instance of theorem 262. “If” is lemma 332.

□

11.7 Riguet's Rectangle Theorem

As mentioned earlier, the purpose of undertaking this exercise was to demonstrate how reasoning with factors is so much more straightforward than reasoning with nested negations. It was a surprise to discover an error in the extant literature. This section is about our attempt to trace the source of the material on difunctional and staircase relations and, in particular, the source of the error.

Riguet introduces the notion of a difunctional relation in [Rig48] and the notion of a staircase relation in [Rig51] — but uses the name "relation de Ferrers". His definition corresponds to property (308). He lists a number of properties related to the ones stated above. Direct comparison is slightly complicated by the fact that he does not make use of factors. For example, he states that R is a "relation de Ferrers" if and only if $R\circ\neg R^{\cup}$ is a "relation de Ferrers". This is a combination of lemma 304 (in particular (305)) and lemma 311.

Riguet does not give a proof of the theorem. Riguet [Rig51] states that there is a striking analogy ("une analogie frappante") between the definitions and properties of "relations de Ferrers" and difunctional relations. He states that the analogy is clarified by[14] a theorem similar to our lemma 182 but does not go into further details. As mentioned earlier, his theorem is that, if R is a staircase relation (a "relation de Ferrers"), then so too is $R\circ\neg R^{\cup}\circ R$ and their "différence" $R\cap\neg(R\circ\neg R^{\cup}\circ R)$ is difunctional. Lemma 182 is stronger than Riguet's difunctionality property because it does not require R to be a staircase relation.

Note that in the case that R is the less-than relation on real numbers, $R\circ\neg R^{\cup}\circ R$ is also the less-than relation and $R\cap\neg(R\circ\neg R^{\cup}\circ R)$ is trivially difunctional (since it is the empty relation). This observation leads one to wonder precisely how the "analogie frappante" is clarified by Riguet's theorem. (We invite the reader to verify the claims we have just made and then work out the difference when "real number" is replaced by "integer".)

As announced earlier, the proof of Riguet's theorem is straightforward[15]:

Theorem 334 (Riguet's theorem) A relation is a staircase relation if and only if it has a linear polar covering.

Proof By mutual implication.

For the "only-if" part, theorem 211 establishes that every relation has a polar covering. So it suffices to show that if R is a staircase relation the covering is linear. Recall the construction of $\mathcal{R}$ in theorem 211. If R is a staircase relation, that the set $\mathcal{R}_{<}$ is linearly ordered by inclusion is immediate from lemma 303.

[14] "Cette analogie s'éclaire par"

[15] This may explain why he didn't provide a proof.

For the "if" part, suppose R of type $A{\sim}B$ has a linear polar covering $\mathcal{R}$. Our task is to show that R is a staircase relation. Aiming to apply lemma 303, we consider points b and b' such that $b \subseteq R{>}$ and $b' \subseteq R{>}$. Our task becomes to show that

$$(R{\circ}b){<} \subseteq (R{\circ}b'){<} \;\vee\; (R{\circ}b'){<} \subseteq (R{\circ}b){<} \quad .$$

This is achieved as follows:

