The Thins Ordering on Relations

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Abstract

Earlier papers [BV22, BV23] introduced the notions of a core and an index of a relation (an index being a special case of a core). A limited form of the axiom of choice was postulated —specifically that all partial equivalence relations (pers) have an index— and the consequences of adding the axiom to axiom systems for point-free reasoning were explored. In this paper, we define a partial ordering on relations, which we call the thins ordering. We show that our axiom of choice is equivalent to the property that core relations are the minimal elements of the thins ordering. We also postulate a novel axiom that guarantees that, when thins is restricted to non-empty pers, equivalence relations are maximal. This and other properties of thins provide further evidence that our axiom of choice is a desirable means of strengthening point-free reasoning on relations.

Although our novel axiom is valid for concrete relations and is a sufficient condition for characterising maximality, we show that it is not a necessary condition in the abstract point-free algebra. This leaves open the problem of deriving a necessary and sufficient condition.

1 Introduction

Earlier papers [BV22, BV23] introduced the notions of a core and an index of a relation (an index being a special case of a core). In [BV23] the focus was on strengthening standard axiom systems for point-free reasoning. A limited form of the axiom of choice was postulated — specifically that all partial equivalence relations (pers) have an index— and the consequences of adding the axiom were explored. The working document [BV22] extends this work to practical applications of the notions.

In this paper, we define a partial ordering on relations, which we call the thins ordering. We begin by defining thins on partial equivalence relations (pers), and then extend the ordering to all relations. We show that our axiom of choice is equivalent to the property that the minimal elements of the thins relation on pers are precisely the indexes of pers. (See theorem 21 for a precise statement.) We then extend the thins ordering to all relations and we show that, assuming our axiom of choice, the minimal elements of the ordering are precisely the core relations. (See theorem 70.) We also show that, when the thins relation is restricted

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to pers, equivalence relations are maximal. This entails introducing a novel, aesthetically pleasing axiom that links the three constants of point-free relation algebra: the empty relation, the identity relation and the universal relation. The new axiom serves to distinguish the lattice and composition structures underlying point-free relation algebra from so-called Kleene algebra, the algebra of regular languages.

Because this paper is an extension of [BV23] we have omitted all introductory material. For ease of reference, we do repeat some key topics from [BV23]. In such cases, we omit proofs of lemmas and theorems. Hints in our calculations often refer to properties proved in earlier publications; in such cases we state the properties within square brackets. (See for example the proof of lemma 8 where the hint in the first step is

$$[P{\approx}\circ P = P = P\circ P{\approx}\ ] \quad .$$

The square brackets should be read as "everywhere". So the stated property is true for all instances of the dummy P, which ranges in this case over pers.) Nevertheless, [BV23] is recommended reading before embarking on the current paper.

Section 2 gives a brief summary of our axiom system. The novel contributions of the paper begin in section 3 with the definition of the thins relation. (At this stage, we don't call it an "ordering" because that property has yet to be established.) We also reproduce the definition of an index and the axiom of choice from [BV23].

Section 4 formulates a number of properties of thins. An important property (specifically, theorem 14) is that the indexes of a per Q are the pers P that thin Q and are coreflexive. This section also includes the proof that the thins relation is an ordering relation on pers.

Section 5 is about pers that are minimal with respect to the thins ordering. We show that coreflexive relations are minimal. The converse property —minimal implies coreflexive— is then shown to be equivalent to our axiom of choice. The conclusion of the section, theorem 21, is that the axiom of choice is equivalent to the conjunction of two properties: firstly, the minimal elements of the thins ordering on pers are precisely the coreflexive relations and, secondly, every per thins to a minimal element.

Section 6 is about maximality. A brief, informal summary of the main theorem (theorem 48) is that the equivalence relations are maximal with respect to the thins ordering on non-empty pers. The theorem we formulate is, in fact, more general than this since it applies to models of point-free relation algebra quite different from the standard set-theoretic binary relations.

In order to prove the theorem, we are obliged to introduce a novel axiom. Our axiom avoids explicit mention of complements (which we prefer to avoid whenever possible). However, in the context of a complete, universally distributive lattice, there is a link, which we formulate in section 7. Specifically, we show in section 7.2 that the novel axiom is valid if the lattice of coreflexives is complemented. This is a weaker requirement than that all complements exist but is valid for concrete relations. (Section 7.1 reviews elementary properties regarding the existence of complements in a distributive lattice.) The axiom is sufficient to characterise the maximality property but, as shown in section 7.3, it is not necessary in the abstract algebra.

Section 8 is where we extend the thins ordering to arbitrary relations (and not just pers). We prove that, assuming our axiom of choice, a relation S is minimal with respect to the thins ordering on arbitrary relations iff S is a core relation.

2 Point-Free Relation Algebras

In this section, we define a *point-free relation algebra*. Such an algebra has three components with interfaces between them: a (typed) monoid structure, a lattice structure, and a converse structure.

Underpinning any relation algebra is a very simple type structure. We assume the existence of a non-empty set $\mathcal{T}$ of so-called *basic types*. A *relation type* is an ordered pair of basic types. We write $A{\sim}B$ for the ordered pair of basic types A and B. We often omit "relation" and refer to $A{\sim}B$ as a "type". The carrier set of a point-free relation algebra is typed in the sense that each element X of the carrier set has a type $A{\sim}B$ for some basic types A and B. A relation of type $A{\sim}A$, for some A, is said to be *homogeneous*. If $\mathcal{T}$ has exactly one element we say that the algebra is *untyped*.

The monoid structure is defined as follows. For each triple of basic types A, B and C, and each element X of type $A{\sim}B$ and each element Y of type $B{\sim}C$, there is an element $X{\circ}Y$ of type $A{\sim}C$. Also for each basic type A there is an element $\mathbb{I}$ of type $A{\sim}A$. The element $X{\circ}Y$ is called the *composition* of X and Y, and $\mathbb{I}$ is called the *identity* of A. Composition is required to be associative, and identities are required to be the units of composition. The composition $X{\circ}Y$ is only defined when X and Y have appropriate types. (Such a typed monoid structure is commonly called a "category".)

In principle, the type of the identities should be made explicit in the notation we use: for example, by writing $\mathbb{I}_A$ for the identity of type $A{\sim}A$. It is convenient for us not to do so, leaving the type information to be deduced from the context. This is also the case for other operators and constants that we introduce below.

For each type $A{\sim}B$, we assume the existence of a complete, universally distibutive lattice (partially) ordered by $\subseteq$. The binary supremum and infimum operators of the lattice are denoted by $\cup$ and $\cap$, respectively. The least and greatest elements of the lattice are denoted by $\perp\!\!\!\perp$ and $\top\!\!\!\top$, respectively. The elements of the carrier set of the lattice are sometimes referred to as relations of type $A{\sim}B$.

("Complete" means that all suprema and infima exist. "Universally distributive" means that for each element X of the carrier set the function $X\cup$ distributes over all infima and the function $X\cap$ distributes over all suprema. Universally distributive is weaker than "completely distributive", which is tantamount to the (standard) axiom of choice.)

The interface between the monoid structure and the lattice structure is the requirement that composition is universally distributive over supremum. Equivalently (given the completeness of the lattice structure), we demand the existence of the two factor operators defined by, for all X, Y and Z of appropriate types,

$$(Y \subseteq X\backslash Z) \;=\; (X{\circ}Y \subseteq Z) \;=\; (X \subseteq Z/Y) \;\;.$$

(Note that this means that $\perp\!\!\!\perp$ is the zero of composition.) The combination of the monoid structure, the lattice structure and the interface between them is called a *regular algebra*.

The converse structure is very simple: for each element X of type $A{\sim}B$ there is an element $X^{\cup}$ of type $B{\sim}A$.

The interface between the lattice structure and the converse structure is the Galois connection: for all X and Y of appropriate types,

$$X^{\cup} \subseteq Y \;\equiv\; X \subseteq Y^{\cup} \;\; .$$

The interface between the monoid structure and the converse structure is: for each identity $\mathbb{I}$,

$$\mathbb{I}^{\cup} = \mathbb{I}$$

and, for all X and Y of appropriate types,

$$(X{\circ}Y)^{\cup} \;=\; Y^{\cup}{\circ}X^{\cup} \;\; .$$

Finally, the *modularity law* acts as an interface between all three components: for all X, Y and Z of appropriate types,

$$X{\circ}Y \cap Z \;\subseteq\; X{\circ}(Y \cap X^{\cup}{\circ}Z) \;\; .$$

In fact, we do not use the modular rule anywhere explicitly in this paper. We do, however, make extensive use of the properties of the domain operators first mentioned in section 3, with which we assume familiarity. The properties of the domain operators rely heavily on the modularity law.

The axioms of point-free relation algebra do not completely characterise all the properties of binary relations and, therefore, admit other models (for example geometric models: see [Fv90, 2.158] and [Voe99, section 3.5]). We use the term *concrete relation* below to refer to binary relations as they are normally understood. That is, a *concrete relation of type* $A{\sim}B$ is an element of the powerset $2^{A\times B}$. (The types A and B do not need to be finite.)

A point-free relation algebra is said to be *unary* if it satisfies the *cone rule*: for all X,

$$\mathbb{T}{\circ}X{\circ}\mathbb{T} = \mathbb{T} \;\;\equiv\;\; X \neq \bot\!\!\!\bot \;\; .$$

(The three occurrences of $\mathbb{T}$ may have different types. The rightmost occurrence is assumed to have the same type as X; the other two are assumed to be homogeneous relations of the appropriate types. The terminology reflects the fact that the cartesian product of two relation algebras is non-unary. See [Voe99, section 3.4.3].)

In [BV22, BV23], much of the focus was on introducing axioms that facilitate pointwise reasoning. To this end, the cone rule was used extensively. In contrast, in this paper the goal is not to facilitate pointwise reasoning but, instead, to strengthen point-free reasoning. So here the cone rule is deemed to be invalid. See the introductory remarks in section 6.

It is, of course, always useful to include examples. The simplest examples —detailed below— are often very illuminating. We return to these in later sections.

Example 1 The simplest examples of point-free relation algebras are all untyped. The simplest of all has just one element: all of the constants $\bot\!\!\!\bot$, $\mathbb{I}$ and $\mathbb{T}$ are defined to be equal. The second simplest has two elements $\bot\!\!\!\bot$ and $\mathbb{T}$; $\mathbb{I}$ is defined to be equal to $\mathbb{T}$ (and different from $\bot\!\!\!\bot$). The third simplest has three elements: the constants $\bot\!\!\!\bot$, $\mathbb{I}$ and $\mathbb{T}$, which are defined to be distinct. (In all three cases, the definitions of the ordering relation, composition and converse can be deduced from the axioms.)

A four-element algebra is obtained by adding a new element $\neg\mathbb{I}$ to the three-element algebra and defining the composition $\neg\mathbb{I}\circ\neg\mathbb{I}$ to be $\mathbb{I}$ and the converse $(\neg\mathbb{I})^{\cup}$ to be $\neg\mathbb{I}$. As suggested by the notation, $\neg\mathbb{I}$ is the complement of $\mathbb{I}$. That is, the lattice structure is as shown in the diagram below.

$$\begin{tikzcd}[column sep=small] & \mathbb{T} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbb{I} \arrow[dr, dash] & & \neg\mathbb{I} \arrow[dl, dash] \\ & \perp\!\!\!\perp & \end{tikzcd}$$

The simplest example is not unary, the other examples are unary. A model of the two-element algebra is formed by the (homogeneous) concrete relations on a set with exactly one element. The other examples do not have such a model since the concrete relations on a set of size n form a power set of size $2^{n\times n}$.
□

3 Basic Definitions

We begin by restricting our study to partial equivalence relations (pers[1]). In this section we recall the definition of an index of a per and our axiom of choice. New is definition 5.

Throughout the paper, P and Q denote pers. For pers, the left and right domains coincide. (I.e. for all pers P, $P{<} = P{>}$.) For this reason, $P{\asymp}$ is used to denote the left/right domain of P. That is, $P{<} = P{\asymp} = P{>}$.

