The Index and Core of a Relation With Applications to the Axiomatics of Relation Algebra and Block-Ordered Relations

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Abstract

We introduce the general notions of an index and a core of a relation. We postulate a limited form of the axiom of choice —specifically that all partial equivalence relations have an index— and explore the consequences of adding the axiom to standard axiom systems for point-free reasoning. Examples of the theorems we prove are that a core/index of a difunction is a bijection, and that the so-called "all or nothing" axiom used to facilitate pointwise reasoning is derivable from our axiom of choice. We reformulate and generalise a number of theorems originally due to Riguet on polar coverings of a relation. We study the properties of the "diagonal" of a relation (called the "différence" by Riguet who introduced the concept in 1951). In particular, we formulate and prove a general theorem relating properties of the diagonal of a relation to block-ordered relations; the theorem generalises a property that Riguet called an "analogie frappante" between the "différence" of a relation and "relations de Ferrers" (a special case of block-ordered relations).

Contents

1 Introduction

Seventy years ago, in a series of publications [Rig48, Rig50, Rig51], Jacques Riguet introduced the notions of a "relation difonctionelle", the "différence" of a relation and "relations de Ferrers". In the case of finite relations, he provided an informal mental picture of a "relation de Ferrers" in the form of a staircase-like structure. But his formal definition of a "relation de Ferrers" bears little or no resemblance to the mental picture and it is difficult to see how the formal corresponds to the informal. The name "relation de Ferrers" also gives little clue as to the practical relevance of the notion. Riguet's definitions, particularly of the "différence" of a relation, use (in our view) over-complicated and outdated formulae involving nested complements that are better formulated using the factor operators (aka division or residual operators). Riguet also relies heavily on natural language justifications of important properties as well as asserting several properties without proof. More recent publications, some of which do not cite Riguet but which copy his definitions, introduce errors by failing to recognise the restrictions that Riguet made clear in his account of the properties of the notions.

The writing of this paper initially began as an exercise in applying modern calculational reasoning to bring Riguet's work up to date and more accessible to a wider audience. In view of the extant errors in relatively recent publications and to try to avoid introducing yet more errors, we decided to include full details of all proofs. In the process, we decided that some changes in terminology were desirable: we call the "différence" of a relation the "diagonal" of the relation and we call "relations de Ferrers" "staircase" relations. We also realised that certain generalisations of Riguet's work were desirable, the primary one being from "staircase" relations to "block-ordered relations": the property of being a "staircase" relation demands a certain total ordering on "blocks" ("rectangles totalement ordonnées par inclusion" [Rig51]), being "block-ordered" does not require the ordering to be total.

As this work continued, we began to realise that substantial improvements could be made by introducing the notion of the "core" of a relation, drawing inspiration from Voerman's [Voe99] notion of the (left- and right-) per domains of a relation. The results were documented by Backhouse in [Bac21].

A significant disadvantage of the general notion is that a "core" of a relation typically has a type that is different from the type of the relation itself. (The complications this involves is particularly evident when one's subject of interest is homogeneous relations because it forces one to introduce type judgements.) Voermans suggested that the notion of "core" could be better replaced by the notion of an "index" of the relation, with the property that an index of relation R is a subset of R, and thus has the same type as R. Here is a simple example.

Example 1 Fig. 1 depicts a relation (on the left) and two instances of cores of the relation (in the middle and on the right). All are depicted as bipartite graphs. The relation R is a relation on blue and red nodes. The middle figure depicts a core as a relation on squares of blue nodes and squares of red nodes, each square being an equivalence class of the left per-domain of R (on the left) or of the right per-domain of R (on the right). The rightmost figure depicts a core as a relation on representatives of the equivalence classes: the relation depicted by the thick green edges. The rightmost figure also depicts an index of the relation; the middle does not: although the relations depicted in the middle and rightmost figures are isomorphic, they have different types.