$$
\begin{array}{ll}
 & (R{\circ}b){<} \subseteq (R{\circ}b'){<} \;\vee\; (R{\circ}b'){<} \subseteq (R{\circ}b){<} \\
= & \{ \qquad R = \cup\mathcal{R} \quad \} \\
 & (\cup\mathcal{R} \circ b){<} \subseteq (R{\circ}b'){<} \;\vee\; (\cup\mathcal{R} \circ b'){<} \subseteq (R{\circ}b){<} \\
= & \{ \qquad \text{distributivity properties} \quad \} \\
 & \langle \forall U : U{\in}\mathcal{R} : (U{\circ}b){<} \subseteq (R{\circ}b'){<} \rangle \;\vee\; \langle \forall U : U{\in}\mathcal{R} : (U{\circ}b'){<} \subseteq (R{\circ}b){<} \rangle \\
= & \{ \qquad \text{lemma 128,} \\
 & \qquad\quad \text{case analyses on } (b' \subseteq U{>} \wedge (U{\circ}b'){<} = U{<}) \vee (U{\circ}b'){<} = \perp\!\!\perp \\
 & \qquad\quad \text{and } (b \subseteq U{>} \wedge (U{\circ}b){<} = U{<}) \vee (U{\circ}b){<} = \perp\!\!\perp \quad \} \\
 & \quad \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : U{<} \subseteq (R{\circ}b'){<} \rangle \\
 & \vee\; \langle \forall U : U{\in}\mathcal{R} \wedge b' \subseteq U{>} : U{<} \subseteq (R{\circ}b){<} \rangle \\
\Leftarrow & \{ \qquad R = \cup\mathcal{R}\text{, monotonicity and lemma 128} \quad \} \\
 & \quad \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b' \subseteq V{>} : U{<} \subseteq V{<} \rangle\rangle \\
 & \vee\; \langle \forall U : U{\in}\mathcal{R} \wedge b' \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b \subseteq V{>} : U{<} \subseteq V{<} \rangle\rangle \\
= & \{ \qquad \text{assumption: } \mathcal{R} \text{ is a polar covering} \\
 & \qquad\quad \text{so } U{<} \subseteq V{<} \;\equiv\; U{>} \supseteq V{>} \quad \} \\
 & \quad \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b' \subseteq V{>} : U{>} \supseteq V{>} \rangle\rangle \\
 & \vee\; \langle \forall U : U{\in}\mathcal{R} \wedge b' \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b \subseteq V{>} : U{>} \supseteq V{>} \rangle\rangle \\
= & \{ \qquad [\, p \vee q \;\equiv\; (\neg q \Rightarrow p) \,] \text{ together with the calculation below} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

The justification of the final step is as follows.

$$
\begin{array}{ll}
 & \neg\langle \forall U : U{\in}\mathcal{R} \wedge b' \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b \subseteq V{>} : U{>} \supseteq V{>} \rangle\rangle \\
= & \{ \qquad \text{predicate calculus (and dummy change: } U,V := V,U \text{)} \quad \} \\
 & \langle \exists V : V{\in}\mathcal{R} \wedge b' \subseteq V{>} : \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : \neg(V{>} \supseteq U{>}) \rangle\rangle \\
= & \{ \qquad \text{assumption: } \mathcal{R} \text{ is a linear polar covering}
\end{array}
$$

in particular, the inclusion ordering on left domains is linear $\}$

$$\langle \exists V : V \in \mathcal{R} \wedge b' \subseteq V_> : \langle \forall U : U \in \mathcal{R} \wedge b \subseteq U_> : V_> \subset U_> \rangle\rangle$$

$\Rightarrow$ $\{$ predicate calculus and $V_> \subset U_> \Rightarrow U_> \supseteq V_>$ $\}$

$$\langle \forall U : U \in \mathcal{R} \wedge b \subseteq U_> : \langle \exists V : V \in \mathcal{R} \wedge b' \subseteq V_> : U_> \supseteq V_> \rangle\rangle \ .$$

□

In the proof of theorem 334 we have chosen a covering that is indexed by points in the *source* of the given relation R. We could, of course, have chosen a covering that is indexed by points in the relation's *target*. Fig. 19 is a mental picture of the different choices.