Definition 2 (Index of a Per) Suppose P is a per. Then an *index* of P is a relation J such that

(a) $J \subseteq P{\asymp}$,

(b) $J\circ P\circ J = J$,

(c) $P\circ J\circ P = P$.
□

Axiom 3 (Axiom of Choice) Every per has an index.
□

Example 4 The three- and four-element algebras detailed in example 1 do not satisfy the axiom of choice since, in both cases, $\mathbb{T}$ does not have an index. The two simplest examples do satisfy the axiom of choice because each element is an index of itself.
□

Definition 5 (Thins) The thins relation on pers is defined by, for all pers P and Q,

$$P \preceq Q \quad\equiv\quad P = P{\asymp}\circ Q\circ P{\asymp} \ \wedge\ Q = Q\circ P{\asymp}\circ Q \ .$$
□

[1]Relation P is a per iff it is reflexive (i.e. $\mathbb{I}\subseteq P$) and transitive (i.e. $P\circ P\subseteq P$).

This paper is about the properties of the $\preceq$ relation. We call it the thins relation. (So $P \preceq Q$ is pronounced P thins Q.) Much of the paper is about the thins relation on pers but we extend it to all relations in section 8.

Informally, the first conjunct in the definition of $P \preceq Q$ states that the equivalence classes of P are subsets of the equivalence classes of Q, and the second conjunct states that, for each equivalence class of P, there is a corresponding equivalence class of Q. Some of the properties stated below are intended to confirm this informal interpretation of the definition.

Example 6 In all four example algebras detailed in example 1, the pers are $\mathbb{⊥}$, $\mathbb{I}$ and $\mathbb{T}$ and the thins relation is discrete. (That is, the thins relation is the equality relation on the pers.)

□

4 Basic Properties

As the title suggests, this section is about basic properties of the thins relation. Theorem 9 establishes that it is a partial ordering on pers. Theorem 14 formulates an alternative definition of an index of a per in terms of thins. Subsequent lemmas anticipate properties needed in later sections.

Obvious from the property that $P{\lessgtr} \subseteq \mathbb{I}$ (for arbitrary P) and monotonicity of composition, applied to the property $P = P{\lessgtr} \circ Q \circ P{\lessgtr}$ is that, for all pers P and Q,

(7) $\quad P \preceq Q \;\Rightarrow\; P \subseteq Q \quad .$

We use this frequently below.

The following lemma is also used on several occasions. Compare the lemma with properties 2(b) and 2(c) of an index.

Lemma 8 For all pers P and Q,

$$P \preceq Q \quad\Rightarrow\quad P = P{\circ}Q{\circ}P \;\wedge\; Q = Q{\circ}P{\circ}Q \quad .$$

Proof Assume $P = P{\lessgtr} \circ Q \circ P{\lessgtr}$. Then

$$
\begin{array}{ll}
 & P{\circ}Q{\circ}P \\
= & \qquad \{ \qquad \text{domains (specifically, } [\, P{\lessgtr} \circ P = P = P \circ P{\lessgtr} \,]) \quad \} \\
 & P \circ P{\lessgtr} \circ Q \circ P{\lessgtr} \circ P \\
= & \qquad \{ \qquad \text{assumption: } P = P{\lessgtr} \circ Q \circ P{\lessgtr} \quad \} \\
 & P{\circ}P{\circ}P \\
= & \qquad \{ \qquad P \text{ is a per, } [\, P = P{\circ}P \,] \quad \} \\
 & P \quad .
\end{array}
$$

Since $P \preceq Q \;\Rightarrow\; P = P{\lessgtr} \circ Q \circ P{\lessgtr}$, it follows that $P \preceq Q \;\Rightarrow\; P = P{\circ}Q{\circ}P$. Also,

$$
\begin{array}{ll}
 & Q{\circ}P{\circ}Q \\
\supseteq & \qquad \{ \qquad [\, P \supseteq P{\lessgtr} \,] \quad \} \\
 & Q \circ P{\lessgtr} \circ Q
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{assumption: } P \preceq Q \text{, so } Q = Q \circ P_{\asymp} \circ Q \quad \} \\
& Q \\
= & \{ \quad Q \text{ is a per, } [\, P = P \circ P \,] \text{ with } P := Q \quad \} \\
& Q \circ Q \circ Q \\
\supseteq & \{ \quad \text{domains (specifically } [\, \mathbb{I} \supseteq P_{\asymp} \,]) \text{ and monotonicity} \quad \} \\
& Q \circ P_{\asymp} \circ Q \circ P_{\asymp} \circ Q \\
= & \{ \quad \text{assumption: } P \preceq Q \text{, so } P = P_{\asymp} \circ Q \circ P_{\asymp} \quad \} \\
& Q \circ P \circ Q \quad .
\end{array}
$$

That is, by anti-symmetry, $P \preceq Q \;\Rightarrow\; Q = Q \circ P \circ Q$.

□

The notation we have chosen suggests that thins is a partial ordering. This is indeed the case:

Theorem 9 The thins relation is a partial ordering on pers.

Proof We must prove that the thins relation is reflexive, transitive and anti-symmetric. Reflexivity is straightforward:

$$
\begin{array}{ll}
& P \preceq P \\
= & \{ \quad \text{definition 5} \quad \} \\
& P = P_{\asymp} \circ P \circ P_{\asymp} \;\wedge\; P = P \circ P_{\asymp} \circ P \\
= & \{ \quad \text{domains (specifically, } [\, P_{\asymp} \circ P = P = P \circ P_{\asymp} \,]) \quad \} \\
& P = P \;\wedge\; P = P \circ P \\
= & \{ \quad \text{reflexivitiy of equality; } P \text{ is a per and } [\, P = P \circ P \,] \quad \} \\
& \text{true} \quad .
\end{array}
$$

Now, suppose P, Q and R are pers, and $P \preceq Q$ and $Q \preceq R$. Applying (7), we have:

(10) $\quad P \subseteq Q \subseteq R$.

To prove transitivity, we must prove that $P \preceq R$. Applying definition 5, we must prove that

$$
P \;=\; P_{\asymp} \circ R \circ P_{\asymp} \quad \wedge \quad R \;=\; R \circ P_{\asymp} \circ R \quad .
$$

We prove that $P = P_{\asymp} \circ R \circ P_{\asymp}$ by anti-symmetry. One inclusion is easy:

$$
\begin{array}{ll}
& P \\
= & \{ \quad \text{domains (specifically, } [\, P_{\asymp} \circ P = P = P \circ P_{\asymp} \,]) \quad \} \\
& P_{\asymp} \circ P \circ P_{\asymp} \\
\subseteq & \{ \quad \text{by (10), } P \subseteq R \text{ ; monotonicity} \quad \} \\
& P_{\asymp} \circ R \circ P_{\asymp} \quad .
\end{array}
$$

For the opposite inclusion, we have:

$$
\begin{array}{ll}
& P_{\asymp} \circ R \circ P_{\asymp} \\
\subseteq & \{ \quad P \text{ is a per and } [\, P_{\asymp} \subseteq P \,] \text{ ; monotonicity} \quad \} \\
& P \circ R \circ P
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{assumption: } P = P_{\asymp} \circ Q \circ P_{\asymp} \quad \} \\
& P_{\asymp} \circ Q \circ P_{\asymp} \circ R \circ P_{\asymp} \circ Q \circ P_{\asymp} \\
\subseteq & \{ \quad [\, P_{\asymp} \subseteq \mathbb{I} \,]\,;\ \text{monotonicity} \quad \} \\
& P_{\asymp} \circ Q \circ R \circ Q \circ P_{\asymp} \\
= & \{ \quad \text{assumption: } Q \preceq R\,,\ \text{lemma 8 with } P{,}Q := Q{,}R \quad \} \\
& P_{\asymp} \circ Q \circ P_{\asymp} \\
= & \{ \quad \text{assumption: } P = P_{\asymp} \circ Q \circ P_{\asymp} \quad \} \\
& P \ .
\end{array}
$$

Combining the two calculations, we have proved that

(11) $\quad P \;=\; P_{\asymp} \circ R \circ P_{\asymp} \ .$

Now we prove that $R \;=\; R \circ P_{\asymp} \circ R$. Again, the proof is by anti-symmetry with one inclusion being straightforward:

$$
\begin{array}{ll}
& R \circ P_{\asymp} \circ R \\
\subseteq & \{ \quad [\, P_{\asymp} \subseteq \mathbb{I} \,] \text{ and monotonicity} \quad \} \\
& R \circ R \\
\subseteq & \{ \quad R \text{ is a per (and hence transitive)} \quad \} \\
& R \ .
\end{array}
$$

For the opposite inclusion, we have:

$$
\begin{array}{ll}
& R \\
= & \{ \quad \text{assumption: } Q \preceq R\,,\ \text{lemma 8 with } P{,}Q := Q{,}R \quad \} \\
& R \circ Q \circ R \\
= & \{ \quad \text{assumption: } P \preceq Q \quad \} \\
& R \circ Q \circ P_{\asymp} \circ Q \circ R \\
\subseteq & \{ \quad \text{assumption: } Q \preceq R\,,\ (7) \text{ and monotonicity} \quad \} \\
& R \circ R \circ P_{\asymp} \circ R \circ R \\
\subseteq & \{ \quad R \text{ is a per (and hence transitive)} \quad \} \\
& R \circ P_{\asymp} \circ R \ .
\end{array}
$$

Combining the two calculations, we have proved that

(12) $\quad R \;=\; R \circ P_{\asymp} \circ R \ .$

Finally, combining (11) and (12) and applying definition 5 (with $P{,}Q := P{,}R$) we have shown that $P \preceq R$. This concludes the proof that the thins relation is transitive.

Finally, we prove that the thins relation is anti-symmetric. Suppose $P \preceq Q \preceq P$. Then, by (7), $P \subseteq Q \subseteq P$. Thus $P = Q$ by the anti-symmetry of the $\subseteq$ relation.
□

We now consider the properties of indexes with respect to the thins ordering. Our first goal is to show that the definition of an index J of a per P can be split into two conjuncts, namely $J \subseteq \mathbb{I}$ and $J \preceq P$.

Lemma 13 If J is an index of P then $J \preceq P$.

Proof

$$
\begin{array}{cl}
 & \mathrm{J} \text{ is an index of } \mathrm{P} \\
= & \{ \quad \text{definition 2} \quad \} \\
 & \mathrm{J} \subseteq \mathrm{P}{\prec} \ \wedge\ \mathrm{J} \circ \mathrm{P} \circ \mathrm{J} = \mathrm{J} \ \wedge\ \mathrm{P} \circ \mathrm{J} \circ \mathrm{P} = \mathrm{P} \\
\Rightarrow & \{ \quad \text{domains (specifically, } [\, \mathrm{P}{\prec} \subseteq \mathbb{I} \,] \text{ and } [\, \mathrm{Q} \subseteq \mathbb{I} \equiv \mathrm{Q} = \mathrm{Q}{\prec} \,] \text{ with } \mathrm{Q} := \mathrm{J}\,) \quad \} \\
 & \mathrm{J} = \mathrm{J}{\prec} \ \wedge\ \mathrm{J} \circ \mathrm{P} \circ \mathrm{J} = \mathrm{J} \ \wedge\ \mathrm{P} \circ \mathrm{J} \circ \mathrm{P} = \mathrm{P} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & \mathrm{J}{\prec} \circ \mathrm{P} \circ \mathrm{J}{\prec} = \mathrm{J} \ \wedge\ \mathrm{P} \circ \mathrm{J}{\prec} \circ \mathrm{P} = \mathrm{P} \\
= & \{ \quad \text{definition 5} \quad \} \\
 & \mathrm{J} \preceq \mathrm{P} \quad .
\end{array}
$$

□

Theorem 14 For all pers P and Q ,

$$\mathrm{P} \subseteq \mathbb{I} \ \wedge\ \mathrm{P} \preceq \mathrm{Q} \quad \equiv \quad \mathrm{P} \text{ is an index of } \mathrm{Q} \quad .$$

Proof The proof is by mutual implication. For ease of reference, we instantiate definition 2 with $\mathrm{J,P} := \mathrm{P,Q}$:

$$(15) \qquad \mathrm{P} \subseteq \mathrm{Q}{\prec} \ \wedge\ \mathrm{P} \circ \mathrm{Q} \circ \mathrm{P} = \mathrm{P} \ \wedge\ \mathrm{Q} \circ \mathrm{P} \circ \mathrm{Q} = \mathrm{Q} \quad .$$

Suppose $\mathrm{P} \subseteq \mathbb{I} \wedge \mathrm{P} \preceq \mathrm{Q}$. We must verify (15). The first conjunct is verified as follows:

$$
\begin{array}{cl}
 & \mathrm{P} \subseteq \mathrm{Q}{\prec} \\
= & \{ \quad \text{assumption: } \mathrm{P} \subseteq \mathbb{I} \text{ ; so } \mathrm{P} = \mathrm{P}{\prec} \quad \} \\
 & \mathrm{P}{\prec} \subseteq \mathrm{Q}{\prec} \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & \mathrm{P} \subseteq \mathrm{Q} \\
\Leftarrow & \{ \quad (7) \quad \} \\
 & \mathrm{P} \preceq \mathrm{Q} \quad .
\end{array}
$$

The second and third conjuncts both follow directly from the assumption $\mathrm{P} \preceq \mathrm{Q}$ by lemma 8.