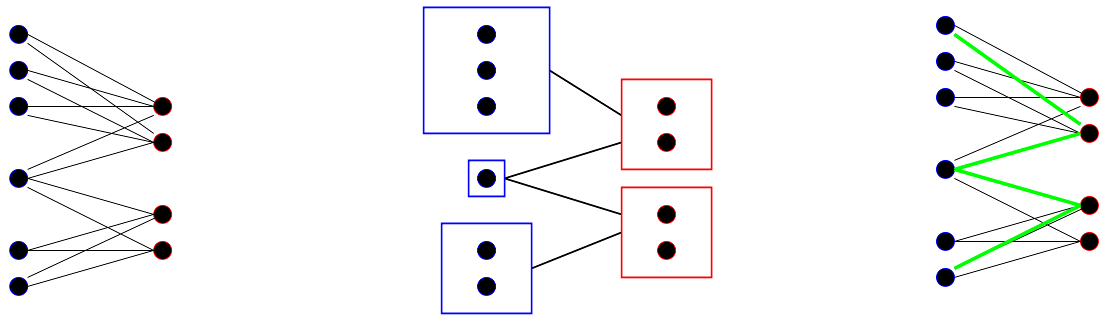


Figure 1: A Relation, a Core and an Index.

□

Further (joint) work led us to fresh insights on relation algebra, in particular on point-free versus pointwise relation algebra, which we report on in this paper.

The paper is divided into three parts. In the first part, we introduce the notions of a "core" and an "index" of a relation in the context of point-free relation algebra. We establish a large collection of properties of these notions which form a basis for parts II and III of the paper. (Because the notions are new, almost all the properties are new. An example of a property that some readers may recognise, albeit expressed differently, is that a difunction has an index that is a bijection.) Part I concludes by the introduction of a restricted form of the axiom of choice: we postulate that every partial equivalence relation has an index. This is the same as saying that it is possible to choose a representative element of every equivalence class of a partial equivalence relation.

Part II examines the consequences of adding our axiom of choice to point-free relation algebra in order to facilitate pointwise reasoning. We show that so doing has surprising and remarkable consequences. One such consequence is that we can derive the so-called "all-or-nothing" rule; this is a rule introduced by Glück [Glü17] also as a means of facilitating pointwise reasoning. (See [BDGv22] for examples of how the rule is used in reasoning about graphs.) The main theorem in part II is that, with the addition of our axiom of choice, the type $A{\sim}B$ of relations is isomorphic to the powerset $2^{A\times B}$ (the set of subsets of the cartesian product of A and B).

Part III applies the results of part II to revise and generalise Riguet's earlier work. We show, for example, that any relation is "covered" by a collection of rectangles with very special properties. (Riguet [Rig51] showed how to construct a ""réunion" of "rectangles" but only for the case of "rectangles de Ferrers".)

A novel result in part III is a generalisation of Riguet's "analogie frappante" between difunctions and "relations de Ferrers". We introduce the notion of a "block-ordered relation" and formulate and prove a theorem which allows one to determine whether or not a given relation is block-ordered by analysing the relation's "diagonal" (its 'différence" in Riguet's terminology). We call the theorem the "analogie frappante" in recognition of Riguet's pioneering insights. Several other properties of the "diagonal", which we believe to be novel, are also presented.

Part I
Point-free Relation Algebra

2 Axiomatisation

In traditional, pointwise reasoning about relations, it is not the relations themselves that are the focus of interest. Rather, a relation R of type $A{\sim}B$ is defined to be a subset of the cartesian product $A{\times}B$ and the focus of interest is the boolean membership property $(a,b){\in}R$ where a and b are elements of type A and B, respectively. Equality of relations R and S is defined in terms of membership (typically in terms of "if and only if"), leading to a lack of concision (and frequently precision). In point-free relation algebra, the membership relation plays no role, and reasoning is truly about properties of relations.

In this section, we give a brief summary of the axioms of point-free relation algebra. For full details of the axioms, see [BDGv22].

2.1 Summary

Point-free relation algebra comprises three layers with interfaces between the layers plus additional axioms peculiar to relations. (It is useful to separate the layers for use in other application areas.)

The axiom system is typed. For types A and B, $A{\sim}B$ denotes a set; the elements of the set are called *(heterogeneous) relations of type* $A{\sim}B$. Elements of type $A{\sim}A$, for some type A, are called *homogeneous relations*.

The first layer axiomatises the properties of a partially ordered set. We postulate that, for each pair of types A and B, $A{\sim}B$ forms a complete, universally distributive lattice. In anticipation of part II, where we add axioms that require $A{\sim}B$ to be a powerset, we use the symbol "$\subseteq$" for the ordering relation, and "$\cup$" and "$\cap$" for the supremum and infimum operators. We assume that this notation is familiar to the reader, allowing us to skip a more detailed account of its properties. However, we use $\perp\!\!\!\perp$ for the least element of the ordering (rather than the conventional $\emptyset$) and $\top\!\!\!\top$ for the greatest element. In keeping with the conventional practice of overloading the symbol "$\emptyset$", both these symbols are overloaded. The symbols "$\perp\!\!\!\perp$" and "$\top\!\!\!\top$" are pronounced "bottom" and "top", respectively. (Strictly we should write something like ${}_{A}\perp\!\!\!\perp_{B}$ and ${}_{A}\top\!\!\!\top_{B}$ for the bottom and top elements of type $A{\sim}B$. Of course, care needs to be taken when overloading operators in this way but it is usually the case that elementary type considerations allow the appropriate type to be deduced.)