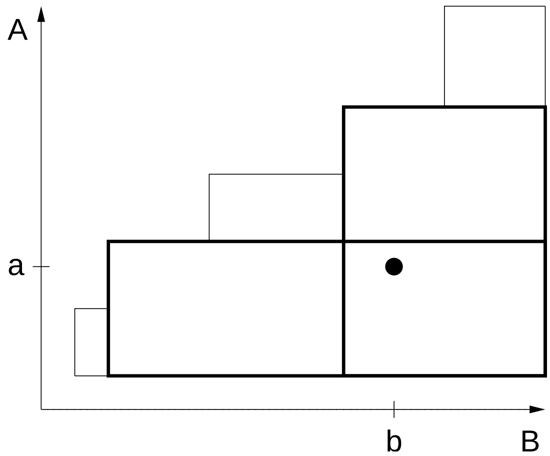


Figure 19: Choices of Polar Covering

Highlighted in fig. 19 are a point —the point $a \circ \top\!\!\top \circ b$ in our formalism— and two rectangles. The (highlighted) long, low rectangle depicts the relation

$$R/R \circ a \circ R \ ,$$

whilst the (highlighted) short, tall rectangle depicts the relation

$$R \circ b \circ R \backslash R \ .$$

Rather than choosing the latter as the elements of the polar covering —as we did— , we could have chosen the former. The (highlighted) corner rectangle depicts the relation

$$R/R \circ a \circ \top\!\!\top \circ b \circ R \backslash R \ .$$

Indeed, for all relations R,

$$R/R \circ a \circ R \cap R \circ b \circ R \backslash R \ = \ R/R \circ a \circ \top\!\!\top \circ b \circ R \backslash R$$

$$\Leftarrow \quad a \circ \top\!\!\top \circ b \subseteq R \ .$$

We leave the proof of this property to the reader. (Hint: use lemma 125.)

11.8 Finite Staircase Relations

As we have seen in theorems 319 and 320, not every staircase relation is block-ordered. However, for a relation to satisfy the assumptions made in theorem 320 it must be infinite. In this section we show that every finite staircase relation is indeed block-ordered.

Theorem 335 Suppose R is a finite relation. (That is, the sets represented by $R<$ and $R>$ are finite.) Then R is a staircase relation equivales R is a linear block-ordered relation.

Proof Lemma 331 shows that R is a staircase relation if it is a linear block-ordered relation (whether or not R is finite). It remains to show that, if R is finite and a staircase relation then it is a linear block-ordered relation.

Given a finite, staircase relation R, our task is to costruct functionals f and g and a provisional ordering T satisfying definition 225. The key is a combination of theorems 283, 334 and 292.

Theorem 283 states that $R = f^{\cup} \circ \sqsubseteq \circ g$, where the ordering $\sqsubseteq$ is as in definition 276, and

$$f> = R< \;\; \wedge \;\; \langle \forall a \; : \; a \subseteq R< \; : \; f.a = R/R \circ a \circ R \rangle$$

and

$$g> = R> \;\; \wedge \;\; \langle \forall b \; : \; b \subseteq R> \; : \; g.b = R \circ b \circ R \backslash R \rangle \quad .$$

To complete our task, we must show that $f< = g<$. That is, we must show that

$$(336) \quad \langle \forall a \; : \; a \subseteq R< \; : \; \langle \exists b : b \subseteq R> : f.a = g.b \rangle \rangle$$

and vice-versa

$$(337) \quad \langle \forall b \; : \; b \subseteq R> \; : \; \langle \exists a : a \subseteq R< : f.a = g.b \rangle \rangle \quad .$$

Riguet's theorem (theorem 334) states that, if R is a staircase relation, both f and g are linear polar coverings of R; although not stated explicitly there, the ordering on the elements of both coverings is the ordering $\sqsubseteq$ introduced in definition 276.

Now, a characteristic feature of a *finite* linear ordering is that the suprema and infima of any non-empty set always exist and are the maxima and minima. That is, for all points a such that $a \subseteq R<$, the minimum

$$\langle \mathsf{MIN}_{\sqsubseteq}\, b' : a \circ \top\!\!\top \circ b' \subseteq R : g.b' \rangle$$

is well-defined. More precisely, the minimum value "witnesses" the existentially quantified dummy b in the property

$$\langle \forall a \; : \; a \subseteq R< \; : \; \langle \exists b : a \circ \top\!\!\top \circ b \subseteq R : \langle \forall b' : a \circ \top\!\!\top \circ b' \subseteq R : g.b \sqsubseteq g.b' \rangle \rangle \rangle$$