For the converse implication, we have:

$$
\begin{array}{cl}
 & \mathrm{P} \text{ is an index of } \mathrm{Q} \\
\Rightarrow & \{ \quad \text{definition 2(a) and lemma 13 (both with } \mathrm{J,P} := \mathrm{P,Q}\,) \quad \} \\
 & \mathrm{P} \subseteq \mathrm{Q}{\prec} \ \wedge\ \mathrm{P} \preceq \mathrm{Q} \\
\Rightarrow & \{ \quad \text{domains (specifically } [\, \mathrm{P}{\prec} \subseteq \mathbb{I} \,] \text{ with } \mathrm{P} := \mathrm{Q}\,) \text{ and transitivity} \quad \} \\
 & \mathrm{P} \subseteq \mathbb{I} \ \wedge\ \mathrm{P} \preceq \mathrm{Q} \quad .
\end{array}
$$

□

The remaining lemmas in this section are not needed elsewhere; they are included in order to give further insight into the nature of the thins ordering on pers.

Central to the notion of the thins relation is that an index of a per is found by successively "thinning" the relation. More precisely, an index of a per is a "thinning" of the per and being an index of a per is invariant under the process of "thinning" the relation. The first of these two properties is lemma 13; the second is formulated in lemma 16.

Lemma 16 For all pers P and Q, and coreflexive J,

$$J \text{ is an index of } Q \;\Leftarrow\; J \text{ is an index of } P \wedge P \preceq Q \;\; .$$

Proof We apply theorem 14:

$$
\begin{array}{ll}
 & J \text{ is an index of } Q \\
= & \quad \{ \quad \text{theorem 14 with } P,Q := J,Q \quad \} \\
 & J \subseteq \mathbb{I} \;\wedge\; J \preceq Q \\
\Leftarrow & \quad \{ \quad \text{theorem 9 (in particular } \preceq \text{ is transitive)} \quad \} \\
 & J \subseteq \mathbb{I} \;\wedge\; J \preceq P \;\wedge\; P \preceq Q \\
= & \quad \{ \quad \text{theorem 14 with } P,Q := J,P \quad \} \\
 & J \text{ is an index of } P \;\wedge\; P \preceq Q \;\; .
\end{array}
$$

□

Our earlier informal interpretation of the first conjunct in the definition of thins is reinforced by the following simple lemma. Specifically, if J is an index of P, $J{\circ}P$ is the functional that maps a point a of P to the point in J that represents the equivalence class containing a. If $P \preceq Q$ then, by lemma 16, J is an index of Q. So, $J{\circ}Q$ is the functional that maps a point b of Q to the point in J that represents the equivalence class containing b. The lemma states that the two functionals agree on points common to both P and Q.

Lemma 17 Suppose J is an index of per P. Then, for all pers Q,

$$J{\circ}P \;=\; J \circ Q \circ P_{\asymp} \quad \Leftarrow \quad P \preceq Q \;\; .$$

Proof

$$
\begin{array}{ll}
 & J{\circ}P \\
= & \quad \{ \quad \text{assumption: } P \preceq Q \text{, definition 5} \quad \} \\
 & J \circ P_{\asymp} \circ Q \circ P_{\asymp} \\
= & \quad \{ \quad J \text{ is an index of } P \text{, so, by definition 2(a), } J \circ P_{\asymp} = J \quad \} \\
 & J \circ Q \circ P_{\asymp} \;\; .
\end{array}
$$

□

The final lemma in this section gives further insight into the relation between the thins relation and indexes.

Lemma 18 Suppose P and Q have a common index and $P \subseteq Q$. Then $P \preceq Q$.

Proof Suppose J is an index of both P and Q, and $P \subseteq Q$. The definition of the thins relation demands that we prove two properties. First,

$$
\begin{array}{ll}
 & Q \\
= & \quad \{ \quad Q \text{ is a per, } [\, P = P{\circ}P \,] \text{ with } P := Q \quad \} \\
 & Q{\circ}Q{\circ}Q \\
\supseteq & \quad \{ \quad \text{assumption: } P \subseteq Q \text{, monotonicity} \quad \} \\
 & Q{\circ}P{\circ}Q \\
\supseteq & \quad \{ \quad P \text{ is a per, so } P_{\asymp} \subseteq P \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & Q \circ P{\asymp} \circ Q \\
\supseteq & \{ \quad \text{assumption: } J \text{ is an index of } P \text{, definition 2(a)} \quad \} \\
 & Q{\circ}J{\circ}Q \\
= & \{ \quad \text{assumption: } J \text{ is an index of } Q \text{, definition 2(c) (with } P{:=}Q \text{)} \quad \} \\
 & Q \quad .
\end{array}
$$

We conclude, by anti-symmetry, that $Q = Q \circ P{\asymp} \circ Q$. Now for the second property,

$$
\begin{array}{ll}
 & P{\asymp} \circ Q \circ P{\asymp} \\
\supseteq & \{ \quad \text{assumption: } P \subseteq Q \text{, monotonicity} \quad \} \\
 & P{\asymp} \circ P \circ P{\asymp} \\
= & \{ \quad \text{domains} \quad \} \\
 & P \\
= & \{ \quad \text{assumption: } J \text{ is an index of } P \text{, definition 2(c)} \quad \} \\
 & P{\circ}J{\circ}P \\
= & \{ \quad \text{assumption: } J \text{ is an index of } Q \text{, definition 2(b) (with } P{:=}Q \text{)} \quad \} \\
 & P{\circ}J{\circ}Q{\circ}J{\circ}P \\
= & \{ \quad Q \text{ is a per, } [\, P{=}P{\circ}P \,] \text{ with } P{:=}Q \quad \} \\
 & P{\circ}J{\circ}Q{\circ}Q{\circ}Q{\circ}J{\circ}P \\
\supseteq & \{ \quad \text{assumption: } P \subseteq Q \text{, monotonicity} \quad \} \\
 & P{\circ}J{\circ}P{\circ}Q{\circ}P{\circ}J{\circ}P \\
= & \{ \quad \text{assumption: } J \text{ is an index of } P \text{, definition 2(c)} \quad \} \\
 & P{\circ}Q{\circ}P \\
\supseteq & \{ \quad P \text{ is a per, } [\, P{\asymp} \subseteq P \,] \quad \} \\
 & P{\asymp} \circ Q \circ P{\asymp} \quad .
\end{array}
$$

We conclude by anti-symmetry that $P{\asymp} \circ Q \circ P{\asymp} = P$. Combining the two calculations, we have shown that $P \preceq Q$. (See definition 5.)

□

5 Minimal Pers

Our goal in this section is to characterise the pers that are minimal with respect to the thins ordering on pers. Section 6 is about characterising the pers that are maximal. The notions of minimality and maximality with respect to an ordering relation are well known. For completeness the definition is given below.

Definition 19 Suppose $\sqsubseteq$ is a partial ordering on some set X. With x and y ranging over elements of X, we say that y is *minimal* with respect to the ordering iff

$$\langle \forall x \,:\, x \sqsubseteq y \,:\, x = y \rangle$$

and we say that x is *maximal* with respect to the ordering iff

$$\langle \forall y \,:\, x \sqsubseteq y \,:\, x = y \rangle \quad .$$

□

We apply definition 19 in this section and in section 6 to the thins ordering on pers; in section 8 we apply the definition to the (yet-to-be-introduced) thins ordering on arbitrary relations.

A straightforward observation is that coreflexives are minimal:

Lemma 20

$$\langle \forall Q \ :: \ \mathsf{minimal}.Q \Leftarrow Q \subseteq \mathbb{I} \rangle \ \ .$$

Proof Suppose $Q \subseteq \mathbb{I}$ and $P \preceq Q$. We prove that $P = Q$.

$$\begin{array}{ll}
 & P = Q \\
= & \quad \{ \quad \text{anti-symmetry} \quad \} \\
 & P \subseteq Q \ \wedge \ Q \subseteq P \\
= & \quad \{ \quad \text{assumption: } P \preceq Q \text{ ; hence, by (7), } P \subseteq Q \quad \} \\
 & Q \subseteq P \\
= & \quad \{ \quad \text{assumption: } P \preceq Q \text{ ; hence by definition 5, } Q = Q \circ P^{\asymp} \circ Q \quad \} \\
 & Q \circ P^{\asymp} \circ Q \subseteq P \\
\Leftarrow & \quad \{ \quad \text{assumption: } Q \subseteq \mathbb{I} \text{ ; monotonicity} \quad \} \\
 & P^{\asymp} \subseteq P \\
= & \quad \{ \quad P \text{ is a per} \quad \} \\
 & \mathsf{true} \ \ .
\end{array}$$

□

Lemma 20 suggests that we explore the circumstances in which all minimal elements are coreflexives. We show that this is equivalent to the axiom of choice introduced in [BV22, BV23].

Theorem 21 The axiom of choice is equivalent to

(22) $\langle \forall P :: \mathsf{minimal}.P \ \equiv \ P \subseteq \mathbb{I} \rangle \ \wedge \ \langle \forall Q \ :: \ \langle \exists P : \mathsf{minimal}.P : P \preceq Q \rangle \rangle$

Proof The proof is by mutual implication. First, assume (22). Suppose Q is an arbitrary per. We prove that Q has an index.

By assumption, there exists a per P such that $P \subseteq \mathbb{I}$ and $P \preceq Q$. Theorem 14 proves that P is an index of Q.

Now assume the axiom of choice. We must prove (22). We begin with the property

(23) $\langle \forall P :: \mathsf{minimal}.P \ \equiv \ P \subseteq \mathbb{I} \rangle$.

By lemma 20, it suffices to prove the implication. Suppose P is minimal. By the axiom of choice, P has an index, J say. Then

$$\begin{array}{ll}
 & \mathsf{true} \\
= & \quad \{ \quad J \text{ is an index of } P\text{; lemma 13} \quad \} \\
 & J \preceq P \\
\Rightarrow & \quad \{ \quad P \text{ is minimal; definition 19} \quad \} \\
 & P = J \\
\Rightarrow & \quad \{ \quad [\, P^{\asymp} \subseteq \mathbb{I} \,]\text{; so, by definition 2(a), } J \subseteq \mathbb{I} \quad \} \\
 & P \subseteq \mathbb{I} \ \ .
\end{array}$$

We have thus proved (23). Now we consider the property

(24) $\langle\forall Q :: \langle\exists P : \mathsf{minimal}.P : P \preceq Q\rangle\rangle$.