It is important to note that there is no axiom stating that a relation is a set, and there is no corresponding notion of membership. (In, for example, [ABH+92] and [Voe99], we used the symbols "$\sqsubseteq$", "$\sqcup$" and "$\sqcap$" and the name "spec calculus" rather than "relation algebra" in order to avoid misunderstanding.) The lack of a notion of membership distinguishes point-free relation algebra from pointwise algebra.

The second layer adds a composition operator. If R is a relation of type $A{\sim}B$ and S is a relation of type $B{\sim}C$, the composition of R and S is a relation of type $A{\sim}C$ which we denote by $R{\circ}S$. Composition is associative and, for each type A, there is an identity relation which we denote by I_A. We often overload the notation for the identity relation, writing just I. Occasionally, for greater clarity, we do supply the type information.

The interface between the first and second layers defines a relation algebra to be an instance of a *regular algebra* [Bac06] (also called a *standard Kleene algebra*, or **S**-*algebra* [Con71]). For this paper, the most important aspect of this interface is the existence and properties of the factor operators. These are introduced in section 2.2. Also, $\bot\!\!\bot$ is a zero of composition: for all R, $\bot\!\!\bot{\circ}R = \bot\!\!\bot = R{\circ}\bot\!\!\bot$.

The completeness axiom in the first layer allows the reflexive-transitive closure R^* of each element R of type $A{\sim}A$, for some type A, to be defined. For practical applications, this is possibly the most important aspect of regular algebra but such applications are not considered in this paper. For this paper, completeness is only relevant when we add axioms to the algebra that model pointwise reasoning. We do require, however, the existence of $R{\cup}S$ and $R{\cap}S$, for all pairs of relations R and S of the same type, and the usual properties of set union and intersection.

The third layer is the introduction of a *converse* operator. If R is a relation of type $A{\sim}B$, the converse of R, which we denote by $R^{\cup}$ (pronounced R "wok") is a relation of type $B{\sim}A$. The interface with the first layer is that converse is a poset isomorphism (in particular, $\bot\!\!\bot^{\cup} = \bot\!\!\bot$ and $\top\!\!\top^{\cup} = \top\!\!\top$), and the interface with the second layer is formed by the two rules $I^{\cup} = I$ and, for all relations R and S of appropriate type, $(R{\circ}S)^{\cup} = S^{\cup} \circ R^{\cup}$.

Additional axioms characterise properties peculiar to relations. The modularity rule (aka Dedekind's rule [Rig48]) is that, for all relations R, S and T,

$$(2) \qquad R{\circ}S \cap T \;\subseteq\; R \circ (S \,\cap\, R^{\cup} \circ T) \quad .$$

The dual property, obtained by exploiting properties of the converse operator, is, for all relations R, S and T,

$$(3) \qquad S{\circ}R \cap T \;\subseteq\; (S \,\cap\, T \circ R^{\cup}) \circ R \quad .$$

The modularity rule is necessary to the derivation of some of the properties we state without proof (for example, the properties of the domain operators given in section 3.1). Another rule is the *cone rule*:

$$(4) \qquad \langle \forall R \;::\; \top\!\!\top{\circ}R{\circ}\top\!\!\top = \top\!\!\top \;\;\equiv\;\; R \neq \bot\!\!\bot \rangle \quad .$$

2.2 Factors

If R is a relation of type $A{\sim}B$ and S is a relation of type $A{\sim}C$, the relation $R\backslash S$ of type $B{\sim}C$ is defined by the Galois connection, for all T (of type $B{\sim}C$),

(5) $\quad T \subseteq R\backslash S \;\equiv\; R{\circ}T \subseteq S \;.$

Similarly, if R is a relation of type $A{\sim}B$ and S is a relation of type $C{\sim}B$, the relation R/S of type $A{\sim}C$ is defined by the Galois connection, for all T,