assuming that the ordering $\sqsubseteq$ is a linear ordering of a finite set. Similarly, for all points b such that $b \subseteq R_{>}$, the maximum

$$\langle \mathsf{MAX}_{\sqsubseteq}\, a' : a' \circ \top\!\top \circ b \subseteq R : f.a' \rangle$$

exists and "witnesses" the existentially quantified dummy a in the property

$$\langle \forall b \,:\, b \subseteq R_{>} \,:\, \langle \exists a : a \circ \top\!\top \circ b \subseteq R : \langle \forall a' : a' \circ \top\!\top \circ b \subseteq R : f.a' \sqsubseteq f.a \rangle\rangle\rangle \quad .$$

With this knowledge, we can now prove (336). Suppose $a \subseteq R_{<}$. Then

$$
\begin{array}{ll}
 & \langle \exists b \,:\, b \subseteq R_{>} \,:\, f.a = g.b \rangle \\
= & \{ \quad \text{definition of } f \text{ and } g \quad \} \\
 & \langle \exists b \,:\, b \subseteq R_{>} \,:\, R/R \circ a \circ R \;=\; R \circ b \circ R\backslash R \rangle \\
\Leftarrow & \{ \quad a \circ \top\!\top \circ b \subseteq R \Rightarrow b \subseteq R_{>} \quad \} \\
 & \langle \exists b \,:\, a \circ \top\!\top \circ b \subseteq R \,:\, R/R \circ a \circ R \;=\; R \circ b \circ R\backslash R \rangle \\
= & \{ \quad \text{theorem 292} \quad \} \\
 & \langle \exists b : a \circ \top\!\top \circ b \subseteq R : \langle \forall b' \,:\, a \circ \top\!\top \circ b' \subseteq R \,:\, R \circ b \circ R\backslash R \sqsubseteq R \circ b' \circ R\backslash R \rangle\rangle \\
= & \{ \quad \text{definition of } g \quad \} \\
 & \langle \exists b : a \circ \top\!\top \circ b \subseteq R : \langle \forall b' : a \circ \top\!\top \circ b' \subseteq R : g.b \sqsubseteq g.b' \rangle\rangle \\
\Leftarrow & \{ \quad \text{definition of } \mathsf{MIN} \quad \} \\
 & \langle \exists b : a \circ \top\!\top \circ b \subseteq R : g.b = \langle \mathsf{MIN}_{\sqsubseteq}\, b' : a \circ \top\!\top \circ b' \subseteq R : g.b' \rangle\rangle \\
= & \{ \quad \text{assumption: } R \text{ is a finite relation} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

We have thus established (336). Property (337) is the converse dual.
□

Although theorem 335 assumes that the relation R is finite, it can of course be applied to the core $|R|$ of the relation R. From the definition of a core (definition 191) and lemma 193, it is easy to establish the equivalence of properties of a relation and properties of its core, in particular, being a staircase relation and being block-ordered. Thus, the theorem is more generally applicable to relations whose core is finite.

12 Discussion

The writing of this paper began after reading a paper by Wolfram Kahl (see [Kah20]) which included a section on "Ferrers-type relations" citing not Riguet [Rig51] (where the notion is introduced) but the textbook by Schmidt and Ströhlein [SS93]. Although Schmidt and Ströhlein also do not cite [Rig51], they do use Riguet's definitions. It was immediately clear that substantial improvements could be made to Schmidt and Ströhlein's calculations by exploiting the properties of the factors of a relation. Further study also revealed an obvious error in their "definition" [SS93, Definition 4.4.11]. (Schmidt and Ströhlein's "definitions" often include what they call "definition variants" which, in most cases, they deem to be obviously equivalent. This is not the case here — see below.) This led to an investigation of the origin of the error which, in turn, led to the discovery of the original paper by [Rig51]. Several more recent publications were also discovered where the opportunity to correct Schmidt and Ströhlein's error is not taken. Intrigued, it was decided to embark on a thorough investigation of the notions introduced in [Rig51]: the notion of the "différence" of a relation and the notion of a "relation de Ferrers" as well as Riguet's "analogie frappante" connecting the two. In the process, it became clear that a more general notion of "block-ordering" was relevant than the total ordering demanded by Riguet. This led to the four goals enumerated in the introduction.