This is established by choosing, for given Q, an index of Q. Indeed, if P is an index of Q, then, by lemma 13, $P \preceq Q$ and, by (23), it is minimal.
□

Example 25 As observed in example 4, the algebras of example 1 with at least three elements do not satisfy our axiom of choice. Consequently, they do not satisfy the minimality property of theorem 21: in both cases, $\mathbb{T}$ is minimal but is not coreflexive. (See example 6.)

□

6 Maximal Pers

We now consider properties that guarantee maximality.

It is relatively straightforward to show that $\perp\!\!\!\perp$ is maximal and all equivalence relations are maximal. This suggests the conjecture that these are the only maximal elements. Rather than formulate a proof that involves a case analysis, we prove a more general property. Specifically, we prove that a per P is maximal iff $\mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T} \subseteq P$. If the cone rule holds, $\mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T} \subseteq P$ is equivalent to $\mathbb{I} \subseteq P \vee P = \perp\!\!\!\perp$. The additional generality comes from instances of relation algebra where the cone rule does not hold.

First, we prove the "if" statement.

Lemma 26 Per P is maximal if $\mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T} \subseteq P$.

Proof Suppose $\mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T} \subseteq P$ and $P \preceq Q$. To show that P is maximal we must show that $P = Q$. We first show that $P_{\asymp} = Q_{\asymp}$.

$$
\begin{array}{ll}
 & Q_{\asymp} \\
= & \{ \quad Q \text{ is a per} \quad \} \\
 & \mathbb{I} \cap Q \\
= & \{ \quad P \preceq Q \text{, lemma 8} \quad \} \\
 & \mathbb{I} \cap Q \circ P \circ Q \\
\subseteq & \{ \quad Q \subseteq \mathbb{T} \text{ and monotonicity} \quad \} \\
 & \mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T} \\
= & \{ \quad \text{domains} \quad \} \\
 & (\mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T})_{\asymp} \\
\subseteq & \{ \quad \text{assumption: } \mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T} \subseteq P \text{ and monotonicity} \quad \} \\
 & P_{\asymp} \\
\subseteq & \{ \quad \text{assumption: } P \preceq Q \text{, (7) and monotonicity} \quad \} \\
 & Q_{\asymp}
\end{array}
$$

Thus $P_{\asymp} = Q_{\asymp}$ follows by anti-symmetry. Now we show that $P = Q$:

$$\begin{array}{ll} & P \\ = & \{ \quad \text{assumption: } P \preceq Q \quad \} \\ & P{\asymp} \circ Q \circ P{\asymp} \\ = & \{ \quad P{\asymp} = Q{\asymp} \text{ (just proved)} \quad \} \\ & Q{\asymp} \circ Q \circ Q{\asymp} \\ = & \{ \quad \text{domains} \quad \} \\ & Q \ . \end{array}$$

□

We now turn to the converse of lemma 26. The key fact is lemma 47 in which we show that, for all pers P, there is a relation Q such that $P \preceq Q$ and $\mathbb{I} \cap \mathbb{T}{\circ}Q{\circ}\mathbb{T} \subseteq Q$. The construction involves exploiting properties of relations of the form $P{\circ}\mathbb{T}{\circ}P$, where P is a per. Anticipating their use in the proof of lemma 47, we begin by recording some elementary properties. Before doing so, note that a relation of the form $P{\circ}\mathbb{T}{\circ}P$, where P is a per (or, in particular, a coreflexive), is a per.

Lemma 27 Suppose P is a per and J is an index of P. Then

$$P{\asymp} \;=\; (P{\circ}\mathbb{T}{\circ}P){\asymp} \quad \wedge \quad (J{\circ}\mathbb{T}{\circ}J){\asymp} \;=\; J \ .$$

Proof

$$\begin{array}{ll} & P{\asymp} \\ = & \{ \quad P \text{ is a per} \quad \} \\ & (P{\circ}P{\circ}P){\asymp} \\ \subseteq & \{ \quad P \subseteq \mathbb{T} \text{ ; monotonicity} \quad \} \\ & (P{\circ}\mathbb{T}{\circ}P){\asymp} \\ \subseteq & \{ \quad \text{domains (specifically } [\,(R{\circ}S){<} \subseteq R{<}\,] \text{ with } R,S := P\,,\,\mathbb{T}{\circ}P\,) \quad \} \\ & P{\asymp} \ . \end{array}$$

That is, by anti-symmetry, $P{\asymp} = (P{\circ}\mathbb{T}{\circ}P){\asymp}$. Hence,

$$\begin{array}{ll} & (J{\circ}\mathbb{T}{\circ}J){\asymp} \\ = & \{ \quad \text{above with } P{:=}J \text{ (noting that } J \text{ is coreflexive and hence a per} \quad \} \\ & J{\asymp} \\ = & \{ \quad J \text{ is an index of a per and, hence, } J \text{ is coreflexive} \quad \} \\ & J \ . \end{array}$$

□

The following three lemmas are used in the proof of lemma 47.

Lemma 28 Suppose P is a per and J is an index of P. Then

$$\mathbb{T}{\circ}J{\circ}\mathbb{T} \;=\; \mathbb{T}{\circ}P{\circ}\mathbb{T} \ .$$

Proof

$\quad\mathbb{T}{\circ}P{\circ}\mathbb{T}$

$=\qquad\{\quad J$ is an index of P, definition 2(c) $\quad\}$

$\quad\mathbb{T}{\circ}P{\circ}J{\circ}P{\circ}\mathbb{T}$

$=\qquad\{\quad$ domains (specifically $[\;\mathbb{T}{\circ}R = \mathbb{T}\circ R{>} \;\wedge\; R{\circ}\mathbb{T} = R{<}\circ\mathbb{T}\;]$
with $R := P\,;\; P{>} = P{\asymp} = P{<}$) $\quad\}$

$\quad\mathbb{T}\circ P{\asymp}\circ J\circ P{\asymp}\circ\mathbb{T}$

$=\qquad\{\quad J$ is an index of P, definition 2(a); composition of coreflexives $\quad\}$

$\quad\mathbb{T}{\circ}J{\circ}\mathbb{T}\quad .$

□

Lemma 29 Suppose P is a per and J is an index of the per $P{\circ}\mathbb{T}{\circ}P$. Then

$$\mathbb{T}{\circ}J{\circ}\mathbb{T} \;=\; \mathbb{T}{\circ}P{\circ}\mathbb{T}\quad .$$

Proof

$\quad\mathbb{T}{\circ}J{\circ}\mathbb{T}$

$=\qquad\{\quad J$ is an index of $P{\circ}\mathbb{T}{\circ}P$, lemma 28 with $J,P := J\,,P{\circ}\mathbb{T}{\circ}P\quad\}$

$\quad\mathbb{T}{\circ}P{\circ}\mathbb{T}{\circ}P{\circ}\mathbb{T}$

$\supseteq\qquad\{\quad \mathbb{T}\supseteq P$ and monotonicity $\quad\}$

$\quad\mathbb{T}{\circ}P{\circ}P{\circ}P{\circ}\mathbb{T}$

$=\qquad\{\quad P$ is a per, so $P{\circ}P = P$ (applied twice) $\quad\}$

$\quad\mathbb{T}{\circ}P{\circ}\mathbb{T}$

$=\qquad\{\quad$ domains (specifically $[\;\mathbb{T}{\circ}R = \mathbb{T}\circ R{>}\;]$ with $R := P\,;\; P{>} = P{\asymp}$) $\quad\}$

$\quad\mathbb{T}\circ P{\asymp}\circ\mathbb{T}$

$\supseteq\qquad\{\quad$ domains (specifically $[\;S{>} \supseteq (R{\circ}S){>}\;]$ with $R,S := P{\circ}\mathbb{T}\,,P\,;\; P{>} = P{\asymp}$) $\quad\}$

$\quad\mathbb{T}\circ(P{\circ}\mathbb{T}{\circ}P){\asymp}\circ\mathbb{T}$

$\supseteq\qquad\{\quad J$ is an index of $P{\circ}\mathbb{T}{\circ}P$, definition 2(a) with $J,P := J\,,P{\circ}\mathbb{T}{\circ}P\quad\}$

$\quad\mathbb{T}{\circ}J{\circ}\mathbb{T}\quad .$

The equality follows by anti-symmetry.

□

Lemma 30 Suppose P is a per and J is an index of P. Suppose a is an index of $J{\circ}\mathbb{T}{\circ}J$. Then

$$a \subseteq J \quad\wedge\quad a{\circ}P{\circ}a \;=\; a = a{\circ}\mathbb{T}{\circ}a\quad .$$

Proof

$\quad a$

$\subseteq\qquad\{\quad$ assumption: a is an index of $J{\circ}\mathbb{T}{\circ}J$, definition 2(a) $\quad\}$

$\quad(J{\circ}\mathbb{T}{\circ}J){\asymp}$

$=\qquad\{\quad$ lemma 27 $\quad\}$

$\quad J\quad .$

Also,

$$
\begin{array}{ll}
 & a\circ P\circ a \\
= & \{ \quad \text{above: } a\subseteq J\text{; both } a \text{ and } J \text{ are coreflexive, so } a\circ J = a = J\circ a \quad \} \\
 & a\circ J\circ P\circ J\circ a \\
= & \{ \quad J \text{ is an index of } P\text{, definition 2(b)} \quad \} \\
 & a\circ J\circ a \\
= & \{ \quad \text{above: } a\subseteq J\text{; both } a \text{ and } J \text{ are coreflexive, so } a\circ J = a = J\circ a \quad \} \\
 & a\circ a \\
= & \{ \quad \text{assumption: } a \text{ is an index of } J\circ\mathbb{T}\circ J\text{, so } a \text{ is coreflexive} \quad \} \\
 & a \ .
\end{array}
$$

Finally,

$$
\begin{array}{ll}
 & a\circ\mathbb{T}\circ a \\
= & \{ \quad \text{above: } a\subseteq J\text{; both } a \text{ and } J \text{ are coreflexive, so } a\circ J = a = J\circ a \quad \} \\
 & a\circ J\circ\mathbb{T}\circ J\circ a \\
= & \{ \quad \text{assumption: } a \text{ is an index of } J\circ\mathbb{T}\circ J\text{;} \\
 & \quad\quad \text{definition 2(b) with } J{,}P := a\,,J\circ\mathbb{T}\circ J \quad \} \\
 & a \ .
\end{array}
$$

□

We now begin the proof of the key lemma, lemma 47. Throughout the remainder of this section we assume that P is an arbitrary per and J, a, f and Q are defined by (31), (32), (33), (34):

(31) J is an index of P ,

(32) a is an index of $J\circ\mathbb{T}\circ J$,

(33) $f \;=\; J\circ P \,\cup\, a\circ\perp\!\!\!\perp/P$,

(34) $Q \;=\; f^{\cup}\circ f$.

These definitions are guided by the way one might prove lemma 47 in a traditional, pointwise, set-theoretic fashion whilst maintaining the generality and economy of point-free calculation. In a traditional setting, the coreflexive J corresponds to a set of representative elements of the equivalence classes of P; the square $J\circ\mathbb{T}\circ J$ combines all these equivalence classes into one so that a is thus one of the elements of J. Finally, f is the characteristic function of the per Q; it maps each element in the domain of P to the representative element of the equivalence class to which it belongs and all the elements not in the right domain of P to the chosen representative a. The lemmas below make these informal statements precise.

Because it is used several times below, it is useful to record that

(35) $f^{\cup} \;=\; P\circ J \,\cup\, P\backslash\perp\!\!\!\perp\circ a$.

(The straightforward proof uses the fact that P, J and a are symmetric and elementary properties of converse.) Lemmas 36 and 37 are also used repeatedly below.