(6) $\quad T \subseteq R/S \;\equiv\; T{\circ}S \subseteq R \;.$

In relation algebra, factors are also known as "residuals". We prefer the term "factor" because it emphasises calculational properties whereas "residual" emphasises an operational understanding (what is left after taking something away). In particular, factors have the *cancellation* properties:

(7) $\quad T\circ T\backslash U \subseteq U \quad\wedge\quad R/S\circ S \subseteq R \;.$

The factor operators (which we pronounce "under" and "over" respectively) are mutually associative. That is

(8) $\quad R\backslash(S/T) \;=\; (R\backslash S)/T \;.$

This means that it is unambiguous to write $R\backslash S/T$ — which we shall do in order to promote the associativity property by making its use invisible (in the same way that the use of the associativity of composition is made invisible).

The relations $R\backslash R$ (of type $B{\sim}B$ if R has type $A{\sim}B$) and R/R (of type $A{\sim}A$ if R has type $A{\sim}B$) play a central role in what follows. As is easily verified, both are *preorders*. That is, both are *transitive*:

(9) $\quad R\backslash R\circ R\backslash R \subseteq R\backslash R \quad\wedge\quad R/R\circ R/R \subseteq R/R$

and both are *reflexive*:

(10) $\quad I \subseteq R\backslash R \quad\wedge\quad I \subseteq R/R \;.$

(The notation "I" is overloaded in the above equation. In the left conjunct, it denotes the identity relation of type $B{\sim}B$ and, in the right conjunct, it denotes the identity relation of type $A{\sim}A$, assuming R has type $A{\sim}B$. We often overload constants in this way. Note, however, that we do not attempt to combine the two inclusions into one.) In addition, for all R,

(11) $\quad R\circ R\backslash R \;=\; R \;=\; R/R\circ R \;,$

(12) $R/(R\backslash R) \;=\; R \;=\; (R/R)\backslash R$,

(13) $(R\backslash R)/(R\backslash R) \;=\; R\backslash R \;=\; (R\backslash R)\backslash(R\backslash R)$ and

(14) $(R/R)\backslash(R/R) \;=\; R/R \;=\; (R/R)/(R/R)$.

In fact, we don't use (12) directly; its relevance is as the initial step in proving the leftmost equations of (13) and (14). We choose not to exploit the associativity of the over and under operators in (13) and (14) —by writing, for example, $(R\backslash R)/(R\backslash R)$ as $R\backslash R/(R\backslash R)$— in order to emphasise their rôle as properties of the preorders $R\backslash R$ and R/R.

Properties (11) thru (14) are also called *cancellation* rules.

3 Domains

In point-free relation algebra, "coreflexives" of a given type represent sets of elements of that type. A *coreflexive of type* A is a relation p such that $p \subseteq I_A$. Frequently used properties are that, for all coreflexives p,

$$p = p^{\cup} = p{\circ}p$$

and, for all coreflexives p and q,

$$p{\circ}q \;=\; p \cap q \;=\; q{\circ}p \quad .$$

(The proof of these properties relies on the modularity rule.) In the literature, coreflexives have several different names, usually depending on the application area in question. Examples are "monotype", "pid" (short for "partial identity") and "test".

3.1 The Domain Operators

The "domain operators" (see eg. [BH93]) play a dominant and unavoidable role. We exploit their properties frequently in calculations, so much so that we assume great familiarity with them.

Definition 15 (Domain Operators) Given relation R of type $A{\sim}B$, the *left domain* $R{<}$ of R is a relation of type A defined by the equation

$$R{<} \;=\; I_A \,\cap\, R \circ R^{\cup}$$

and the *right domain* $R{>}$ of R is a relation of type B is defined by the equation

$$R{>} \;=\; I_B \,\cap\, R^{\cup} \circ R \quad .$$

□

The name "domain operator" is chosen because of the fundamental properties: for all R and all coreflexives p,

(16) $\quad R = R{\circ}p \;\equiv\; R{>} = R{>}\circ p$

and

(17) $\quad R = p{\circ}R \;\equiv\; R{<} = p\circ R{<}$.

A simple, often used consequence of (16) and (17) is the property:

(18) $\quad R{<}\circ R \;=\; R \;=\; R\circ R{>}$.

In words, $R{>}$ is the least coreflexive p such that restricting the "domain" of R on the right has no effect on R. It is in this sense that $R{<}$ and $R{>}$ represent the set of points on the left and on the right on which the relation R is "defined", i.e. its left and right "domains".