The need for the first two goals is clear from a study of Riguet's paper. Although his work is comprehensive (in particular [Rig48]), the typography of publications written 70 years ago makes them difficult to read; the notation chosen by Riguet is also often rather quaint (and in some cases impossible to reproduce!). Ironically in a paper about "correspondances de Galois", Riguet does not introduce the Galois connection defining the factors of a relation and, instead, makes copious use of (nested) complements. Also, Riguet states many properties without proof: for example, [Rig51] lists ten definitions of a "relation de Ferrers" with justification that it is easy to see ("il est facile de voir") that they are all equivalent. Moreover, subsequent literature leaves many gaps. For example, we have been unable to find any proof of theorem 335, even though we have seen several publications that assume the theorem (correctly in the case of finite relations).

Experience shows that the most important concepts —the ones with wide applicability— tend to be discovered and rediscovered, often quite independently, in several different and apparently unrelated contexts. Different formulations, that turn out to be equivalent, and different terminology, reflecting particular application areas, is introduced, making the task of proper attribution almost impossible. All that an author can be expected to do is to cite the publications that have had a significant influence on their own work — which is what we have done here.

For the reasons given above, the initial steps in the writing of this document were influenced by section 4.4 of the textbook by Schmidt and Ströhlein [SS93]. Like Riguet, Schmidt and Ströhlein do not introduce the factor operators and, implicitly, use the equivalent definition in terms of nested complements. (See lemma 32.) The longest and arguably most opaque calculation in this section of Schmidt and Ströhlein's book is their proof of proposition 4.4.13(ii). Aside from its extensive use of nested complements, it fails to make clear what is being proved, why it is being proved and where and when assumptions are invoked (at least in our view). The proposition is formulated in theorem 234. Various properties are used in their proof which we have formulated and proven in lemma 248 in terms of factors. Properties (249) and —more significantly— (250) are not observed by Schmidt and Ströhlein. Their derivation of (251) is asymmetric in f and g and involves several unexplained steps.

We have not been able to avoid the use of complements altogether. As pointed out at the time, the equivalence of several different formulations of the notion of a staircase relation formulated in lemma 304 uses the definition of factors in terms of nested complements. Also, for concrete examples of (small) finite relations, such as examples 223, 224 and 284, the use of complements often makes calculations easier. Nevertheless our use of complements has been minimal.

We have attributed the two principal concepts of a "relation difonctionelle" and a "relation de Ferrers" to Riguet ([Rig48] and [Rig51], respectively) but we have not explored any publications prior to Riguet's. Riguet himself cites two papers by Norbert Wiener, dated 1912–1914 and 1914-1916, as giving an equivalent definition of a "relation de Ferrers" but no other indication of their content is provided (not even their titles). We have also been unable to find publications on either topic in the forty or so years following their publication. (Riguet [Rig51] announces a "prochaine Note" that will make precise a correspondence between "relations equivalence conjuguées" and "relations de Ferrers" but we have not been able to find the publication.) So the current work should not be regarded as a history of the concepts.

The notion of a difunctional relation is now generally attributed to Riguet [Rig48]; Jaoua et al [JMBD91] use the name "regular relation" but later publications [KGJ00] use the name "difunctional relation". Voermans [Voe99] emphasises their importance in developing a theory of datatypes with laws; Oliveira [Oli18] argues that difunctional relations are "metaphors" for program specification. Much of our presentation on difunctional relations and non-redundant polar coverings is influenced by the goal of gaining a complete understanding of Riguet's "analogie frappante" [Rig51].