Lemma 36 For all R,

$$R\backslash\perp\!\!\!\perp \;=\; R{>}\backslash\perp\!\!\!\perp \quad .$$

Proof The proof is by indirect equality: for all S,

$$\begin{array}{ll}
 & S \subseteq R\backslash\perp\!\!\!\perp \\
= & \quad \{ \quad \text{factors} \quad \} \\
 & R{\circ}S \subseteq \perp\!\!\!\perp \\
= & \quad \{ \quad \text{domains} \quad \} \\
 & R{>}\circ S \subseteq \perp\!\!\!\perp \\
= & \quad \{ \quad \text{factors} \quad \} \\
 & S \subseteq R{>}\backslash\perp\!\!\!\perp \quad .
\end{array}$$

□

Lemma 37

$$J\circ P\backslash\perp\!\!\!\perp \;=\; \perp\!\!\!\perp \;=\; \perp\!\!\!\perp/P\circ J \quad .$$

Proof For the first equality, we have:

$$\begin{array}{ll}
 & J\circ P\backslash\perp\!\!\!\perp \\
= & \quad \{ \quad \text{lemma 36 with } R{:=}P \;;\; P{>}=P{\asymp} \quad \} \\
 & J\circ P{\asymp}\backslash\perp\!\!\!\perp \\
\subseteq & \quad \{ \quad J \text{ is an index of } P\text{, definition 2(a) and monotonicity} \quad \} \\
 & P{\asymp}\circ P{\asymp}\backslash\perp\!\!\!\perp \\
\subseteq & \quad \{ \quad \text{cancellation of factors} \quad \} \\
 & \perp\!\!\!\perp \quad .
\end{array}$$

The equality follows because $\perp\!\!\!\perp$ is the least relation. A symmetric calculation proves the second equality.

□

Lemma 38

$$f\circ f^{\cup} = J \quad .$$

Hence Q is a per.

Proof

$$\begin{array}{ll}
 & f\circ f^{\cup} \\
= & \quad \{ \quad (33) \text{ and } (35) \quad \} \\
 & (J{\circ}P \,\cup\, a\circ\perp\!\!\!\perp/P)\circ(P{\circ}J \,\cup\, P\backslash\perp\!\!\!\perp\circ a) \\
= & \quad \{ \quad \text{distributivity} \quad \} \\
 & J{\circ}P{\circ}P{\circ}J \;\cup\; J\circ P\circ P\backslash\perp\!\!\!\perp\circ a \;\cup\; a\circ\perp\!\!\!\perp/P\circ P\circ J \;\cup\; a\circ\perp\!\!\!\perp/P\circ P\backslash\perp\!\!\!\perp\circ a \\
= & \quad \{ \quad P \text{ is a per and } J \text{ is an index of } P\text{, definition 2(b);} \\
 & \qquad\quad P\circ P\backslash\perp\!\!\!\perp = \perp\!\!\!\perp = \perp\!\!\!\perp/P\circ P \quad \} \\
 & J \;\cup\; a\circ\perp\!\!\!\perp/P\circ P\backslash\perp\!\!\!\perp\circ a \quad .
\end{array}$$

But

$$
\begin{array}{cl}
 & a\circ \perp\!\!\!\perp/P\circ P\backslash\perp\!\!\!\perp\circ a \\
\subseteq & \{\quad \top\!\!\!\top \text{ is greatest element, monotonicity} \quad\} \\
 & a\circ\top\!\!\!\top\circ a \\
= & \{\quad \text{lemma 30} \quad\} \\
 & a \\
\subseteq & \{\quad \text{(32), lemma 27 and definition 2(a)} \quad\} \\
 & J\ .
\end{array}
$$

Combining the two calculations, we conclude that $f\circ f^{\cup} = J$. That Q is a per follows by an elementary calculation.

□

We now prove that P thins Q by applying lemma 18. Two properties are needed. The first is elementary:

Lemma 39 $P\subseteq Q$.

Proof

$$
\begin{array}{cl}
 & Q \\
\supseteq & \{\quad \text{by (33), } f\supseteq J\circ P\text{; } J \text{ and } P \text{ are symmetric} \quad\} \\
 & P\circ J\circ J\circ P \\
= & \{\quad J \text{ is an index of } P\text{, definition 2(a) and (c)} \quad\} \\
 & P\ .
\end{array}
$$

□

Lemma 40 J is an index of Q.

Proof Property 2(a) is immediate from the definition of J, lemma 39 and monotonicity properties of the domain operator. For property 2(b) (with $J,P := J,Q$), we have:

$$
\begin{array}{cl}
 & f\circ J \\
= & \{\quad \text{definition (33), distributivity} \quad\} \\
 & J\circ P\circ J \cup a\circ\perp\!\!\!\perp/P\circ J \\
= & \{\quad J \text{ is an index of } P\text{, definition 2(b)} \quad\} \\
 & J \cup a\circ\perp\!\!\!\perp/P\circ J \\
= & \{\quad \text{lemma 37 and properties of } \perp\!\!\!\perp \quad\} \\
 & J\ .
\end{array}
$$

That is (above and exploiting properties of converse),

(41) $f\circ J = J = J\circ f^{\cup}$.

So

$$\begin{array}{ll} & J\circ Q\circ J \\ = & \{ \quad (34) \quad \} \\ & J\circ f^{\cup}\circ f\circ J \\ = & \{ \quad (41) \quad \} \\ & J\circ J \\ = & \{ \quad J \text{ is an index of } P\text{, so is coreflexive} \quad \} \\ & J \ . \end{array}$$

Now for 2(c) (with $J,P := J,Q$):

$$\begin{array}{ll} & Q\circ J\circ Q \\ = & \{ \quad \text{definition (34)} \quad \} \\ & f^{\cup}\circ f\circ J\circ f^{\cup}\circ f \\ = & \{ \quad \text{(41) (applied twice)} \quad \} \\ & f^{\cup}\circ J\circ f \\ = & \{ \quad \text{lemma 38} \quad \} \\ & f^{\cup}\circ f\circ f^{\cup}\circ f \\ = & \{ \quad \text{definition (34) and } Q \text{ is a per (lemma 38)} \quad \} \\ & Q \ . \end{array}$$

□

We now have all the ingredients of the proof that P thins Q.

Lemma 42

$$P \preceq Q \ .$$

Proof Combination of lemmas 39, 40 and 18.

□

Only one more lemma is needed to establish the second crucial property of Q:

Lemma 43

$$\mathbb{T}\circ Q\circ\mathbb{T} \ = \ \mathbb{T}\circ a\circ\mathbb{T} \ .$$

Proof Because it is needed below, we first note that

(44) $\quad J\circ f \ \subseteq \ P \ \cup \ a\circ\bot\!\!\!\bot/P$

since

$$\begin{array}{ll} & J\circ f \\ = & \{ \quad \text{definition (33) and distributivity} \quad \} \\ & J\circ J\circ P \ \cup \ J\circ a\circ\bot\!\!\!\bot/P \\ \subseteq & \{ \quad J \text{ is coreflexive (i.e. } J\subseteq\mathbb{I}\text{) and monotonicity} \quad \} \\ & P \ \cup \ a\circ\bot\!\!\!\bot/P \ . \end{array}$$

So

$\quad\ \mathbb{T}\circ Q\circ\mathbb{T}$
$=\quad\{\quad$ definition: (34) and (35), $P=P^{\cup}$ and converse $\quad\}$
$\quad\ \mathbb{T}\circ(P\circ J\ \cup\ P\backslash\perp\!\!\!\perp\circ a)\circ f\circ\mathbb{T}$
$=\quad\{\quad$ distributivity $\quad\}$
$\quad\ \mathbb{T}\circ P\circ J\circ f\circ\mathbb{T}\ \cup\ \mathbb{T}\circ P\backslash\perp\!\!\!\perp\circ a\circ f\circ\mathbb{T}$
$\subseteq\quad\{\quad$ (44) and distributivity $\quad\}$
$\quad\ \mathbb{T}\circ P\circ\mathbb{T}\ \cup\ \mathbb{T}\circ a\circ\perp\!\!\!\perp/P\circ\mathbb{T}\ \cup\ \mathbb{T}\circ P\backslash\perp\!\!\!\perp\circ a\circ f\circ\mathbb{T}$
$\subseteq\quad\{\quad$ $[\,R\circ a\circ S\subseteq\mathbb{T}\circ a\circ\mathbb{T}\,]$ (applied twice) $\quad\}$
$\quad\ \mathbb{T}\circ P\circ\mathbb{T}\ \cup\ \mathbb{T}\circ a\circ\mathbb{T}$
$=\quad\{\quad$ by lemma 28, $\mathbb{T}\circ P\circ\mathbb{T}=\mathbb{T}\circ J\circ\mathbb{T}$
$\qquad\qquad$ and, by lemma 29 (with $J,P:=a,J$), $\mathbb{T}\circ J\circ\mathbb{T}=\mathbb{T}\circ a\circ\mathbb{T}$ $\quad\}$
$\quad\ \mathbb{T}\circ a\circ\mathbb{T}$
$\subseteq\quad\{\quad$ $a\subseteq J\subseteq P\subseteq Q$ (by definition of a and J and lemma 39) $\quad\}$
$\quad\ \mathbb{T}\circ Q\circ\mathbb{T}$.

The lemma follows by anti-symmetry.
□

In order to prove the next lemma, we find it necessary to introduce a novel axiom. The axiom we propose is discussed in detail in section 7. An equivalent formulation of the axiom is the property

(45) $\quad\langle\forall p\ :\ p\subseteq\mathbb{I}\ :\ \mathbb{I}\subseteq p\backslash\perp\!\!\!\perp\cup p\rangle$.

(See section 7 for the reason we don't propose (45) as the novel axiom.)

Lemma 46

$$\mathbb{I}\cap\mathbb{T}\circ Q\circ\mathbb{T}\ \subseteq\ Q\ .$$

Proof

$\quad\ \mathbb{I}\cap\mathbb{T}\circ Q\circ\mathbb{T}\ \subseteq\ Q$
$\Leftarrow\quad\{\quad$ (45) with $p:=P{\between}$ $\quad\}$
$\quad\ \mathbb{I}\cap(P{\between}\backslash\perp\!\!\!\perp\ \cup\ P{\between})\cap\mathbb{T}\circ Q\circ\mathbb{T}\ \subseteq\ Q$
$=\quad\{\quad$ distributivity, definition of supremum $\quad\}$
$\quad\ \mathbb{I}\cap P{\between}\backslash\perp\!\!\!\perp\cap\mathbb{T}\circ Q\circ\mathbb{T}\ \subseteq\ Q\quad\wedge\quad P{\between}\cap\mathbb{T}\circ Q\circ\mathbb{T}\ \subseteq\ Q$.

We verify each conjunct separately. First,

$\quad\ \mathbb{I}\cap P{\between}\backslash\perp\!\!\!\perp\cap\mathbb{T}\circ Q\circ\mathbb{T}$
$\subseteq\quad\{\quad$ $[\,p\cap R\subseteq p\circ R\circ p\ \Leftarrow\ p\subseteq\mathbb{I}\,]$ with $p,R\ :=\ \mathbb{I}\cap P{\between}\backslash\perp\!\!\!\perp\ ,\ \mathbb{T}\circ Q\circ\mathbb{T}$ $\quad\}$
$\quad\ (\mathbb{I}\cap P{\between}\backslash\perp\!\!\!\perp)\circ\mathbb{T}\circ Q\circ\mathbb{T}\circ(\mathbb{I}\cap P{\between}\backslash\perp\!\!\!\perp)^{\cup}$
$\subseteq\quad\{\quad$ monotonicity and $[\,(R\backslash S)^{\cup}=S^{\cup}/R^{\cup}\,]$ with $R,S:=P{\between},\perp\!\!\!\perp$ $\quad\}$
$\quad\ P{\between}\backslash\perp\!\!\!\perp\circ\mathbb{T}\circ Q\circ\mathbb{T}\circ\perp\!\!\!\perp/P{\between}$
$=\quad\{\quad$ lemma 43 $\quad\}$
$\quad\ P{\between}\backslash\perp\!\!\!\perp\circ\mathbb{T}\circ a\circ\mathbb{T}\circ\perp\!\!\!\perp/P{\between}$
$=\quad\{\quad$ for all coreflexives p, $p\backslash\perp\!\!\!\perp\circ\mathbb{T}\ =\ p\backslash\perp\!\!\!\perp$ (proved by anti-symmetry);

$$
\begin{array}{ll}
 & \qquad \text{symmetrically, } \mathbb{T}\circ\bot\!\!\!\bot/p \;=\; \bot\!\!\!\bot/p \quad \} \\
 & P_{\asymp}\backslash\bot\!\!\!\bot\circ a\circ\bot\!\!\!\bot/P_{\asymp} \\
= & \quad \{ \qquad a \text{ is coreflexive, so } a = a\circ a \ ; \\
 & \qquad \text{lemma 36 with } R := P \ (\text{and } P_{>} = P_{\asymp} = P_{<}) \quad \} \\
 & P\backslash\bot\!\!\!\bot\circ a\circ a\circ\bot\!\!\!\bot/P \\
\subseteq & \quad \{ \qquad (34), (33) \text{ and } (35) \quad \} \\
 & Q \ .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & P_{\asymp}\cap\mathbb{T}\circ Q\circ\mathbb{T} \ \subseteq \ Q \\
\Leftarrow & \quad \{ \qquad P_{\asymp}\cap\mathbb{T}\circ Q\circ\mathbb{T}\subseteq P_{\asymp}, \text{ transitivity} \quad \} \\
 & P_{\asymp} \subseteq Q \\
= & \quad \{ \qquad \text{domains: } P_{\asymp}\subseteq P, \text{ and, by lemma 39, } P\subseteq Q \text{; transitivity} \quad \} \\
 & \text{true} \ .
\end{array}
$$

□

Lemma 47 Suppose P is an arbitrary per. Then, assuming the axiom of choice and (45), there is a per Q such that $\mathbb{I}\cap\mathbb{T}\circ Q\circ\mathbb{T}\subseteq Q$ and $P\preceq Q$.

Proof Given per P, define J, a, f and Q by the equations (31), (32), (33), (34). Then the lemma is a direct consequence of lemmas 42 and 46.
□

Theorem 48 Assuming the axiom of choice and (45), a per P is maximal with respect to the thins ordering iff $\mathbb{I}\cap\mathbb{T}\circ P\circ\mathbb{T}\subseteq P$.

Proof We have shown (lemma 26) that P is maximal if $\mathbb{I}\cap\mathbb{T}\circ P\circ\mathbb{T}\subseteq P$. Only-if follows from lemma 47. Specifically, suppose P is a per. By lemma 47, there is a per Q such that $\mathbb{I}\cap\mathbb{T}\circ Q\circ\mathbb{T}\subseteq Q$ and $P\preceq Q$. So, by definition of maximal, if P is maximal, $P=Q$. That is, by Leibniz's rule, if P is maximal, $\mathbb{I}\cap\mathbb{T}\circ P\circ\mathbb{T}\subseteq P$.
□

It is remarkable that the proof of theorem 48 does not rely in any way on saturation properties of the lattices of coreflexives or of relations in general. Although our notation might suggest otherwise, the index a is not assumed to be a point; all that is needed are the properties stated in lemma 30. Lemma 47 does assume (45) but that is separate from saturation.

It is also worth emphasising that we have avoided the use of the cone rule and, in so doing, have avoided a case analysis in the statement of theorem 48. This means that the theorem is also applicable for non-unary relation algebras. For concrete relations (where the cone rule does apply), the interpretation of theorem 48 is that a per P is maximal iff it is empty or is an equivalence relation. (This is because, by applying the cone rule, the property $\mathbb{I}\cap\mathbb{T}\circ P\circ\mathbb{T}\subseteq P$ simplifies to $P=\bot\!\!\!\bot \vee I\subseteq P$.)

We have not avoided a case analysis entirely: this is the topic of the next section.

7 A Novel Axiom

In the proof of lemma 46 we postulated property (45). An alternative is to postulate the existence of complements in the lattice of coreflexives of a given type. Specifically, we could postulate the existence of $\sim(P_{\asymp})$ with the property that

$$(49) \qquad \mathbb{I} \;=\; P_{\asymp} \cup \sim(P_{\asymp}) \quad \wedge \quad \perp\!\!\perp \;=\; P_{\asymp} \circ \sim(P_{\asymp}) \quad .$$

Where in the proof of lemma 46 there is, essentially, a case analysis on $P_{\asymp} \setminus \perp\!\!\perp$ and $P_{\asymp}$, an alternative proof uses (49) to formulate a case analysis on $\sim(P_{\asymp})$ and $P_{\asymp}$. This section explores links between the two approaches. In so doing, we postulate a novel axiom, specifically (58), which we argue is aesthetically preferable to alternative (but equivalent) postulates.

7.1 Adding Complements

We begin by reviewing some basic facts about the existence of complements in a lattice. Suppose X is a set ordered by the relation $\sqsubseteq$ and suppose X forms a complete lattice under this ordering. In what follows, variables x, y and z range over X, $\perp$ and $\top$ denote, respectively, the least and greatest elements, and $\sqcap$ and $\sqcup$ denote, respectively, the infimum and supremum operators of the lattice. We begin with a definition.

Definition 50 The element x is said to be a *pseudo-complement* of y if $x \sqcap y = \perp$; x is said to be a *pseudo-supplement* of y if $x \sqcup y = \top$; x is said to be a *complement* of y if it is both a pseudo-complement and a pseudo-supplement of y.
□

In a distributive lattice, a pseudo-complement is always contained in a pseudo-supplement. Formally,

Lemma 51 Suppose that

$$x \sqcap z = \perp \quad \wedge \quad y \sqcup z = \top \quad .$$

Then, in a distributive lattice, $x \sqsubseteq y$.

Proof We show that $x = x \sqcap y$.

$$\begin{array}{cl} & x \\ = & \{ \quad y \sqcup z = \top \quad \} \\ & x \sqcap (y \sqcup z) \\ = & \{ \quad \text{distributivity} \quad \} \\ & (x \sqcap y) \sqcup (x \sqcap z) \\ = & \{ \quad x \sqcap z = \perp \quad \} \\ & x \sqcap y \quad . \end{array}$$

□

An immediate corollary is that complements are unique. Formally:

Lemma 52 Suppose x and y are both complements of z. That is, suppose that

$$x\sqcap z=\bot \;\wedge\; x\sqcup z=\top \;\wedge\; y\sqcap z=\bot \;\wedge\; y\sqcup z=\top \;\;.$$

Then, in a distributive lattice, $x=y$.

Proof Applying lemma 51, we get $x\sqsubseteq y$; applying lemma 51 with x and y interchanged, we get $y\sqsubseteq x$. So, by anti-symmetry, $x=y$.
□

Now, suppose that X forms a complete, universally distributive lattice under the $\sqsubseteq$ ordering. Then, the property that infimum distributes universally over supremum is equivalent to the existence of a Galois connection: for all x, y and z,

$$(53) \qquad x\sqcap y \sqsubseteq z \;\equiv\; x \sqsubseteq y{\rightarrow}z \;\;.$$

The dual property that supremum distributes universally over infimum is equivalent to the existence of a Galois connection: for all x, y and z,

$$(54) \qquad x{\leftarrow}y \sqsubseteq z \;\equiv\; x \sqsubseteq y\sqcup z \;\;.$$

Readers may recognise (53) as the property in intuitionistic logic connecting conjunction and (constructive) implication; functional programmers may also recognise it as the adjunction sometimes known as “Currying/unCurrying”. Its dual, property (54), is possibly less familiar.

Defining $\sim y$ by $\sim y = y{\rightarrow}\bot$ and $\backsim y$ by $\backsim y = \top{\leftarrow}y$, one easily shows that

$$(55) \qquad y\sqcap{\sim}y = \bot \;\wedge\; y\sqcup{\backsim}y = \top \;\;.$$

(Just take the instantiation $x,y,z := {\sim}y\,,y\,,\bot$ in (53) and the instantiation $x,y,z := \top\,,y\,,{\backsim}y$ in (54).) That is, ${\sim}y$ is a pseudo-complement and ${\backsim}y$ is a pseudo-supplement of y. If the two operators coincide (that is, for all y, ${\sim}y={\backsim}y$), then ${\sim}y$ (equally ${\backsim}y$) is called the *complement* of y and property (55) specialises to (49). The question then becomes: under what conditions do the two operators coincide? An answer is given in the next lemma.

Lemma 56 Suppose that ${\sim}z$ and ${\backsim}z$ denote, respectively, $z{\rightarrow}\bot$ and $\top{\leftarrow}z$. Then equivalent are:

(a) $\quad {\sim}z={\backsim}z \;\;,$

(b) $\quad {\backsim}z\sqsubseteq{\sim}z \;\;,$

(c) $\quad {\backsim}z\sqcap z = \bot$ and

(d) $\quad {\sim}z\sqcup z = \top \;\;.$

Proof Properties (a) and (b) are equivalent by anti-symmetry and lemma 51 (with the instantiation $x,y := {\sim}z\,,{\backsim}z$). Properties (c) and (d) are clearly implied by (a) and (55). Also, (c) equivales (b) because

$$
\begin{array}{ll}
 & \backsim z \sqcap z = \bot \\
= & \{ \quad \bot \text{ is least element } \} \\
 & \backsim z \sqcap z \sqsubseteq \bot \\
= & \{ \quad (53) \quad \} \\
 & \backsim z \sqsubseteq z \rightarrow \bot \\
= & \{ \quad z \rightarrow \bot = \sim z \quad \} \\
 & \backsim z \sqsubseteq \sim z \ .
\end{array}
$$

Dually, (d) equivales (b).
□

7.2 Proposed Axiom

A fundamental assumption in our axiomatisation of point-free relation algebra is that the underlying lattice structure is a complete, universally distributive lattice. So the above theory is applicable. In fact, it is applicable in two ways. The first, is that it is assumed that relations of a given type form a complete, universally distributive lattice under the subset ordering. The greatest element in this ordering is $\top\!\!\top$. The second is that the coreflexives of a given type form a complete, universally distributive lattice under the subset ordering. The greatest element in this ordering is $\mathbb{I}$ (the identity relation of the relevant type).

In order to postulate the existence of complements in point-free relation algebra, it therefore suffices to add as axiom one of the four properties listed in lemma 56. However, there is a choice: we may postulate that the entire algebra is complemented, or we may postulate that just the coreflexives (of a given type) are complemented.

The latter choice does not imply that every relation has a complement. For example, the three-element algebra detailed in example 1 is not complemented because $\mathbb{I}$ does not have a complement: its least pseudo-supplement is $\top\!\!\top$ and its greatest pseudo-complement is $\bot\!\!\bot$, which are different. However, the coreflexives are $\bot\!\!\bot$ and $\mathbb{I}$. So the only pseudo-supplement of $\mathbb{I}$ in the lattice of coreflexives is $\bot\!\!\bot$, which is also its only pseudo-complement. So $\bot\!\!\bot$ is a complement of $\mathbb{I}$ — in the lattice of coreflexives. Vice-versa, $\mathbb{I}$ is a complement of $\bot\!\!\bot$. Thus, in the three-element algebra, the lattice of coreflexives is complemented but the lattice of relations is not complemented.