The notions of a rectangle and completely disjoint rectangles, and elementary facts about difunctional relations, in particular theorems 141 and 161, are discussed by Riguet [Rig48]. The corresponding properties of pers are well-known. The construction given in section 6.3.3 is not made explicit in [Rig50] but was possibly the basis of Riguet's statement that the characterisation of difunctional relations as a pair of functional relations (theorem 161) is a generalisation of the theorem that a partial equivalence relation is characterised by a single functional relation (theorem 143). (Evidence for this is that Riguet effectively states lemma 174.)

Theorem 161 is also stated in [JMBD91, Proposition 4.12] and a proof given. Their proof assumes the relation is homogeneous; the proof of theorem 173 is inspired by their proof whilst avoiding the assumption. Winter [Win04] assumes theorem 144 and then uses it to prove theorem 161 (thus making precise Riguet's generalisation). His (very short and elegant) proof, which we have reproduced here, gives different —albeit isomorphic— characterisations of a difunctional relation. Our contribution has been to compare different algebraic proofs of the theorem: point-free and pointwise proofs. Perhaps surprisingly, our conclusion is that the pointwise proof is preferable to the proof that exploits a point-free characterisation of power transpose. This is because of the simplicity of the step from the elementary characterisation of difunctional relations (theorem 160) to a set of rectangles ("réunions de rectangles"): see section 6.3.1.

Theorem 166 is Schmidt and Ströhlein's proposition 4.4.10(ii) . Their statement of the theorem is unclear: it appears to state that a difunctional relation has exactly one representation as a pair of functional, surjective relations but they only prove that there is at most one such representation. (Both here and in the statement of proposition proposition 4.4.13(ii) they use the phrase "may be achieved in essentially one fashion". The English is ambiguous: "may be achieved" suggests "at least one" and "in essentially one fashion" suggests "at most one", the combination being exactly one. But they only prove at most one.) Lemma 164 is novel and permits a subtle difference in presentation, in particular of theorem 166.

There is much in common between our section 8 and Khchérif, Gammoudi and Jaoua [KGJ00]. Khchérif, Gammoudi and Jaoua [KGJ00] correctly attribute the concept of the diagonal to Riguet but do not cite [Rig51]; like Riguet, they define the diagonal in terms of nested complements and do not exploit factors. Their notion of a covering specifies the rectangles to be "maximal". This is the property of not being "obviously redundant" as discussed immediately following definition 209. Slightly confusingly[16], Khchérif, Gammoudi and Jaoua [KGJ00] define two rectangles to be "disjoint" when they are what we call "completely disjoint". With this caveat, they list theorem 163 as a property of difunctional relations. They do not seem to be aware of theorem 211. Their focus is on what they call "minimal" coverings and "isolated points"; "minimal" coverings appear to correspond to what we call "non-redundant" coverings whilst "isolated points" appear to correspond to the points of a definiens of a relation. They seem to suggest a dichotomy: for each relation R, either $(\Delta R)_< = R_<$ and $(\Delta R)_> = R_>$, or $\Delta R = \perp\!\!\perp$. (See

[16]The term "disjoint" is commonly used to describe sets with an empty intersection; the confusion arises because relations are sets of pairs.

[KGJ00, p.161, **Problem**].) Example 223 shows that this is not the case: it is indeed possible to construct a non-redundant covering of a relation R where $(\Delta R)_< \neq R_<$ so long as $(\Delta R)_> = R_>$ (and, of course, dually when $(\Delta R)_> \neq R_>$ so long as $(\Delta R)_< = R_<$). The statements of theorems 1 and 2 in [KGJ00] are unclear (in my view), making them difficult to verify or refute.