For our purposes, a sufficient property is (49) which is a property of the coreflexive $P_{\asymp}$. This suggests that we add the axiom that

(57) $\quad \langle \forall p : p \subseteq \mathbb{I} : p \cap \backsim p = \bot\!\!\bot \rangle \ .$

An alternative approach, which we prefer, is to add as axiom the property:

(58) $\quad \langle \forall R :: \mathbb{I} \subseteq R \backslash \bot\!\!\bot \cup \top\!\!\top \circ R \rangle \ .$

We now establish the equivalence of (57) and (58). We begin by showing that (57) is equivalent to (45). This we do as follows. For all coreflexives p,

$$
\begin{array}{ll}
 & p \cap \backsim p = \bot\!\!\bot \\
= & \{ \quad \backsim p = \mathbb{I} \leftarrow p \subseteq \mathbb{I}, \text{ infimum of coreflexives is composition} \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & p\circ(\mathbb{I}{\leftarrow}p) \subseteq \perp\!\!\!\perp \\
= & \quad\{ \quad \text{factors} \quad \} \\
 & \mathbb{I}{\leftarrow}p \subseteq p\backslash\perp\!\!\!\perp \\
= & \quad\{ \quad (54) \quad \} \\
 & \mathbb{I} \;\subseteq\; p\backslash\perp\!\!\!\perp \cup p \quad .
\end{array}
$$

We now show that (45) is equivalent to (58).

$$
\begin{array}{ll}
 & \langle\forall R :: \mathbb{I} \subseteq R\backslash\perp\!\!\!\perp \cup \top\!\!\!\top\circ R\rangle \\
\Rightarrow & \quad\{ \quad \text{restriction to coreflexives: } R{:=}p \text{ where } p\subseteq\mathbb{I} \quad \} \\
 & \langle\forall p : p\subseteq\mathbb{I} : \mathbb{I} \subseteq p\backslash\perp\!\!\!\perp \cup \top\!\!\!\top\circ p\rangle \\
= & \quad\{ \quad [\, R\subseteq S\cup T \;\equiv\; R \subseteq S\cup(R\cap T) \,] \text{ with } R,S,T := \mathbb{I}, p\backslash\perp\!\!\!\perp, \top\!\!\!\top\circ p \quad \} \\
 & \langle\forall p : p\subseteq\mathbb{I} : \mathbb{I} \subseteq p\backslash\perp\!\!\!\perp \cup (\mathbb{I} \cap \top\!\!\!\top\circ p)\rangle \\
= & \quad\{ \quad \text{domains: } [\, R{>} = \mathbb{I} \cap \top\!\!\!\top\circ R \,] \text{ with } R{:=}p \text{ and } [\, p\subseteq I \Rightarrow p{>}{=}p \,] \quad \} \\
 & \langle\forall p : p\subseteq\mathbb{I} : \mathbb{I} \subseteq p\backslash\perp\!\!\!\perp \cup p\rangle \\
\Rightarrow & \quad\{ \quad \text{specialising } p \text{ to } R{>} \quad \} \\
 & \langle\forall R :: \mathbb{I} \subseteq R{>}\backslash\perp\!\!\!\perp \cup \top\!\!\!\top\circ R{>}\rangle \\
= & \quad\{ \quad \text{lemma 36 and domains (specifically } [\, \top\!\!\!\top\circ R{>} = \top\!\!\!\top\circ R \,]) \quad \} \\
 & \langle\forall R :: \mathbb{I} \subseteq R\backslash\perp\!\!\!\perp \cup \top\!\!\!\top\circ R\rangle \quad .
\end{array}
$$

Returning to the main calculation, we have established, by mutual implication, that (58) is equivalent to (45). We conclude that (45), (57) and (58) are all equivalent — in point-free relation algebra.

Our proposal is to add (58) as a new axiom. Our argument for making this choice is that (45), (57) and (58) are *not* equivalent in substructures of point-free relation algebra; in particular they are not equivalent in regular algebra. The detailed argument follows.

As detailed in section 2, our preferred axiomatisation of point-free relation algebra is a combination of three structures with interfaces between the structures. The monoid and lattice structures together with the interface between them form what Conway [Con71] calls a "standard Kleene algebra". Alternatively, the terminology is "regular algebra" because it is the algebra of regular languages.

Axiom (58) distinguishes point-free relation algebra from regular algebra because it is meaningful in both but it is *not* valid for regular languages. Specifically, suppose T denotes a finite, non-empty "alphabet". Let ε denote the empty word (the word of length 0). Then in the regular algebra defined by T, the constants $\perp\!\!\!\perp$, $\mathbb{I}$ and $\top\!\!\!\top$ are, respectively, $\emptyset$ (the empty set), $\{\varepsilon\}$ (the singleton set containing just the empty word) and T^* (the set of all words over the alphabet T). Moreover, for all languages R, $R\backslash\emptyset$ is either $\emptyset$ (when $R\neq\emptyset$) or T^* (when $R=\emptyset$). Thus, (58) is not valid in general; in particular,

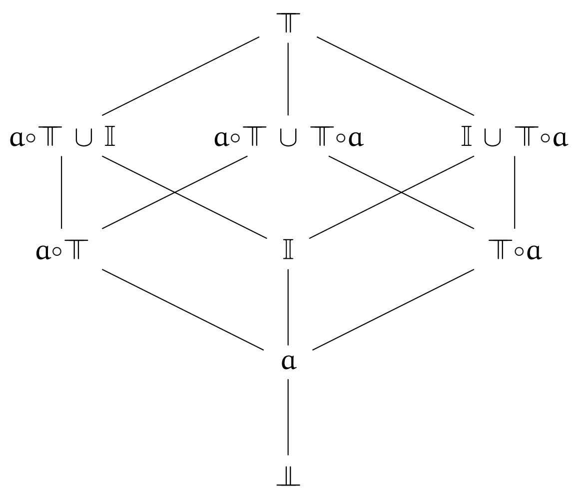
$$\{\varepsilon\} \subseteq T^+ \cup T^+\backslash\emptyset$$

is false. (As usual, T^+ denotes the set of all words different from the empty word.)

Of course, the complement of a language is well-defined. (A language is a subset of T^*.) So a "standard Kleene algebra" is an example of an algebra for which the existence of complements is not equivalent to (58). Scrutiny of the above proof of their equivalence reveals great reliance on properties of the domain operators in relation algebra which, in turn, rely on the modularity law — a law that acts as an interface between all three structures in a point-free relation algebra and is thus meaningless in regular algebra. We prefer (58) over an axiom postulating the existence of complements in the lattice of coreflexives (for example, (57)) because it establishes an interface between the constants $\bot\!\!\!\bot$, $\mathbb{I}$ and $\top\!\!\!\top$; it thus specialises the standard interface between the monoid structure and the lattice structure to one that better captures properties specific to relations.

7.3 Non-necessity

Our proof of theorem 48 shows that our novel axiom (property (58)) is a sufficient condition for the characterisation of maximality. Also, our proof in section 7.2 that (58) is equivalent to the existence of complements in the lattice of coreflexives shows that it is a valid property of concrete relations. However, we are unable to show that the axiom is a necessary condition for a per P to be maximal when $\mathbb{I}\cap\top\!\!\!\top\circ P\circ\top\!\!\!\top \subseteq P$. Indeed, the axiom is *not* necessary. Consider the algebra defined by the nine elements shown in the diagram below.



The diagram defines the lattice structure. The composition and converse structures are defined by specifying that the algebra is unary and a is a point. That is, $\top\!\!\!\top\circ a\circ\top\!\!\!\top = \top\!\!\!\top$, $a\circ\top\!\!\!\top\circ a = a$ and $a^{\cup} = a$. (These, together with the distributivity laws, completely define the algebra.)

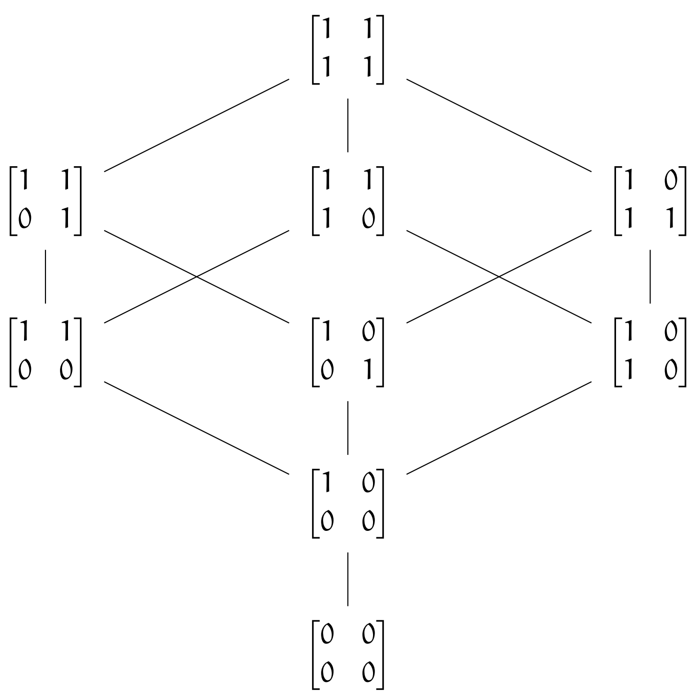
The pers in this algebra are $\bot\!\!\!\bot$, a, $\mathbb{I}$ and $\top\!\!\!\top$. (The element $a\circ\top\!\!\!\top\cup\top\!\!\!\top\circ a$ is not a per because it is not transitive. The remaining elements are not symmetric.) The axiom of choice is valid. Indeed, all four pers have a unique index: $\bot\!\!\!\bot$ and $\mathbb{I}$ are each the unique indexes of themselves and a is the unique index of itself and $\top\!\!\!\top$.

The reflexive-transitive reduction of the thins relation is given by $a \preceq \top\!\!\!\top$. So the maximal elements of the thins relation are $\bot\!\!\!\bot$, $\mathbb{I}$ and $\top\!\!\!\top$. Instantiating P to each of these, it is easily verified that in each case $\mathbb{I}\cap\top\!\!\!\top\circ P\circ\top\!\!\!\top \subseteq P$.

There are two ways that we can show that (58) is not valid in this algebra. The direct way is to observe that $a\backslash\bot\!\!\!\bot = \bot\!\!\!\bot$ and $\mathbb{I} \subseteq \top\!\!\!\top\circ a$ is false. It thus follows that $R := a$ is a counterexample to (58). The second is to note that the coreflexives are $\bot\!\!\!\bot$, a and $\mathbb{I}$; thus, by inspection of the above diagram, the lattice of coreflexives is not complemented. This contradicts (57) which we have shown to be equivalent to (58).

Aside A heuristic for constructing algebras like the one above is to consider a concrete model, reduce the carrier set to some proper subset, $\mathcal{A}$ say, and then close the chosen set under the supremum, composition and converse operations (including, of course, the constants $\perp\!\!\!\perp$, $\mathbb{I}$ and $\top\!\!\!\top$). The algebra so constructed is called the (point-free relation) algebra *generated by* $\mathcal{A}$ and $\mathcal{A}$ is called the *generating set*.

The three-element algebra detailed in example 1 is the algebra generated by the empty set. The algebra above is constructed by considering the concrete homogeneous relations on a set with at least two elements and choosing a to be the identity relation on one of those elements. (The generating set is the singleton set containing the point a.) The diagram below shows the interpretation of each of the elements as a concrete relation on a two-element set.



We use the word "heuristic" because we have not put the effort into formulating the construction precisely. For our purposes, it is sufficient to employ the heuristic to construct ad hoc examples which we then check for validity and usefulness. (Nevertheless we believe that a theorem establishing the validity of the construction can be formulated with relative ease.)

In fact, the above algebra was constructed by giving Prover9/Mace4 [McC] the task of constructing a point-free relation algebra that does not have complements. The output was a set of operation tables on nine elements from which the above matrices were derived.
End of Aside

8 Extending thins to arbitrary relations

In this section, we extend the thins ordering to arbitrary relations. The section is concluded by theorem 70 which states that the minimal elements of the extended ordering are exactly the core relations introduced in [BV22, BV23].

Definition 59 (Thins) For arbitrary relations R and S of the same type, the relation $R \preceq S$ is defined by

$$R \preceq S \quad \equiv \quad R_{\prec} \preceq S_{\prec} \;\wedge\; R_{\succ} \preceq S_{\succ} \;\wedge\; R = R_{<} \circ S \circ R_{>} \quad .$$

□

The symbol "$\preceq$" is overloaded in definition 59. If R and S have type $A \sim B$, the leftmost occurrence is a relation on relations of type $A \sim B$, the middle occurrence is a relation on pers of type A and the rightmost occurrence is a relation on pers of type B.

Lemma 60 The thins relation on arbitrary relations is an ordering relation.

Proof The thins relation on arbitrary relations is clearly reflexive. Transitivity is also easy to prove. Anti-symmetry is proven below.

$$\begin{array}{ll}
 & R \preceq S \;\wedge\; S \preceq R \\
\Rightarrow & \{ \quad \text{definition 59 with } R,S := R,S \text{ and } R,S := S,R \quad \} \\
 & R = R_{<} \circ S \circ R_{>} \;\wedge\; S = S_{<} \circ R \circ S_{>} \\
\Rightarrow & \{ \quad \text{domains} \quad \} \\
 & (R_{<} \subseteq S_{<} \;\wedge\; R_{>} \subseteq S_{>}) \wedge (S_{<} \subseteq R_{<} \;\wedge\; S_{>} \subseteq R_{>}) \\
= & \{ \quad \text{rearranging and anti-symmetry of } \subseteq \quad \} \\
 & R_{<} = S_{<} \;\wedge\; R_{>} = S_{>} \quad .
\end{array}$$

So

$$\begin{array}{ll}
 & R \preceq S \;\wedge\; S \preceq R \\
\Rightarrow & \{ \quad \text{definition 59 and above} \quad \} \\
 & R = R_{<} \circ S \circ R_{>} \;\wedge\; R_{<} = S_{<} \;\wedge\; R_{>} = S_{>} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & R = S_{<} \circ S \circ S_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & R = S \quad .
\end{array}$$

□

The definition of "minimal" and "maximal" with respect to the thins relation on arbitrary relations is the same as definition 19 except that the dummies in the universal quantifications range over arbitrary relations (of appropriate type).

We recall the definition of a core relation [BV22, BV23].

Definition 61 (Core Relation) A relation R is a *core relation* iff $R_{<} = R_{\prec}$ and $R_{>} = R_{\succ}$.

□

Lemma 62 A core relation is minimal with respect to the thins ordering on arbitrary relations.

Proof Suppose S is a core relation. Then, for all R,

$$
\begin{array}{ll}
 & R \preceq S \\
= & \{ \quad \text{definition 59} \quad \} \\
 & R_{\prec} \preceq S_{\prec} \;\wedge\; R_{\succ} \preceq S_{\succ} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
\Rightarrow & \{ \quad \text{assumption: } S \text{ is a core relation i.e. } S_{<} = S_{\prec} \text{ and } S_{>} = S_{\succ}\,, \\
 & \quad\;\; \text{so, by lemma 20, } S_{\prec} \text{ and } S_{\succ} \text{ are minimal; definition 19} \quad \} \\
 & R_{\prec} = S_{\prec} \;\wedge\; R_{\succ} = S_{\succ} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
= & \{ \quad S_{<} = S_{\prec} \text{ and } S_{>} = S_{\succ}\,, \text{ Leibniz} \quad \} \\
 & R_{\prec} = S_{<} \;\wedge\; R_{\succ} = S_{>} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & (R_{\prec})_{<} = (S_{<})_{<} \;\wedge\; (R_{\succ})_{>} = (S_{>})_{>} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
= & \{ \quad \text{per domains and domains} \\
 & \quad\;\; \text{(specifically, for left domains: } [\,(R_{\prec})_{<} = R_{<}\,]\,,\; [\,(R_{<})_{<} = R_{<}\,]\,, \\
 & \quad\;\; \text{similarly for right domains)} \quad \} \\
 & R_{<} = S_{<} \;\wedge\; R_{>} = S_{>} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
\Rightarrow & \{ \quad \text{Leibiz and domains} \quad \} \\
 & R = S \quad .
\end{array}
$$

Thus, by definition, S is minimal with respect to the thins ordering on arbitrary relations.
□

For reference, we include the definition of an index of an arbitrary relation and several of its properties. Proofs are given in [BV22, BV23].

Definition 63 (Index) An *index* of a relation R is a relation J that has the following properties:

(a) $J \subseteq R$,

(b) $R_{\prec} \circ J \circ R_{\succ} = R$,

(c) $J_{<} \circ R_{\prec} \circ J_{<} = J_{<}$,

(d) $J_{>} \circ R_{\succ} \circ J_{>} = J_{>}$.

□

Lemma 64 If J is an index of the relation R then

$$J_{\prec} \subseteq R_{\prec} \;\wedge\; J_{\succ} \subseteq R_{\succ} \quad .$$

It follows that

$$J_{<} = J_{\prec} \;\wedge\; J_{>} = J_{\succ} \quad .$$

That is, an index is a core relation.
□

Lemma 65 Suppose J is an index of R. Then

(a) $R_{\prec} \circ J_{<} \circ R_{\prec} \;=\; R_{\prec}$,

(b) $R_{\succ} \circ J_{>} \circ R_{\succ} \;=\; R_{\succ}$.

□

Theorem 66 Suppose J is an index of R. Then $J_{<}$ is an index of $R_{\prec}$ and $J_{>}$ is an index of $R_{\succ}$.

□

We now resume the study of the extended thins ordering.

Lemma 67 If J is an index of R then $J \preceq R$.

Proof Suppose that J is an index of R. By definition 59, we have to prove that $J_{\prec} \preceq R_{\prec}$, $J_{\succ} \preceq R_{\succ}$ and $J = J_{<} \circ R \circ J_{>}$. For the first property, we have:

$$
\begin{array}{cl}
 & J_{\prec} \preceq R_{\prec} \\
= & \quad \{ \qquad \text{definition 5} \quad \} \\
 & J_{\prec} \;=\; (J_{\prec})^{\asymp} \circ R_{\prec} \circ (J_{\prec})^{\asymp} \quad \wedge \quad R_{\prec} \;=\; R_{\prec} \circ (J_{\prec})^{\asymp} \circ R_{\prec} \\
= & \quad \{ \qquad \text{domains (specifically } [\,(R_{\prec})^{\asymp} = R_{<}\,] \text{ with } R{:=}J\,) \quad \} \\
 & J_{\prec} \;=\; J_{<} \circ R_{\prec} \circ J_{<} \quad \wedge \quad R_{\prec} \;=\; R_{\prec} \circ J_{<} \circ R_{\prec} \\
= & \quad \{ \qquad J \text{ is an index of } R\text{: theorem 66 and definition 63(c) with } J,R := J_{<}, R_{\prec}\,; \\
 & \qquad\qquad J \text{ is an index of } R\text{: lemma 65(a)} \quad \} \\
 & J_{\prec} = J_{<} \quad \wedge \quad \text{true} \\
= & \quad \{ \qquad \text{lemma 64} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

By symmetry, $J_{\succ} \preceq R_{\succ}$. The third property is straightforward:

$$
\begin{array}{cl}
 & J_{<} \circ R \circ J_{>} \\
= & \quad \{ \qquad \text{assumption: } J \text{ is an index of } R\text{, definition 63(b)} \quad \} \\
 & J_{<} \circ R_{\prec} \circ J \circ R_{\succ} \circ J_{>} \\
= & \quad \{ \qquad \text{domains} \quad \} \\
 & J_{<} \circ R_{\prec} \circ J_{<} \circ J \circ J_{>} \circ R_{\succ} \circ J_{>} \\
= & \quad \{ \qquad \text{assumption: } J \text{ is an index of } R\text{, definition 63(c) and 63(d)} \quad \} \\
 & J_{<} \circ J \circ J_{>} \\
= & \quad \{ \qquad \text{domains} \quad \} \\
 & J \quad .
\end{array}
$$

□

Lemma 68 If relation S is minimal with respect to the thins ordering on arbitrary relations and J is an index of S then $J = S$.

Proof Immediate from lemma 67 and the definition of minimal.

□

Lemma 69 Assuming axiom 3 (our axiom of choice), if relation S is minimal with respect to the thins ordering on arbitrary relations then $S\prec$ and $S\succ$ are mimimal with respect to the thins ordering on pers.

Proof

$$
\begin{array}{ll}
& S \text{ is minimal} \\
\Rightarrow & \{ \quad \text{lemma 68} \quad \} \\
& S \text{ is an index of } S \\
\Rightarrow & \{ \quad \text{lemma 64 and theorem 66} \quad \} \\
& S< = S\prec \;\;\wedge\;\; S< \text{ is an index of } S\prec \\
\Rightarrow & \{ \quad \text{lemma 20} \quad \} \\
& S\prec \text{ is minimal} \quad .
\end{array}
$$

Symmetrically, if S is minimal then $S\succ$ is minimal.
□

Note that the axiom of choice is invoked implicitly in the proof of lemma 69: the application of lemma 68 in the first step assumes that S has an index and that this is so is a consequence of the axiom of choice [BV22, BV23]. Consequently, the axiom of choice is also implicit in the proof of the main theorem of this section:

Theorem 70 Assuming axiom 3 (our axiom of choice), a relation S is minimal with respect to the thins ordering on arbitrary relations iff S is a core relation.

Proof "If" is lemma 62. "Only if" is a combination of lemma 69 and theorem 21:

$$
\begin{array}{ll}
& S \text{ is minimal} \\
\Rightarrow & \{ \quad \text{lemma 69} \quad \} \\
& S\prec \text{ is minimal} \\
\Rightarrow & \{ \quad \text{theorem 21} \quad \} \\
& S\prec \subseteq \mathbb{I} \\
= & \{ \quad \text{domains} \quad \} \\
& (S\prec)< = S\prec \\
= & \{ \quad \text{per domains} \quad \} \\
& S< = S\prec \quad .
\end{array}
$$

That is,

$$S \text{ is minimal} \;\;\Rightarrow\;\; S< = S\prec \quad .$$

Symmetrically,

$$S \text{ is minimal} \;\;\Rightarrow\;\; S> = S\succ \quad .$$

Thus, by definition 61,

$$S \text{ is minimal} \;\Rightarrow\; S \text{ is a core relation} \quad .$$

□

9 Conclusion

For us, the primary purpose of point-free relation algebra is to enable precise and concise reasoning about binary relations. Therefore, the axiomatisation does not capture all the properties of concrete relations and sometimes it is necessary to add axioms in order to facilitate such reasoning. (For example, it is sometimes necessary to add the cone rule to the axiom system [Voe99].) Earlier work [BV22, BV23] focused on facilitating pointwise reasoning —whilst not compromising the concision and precision of point-free reasoning— by adding axioms expressing the powerset properties of relations of a given type. To this end, [BV22, BV23] proposed a relatively weak axiom of choice (axiom 3) together with a saturation axiom (the axiom that the lattice of coreflexives of a given type is saturated by points). In this paper, the focus has been on just the axiom of choice.

The main contribution has been to provide further insight into the notion of a core relation introduced in [BV22, BV23]. Theorem 70 shows that the relations that are minimal with respect to the thins relation are precisely the core relations.

Theorem 48, which characterises pers that are maximal with respect to the thins ordering, is undoubtedly harder to prove and has required us to add a novel axiom (property (58)). This axiom differentiates the properties of concrete relations from, for example, regular languages in the monoid and lattice substructure, as discussed in section 7. Our proof of theorem 48 shows that the axiom is a sufficient condition for the characterisation of maximality but, as we have shown, it is not a necessary condition. Finding a better alternative to theorem 48 remains an open question.

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References

[BV22] Roland Backhouse and Ed Voermans. The index and core of a relation. With application to the axiomatics of relation algebra and block-ordered relations. Available at ResearchGate, April 2022.

[BV23] Roland Backhouse and Ed Voermans. The index and core of a relation. With application to the axiomatics of relation algebra. Submitted for publication. Available at ResearchGate and `http://arXiv.org/abs/2309.02017`, September 2023.

[Con71] J.H. Conway. *Regular Algebra and Finite Machines*. Chapman and Hall, London, 1971.

[Fv90] P.J. Freyd and A. Ščedrov. *Categories, Allegories*. North-Holland, 1990.

[McC] W. McCune. Prover9 and Mace4. `http://www.cs.unm.edu/~mccune/mace4/`.

[Voe99] Ed (Theodorus Sebastiaan) Voermans. *Inductive Datatypes with Laws and Subtyping. A Relational Model*. Technische Universiteit Eindhoven, Department of Mathematics and Computer Science, 1999. Ph.D. thesis. DOI:10.6100/IR51.