Schmidt and Ströhlein [SS93, p.80] cite the paper by Jacques Riguet [Rig50] with the word "difonctionelle" in the title; they also use the same definition of a "Ferrers type relation" as Riguet but do not cite [Rig51]. (They do cite [Rig48] earlier in the text but not in connection with difunctional relations.) Schmidt and Ströhlein appear to claim that "staircase" and "linearly block-ordered" are equivalent properties of a relation: Their definition of "Ferrers type" [SS93, Definition 4.4.11] comprises five properties connected by the symbol "$\Leftrightarrow$". Presumably the symbol denotes logical equivalence (an implicit universal quantification over all free variables combined with boolean equality) but it is nowhere defined[17]. From definition 2.1.3, and experience with common mathematical practice, one infers that Schmidt and Ströhlein use the keyword "**Definition**" to simultaneously introduce a definition and to state properties of the defined entity that are deemed to be obvious. The problem is that the equality of the predicates "staircase" and "linearly block-ordered" is far from obvious and, as we have shown in theorem 319, it is just not true! Other papers that cite Riguet assume that the relations under consideration are finite —in which case the equivalence is valid (see lemma 331 and theorem 335)— ; consequently, it would appear that the erroneous claim was introduced by Schmidt and Ströhlein.

Winter restates the erroneous claim made by Schmidt and Ströhlein [SS93, Definition 4.4.11]:

> A concrete relation of Ferrers type may be written as a Boolean matrix in staircase block form by suitably rearranging rows and columns.

There does not appear to be a definition of the word "concrete" in the paper; the use of the word "matrix" suggests that "concrete" means "finite". In this case, the claim is a special case of theorem 335. However, we have been unable to find any proof of the theorem in the published literature: Riguet [Rig51] states the theorem but does not provide a proof; he does make very clear that his definition of a "relation de Ferrers" extends to infinite relations, specifically by giving a concrete example. (In addition to finiteness, Riguet [Rig51] adds a second condition that we do not understand.)

[17] Page 1 introduces set notation and properties of sets. It uses the symbol "$\Rightarrow$" —presumably meaning "only if"— but the symbol is also nowhere defined. The symbol "$\Leftrightarrow$" first appears on p.7 and continued equivalences first appear on p.8 in definition 2.1.3 (reflexive and irreflexive relations). No explanation is given of how a continued equivalence is to be read. (Boolean equality is associative and transitive. So a continued equivalence could be read associatively or conjunctionally.)

Winter is clearly aware that the claim is invalid in general because immediately afterwards [Win04, lemma 5] states that the claim is invalid for "dense" relations. (Winter formulates a property of " dense linear strict-orderings" that is essentially theorem 320.) Winter does not, however, give the most obvious example of a "dense" relation — the less-than relation on real numbers. Schmidt [Sch08] does observe that the less-than relation is "dense" but does not take the opportunity to correct the error in [SS93, Definition 4.4.11].

As previously stated, the notion of the diagonal of a relation is due to Riguet [Rig51]; Riguet called it the "différence". (See the discussion immediately following lemma 316.) The notion of a "polar" covering was also introduced by Riguet in [Rig51], albeit with a slightly stricter definition to fit the topic of his paper ("relations de Ferrers"): he requires the subset ordering on domains to be total ("linear" in the terminology used here).

Winter [Win04] does not give the diagonal function a name but denotes the "différence" of relation R by R^d (as do Khchérif, Gammoudi and Jaoua [KGJ00]); Winter cites [Rig51] but does not ascribe the concept to Riguet. Schmidt [Sch08] calls it the "fringe" of the relation; Schmidt [Sch08] does cite Winter [Win04] but does not cite Riguet [Rig51]. Berghammer and Winter [BW12, p.8] state that Riguet's notion of the "différence" of a relation was "introduced" by Winter [Win04] and Schmidt [Sch08]; like Schmidt [Sch08], Berghammer and Winter [BW12] do not cite Riguet [Rig51]. Although Winter [Win04] and Berghammer and Winter [BW12] define the "différence" using residuals, they frequently use Riguet's definition in terms of nested complements.

Theorem 317 introduces two constraints slightly weaker than those imposed by Schmidt and Ströhlein in their proposition 4.4.13(i); it is also stronger because it states an equality rather than an implication. Lemma 309, in combination with lemma 314 also yields a stronger theorem than their proposition 4.4.13(i). (No constraints are imposed on the parameters f and g.)

The primary novel contribution of this paper is the introduction and exploitation of the notion of the core of a relation. (See definition 191.) Section 9.1 has been included partly to make Hartmanis and Stearn's [HS66] pioneering contribution to information science better known. Their theory of "pair algebras" anticipates results in what has since become known as "concept analysis" [DP90], as discussed in section 10. Some of the properties of grips presented in section 10 may be novel but, as mentioned in the introduction to the section, we have not been able to determine whether or not this is the case. Much emphasis has been placed on illustrative examples which we hope will make the theory more accessible.

Finally, a few words on notation. The very rich algebraic properties of the converse of a relation mean that many notions and properties come in pairs, each element of the pair being the dual mirror-image of the other. For example, we have defined both the left domain and right domain of a relation; lemma 55 is an example of mirror-image properties of the relations. Some authors emphasise such mirroring by their choice of notation. Freyd and Ščedrov [Fv90], for example, denote the source and target of a relation R by $\Box R$ and $R\Box$, respectively.

A consequence of this is that it is possible to get away with defining just one of a pair of operators, leaving its mirror image to have an "obvious" definition in terms of relational converse. For example, in section 3.7 we gave only the definition of the "left" power transpose of a relation, leaving the definition of the "right" power transpose to the reader. Doing this systematically would mean introducing the notation $R{<}$ for the left domain of relation R and then using the notation $(R^{\cup}){<}$ to denote the right domain of R. Similarly, one might introduce just the left factor R/S and then write $(S^{\cup}/R^{\cup})^{\cup}$ for the right factor $R\backslash S$. This is, of course, very undesirable because then the associativity of the operators (the rule that $R\backslash(S/T)$ and $(R\backslash S)/T$ are equal, which we exploit by using the notation $R\backslash S/T$) becomes the very cumbersome

$$((S/T)^{\cup}/R^{\cup})^{\cup} \;=\; (S^{\cup}/R^{\cup})^{\cup}/T \;\; .$$

Even worse is when a symmetric notation is used for an operator that has both left and right variants — as is done by both Freyd and Ščedrov [Fv90] and Schmidt and Ströhlein [SS93, p.80] in the case of the so-called "symmetric division/quotient" of a relation. By writing $\frac{R}{S}$ (or $R{\div}S$), the reader may be misled into supposing that either the operator has no mirror image or that the mirror image is $\frac{S}{R}$ (or $S{\div}R$). The main drawback, however, is that the notation gives —literally and figuratively— a one-sided view of relation algebra that inhibits progress. The notion of the "core" of a relation introduced in section 7.3 is, so far as we know, novel; the insight leading to its introduction is the simple formula

$$R \;=\; R{\prec} \circ R \circ R{\succ}$$

combined with the well-known characterisation of a partial equivalence relation as $f^{\cup} \circ f$ for some functional relation f. It is, in our view, a striking example of the sort of insight that is obscured using Freyd and Ščedrov's or Schmidt and Ströhlein's notation.

Acknowledgement Many thanks to Jules Desharnais for helping to locate Riguet's publications and to José Nuno Oliveira for the proof of theorem 173.

I would also like to express heartfelt thanks to John Horton Conway who sadly died just as I was completing the first draft of this report. As a young PhD student in the early 1970s I chanced upon his little book on Regular Algebra and Finite Machines. It was like a breath of fresh air. I was particularly impressed by Chapter 6 on Factors and the Factor Matrix and its subsequent exploitation in Chapter 7 on biregulators (in particular the later entries in the table of biregulators). Here was an excellent example of the power of algebra! Conway's factor theory has influenced much of my work since then, as should be evident from this report.

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