For readers unfamiliar with the domain operators, we summarise their properties below. We restrict our attention here to the right-domain operator. The reader is requested to dualise the results to the left-domain operator.

The intended interpretation of $R{>}$ (read R "right") for relation R is $\{x \mid \langle\exists y :: y[\![R]\!]x\rangle\}$. Two ways we can reformulate this requirement without recourse to points are formulated in the following theorem.

Theorem 19 (Right Domain) For all relations R and coreflexives p,

(20) $\quad R{>} \subseteq p \;\equiv\; R \subseteq \top\!\!\top{\circ}p$

and

(21) $\quad R{>} \subseteq p \;\equiv\; R = R{\circ}p$.

□

The characterisations (20) and (21) predict a number of useful calculational properties of the right domain operator. Some are immediate, some involve a little bit of work for their verification. Immediate from (20) —a Galois connection— is that the right domain operator is universally $\cup$-junctive, and $(\top\!\!\top\circ)$ is universally distributive over infima of coreflexives. In particular,

$$\top\!\!\top\circ(p\cap q) = (\top\!\!\top\circ p)\cap(\top\!\!\top\circ q) \quad ,$$

$$(R\cup S){>} = R{>}\cup S{>} \quad ,$$

and

$$\bot\!\!\bot{>} = \bot\!\!\bot \quad .$$

The last of these can in fact be strengthened to

(22) $\quad R{>} = \perp\!\!\perp \;\equiv\; R = \perp\!\!\perp \quad .$

The property is obtained by instantiating p to $\perp\!\!\perp$ in (16).

From (20) we may also deduce a number of cancellation properties. But, in combination with the modularity rule, the cancellation properties can be strengthened. We leave their proofs together with a couple of other interesting applications of Galois connections as exercises.

Theorem 23 For all relations R, S and T

(a) $\top\!\!\top \circ R{>} = \top\!\!\top \circ R \quad ,$

(b) $R \cap S \circ \top\!\!\top \circ T \;=\; S{<} \circ R \circ T{>} \quad ,$

(c) $(R^{\cup}){>} = R{<} \quad ,$

(d) $(R \cap S \circ T){>} = (S^{\cup} \circ R \cap T){>} \quad ,$

(e) $(R \circ \top\!\!\top \circ S){>} = S{>} \;\Leftarrow\; R \neq \perp\!\!\perp \quad ,$

(f) $(R \circ S){>} = (R{>} \circ S){>} \quad ,$

(g) $(R \circ S){<} = (R \circ S{<}){<} \quad ,$

□

3.2 Pers and Per Domains

Given relations R of type $A{\sim}B$ and S of type $A{\sim}C$, the symmetric *right-division* is the relation $R\backslash\!\!\backslash S$ of type $B{\sim}C$ defined in terms of *right* factors as

(24) $\quad R\backslash\!\!\backslash S \;=\; R\backslash S \cap (S\backslash R)^{\cup} \quad .$

Dually, given relations R of type $B{\sim}A$ and S of type $C{\sim}A$, the symmetric *left-division* is the relation $R/\!\!/S$ of type $B{\sim}C$ defined in terms of left factors as

(25) $\quad R/\!\!/S \;=\; R/S \cap (S/R)^{\cup} \quad .$

The relation $R\backslash\!\!\backslash R$ is an equivalence relation[1]. Voermans [Voe99] calls it the "greatest right domain" of R. Riguet [Rig48] calls $R\backslash\!\!\backslash R$ the "noyau" of R (but defines it using nested complements). Others (see [Oli18] for references) call it the "kernel" of R.

[1]This is a well-known fact: the relation $R\backslash\!\!\backslash R$ is the symmetric closure of the preorder $R\backslash R$. The easy proof is left to the reader.

As remarked elsewhere [Oli18], the *symmetric left-division* inherits a number of (left) cancellation properties from the properties of factorisation in terms of which it is defined. For our purposes, the only cancellation property we use is the following (inherited from the property $R \circ R\backslash R = R$). For all R,

(26) $\quad R \circ R\backslash\!\backslash R = R \quad .$

In this section the focus is on the left and right "per-domains" introduced by Voermans [Voe99].

Definition 27 (Right and Left Per Domains) The *right per-domain* of relation R, denoted $R\succ$, is defined by the equation

(28) $\quad R\succ \;=\; R{>} \circ R\backslash\!\backslash R \quad .$

Dually, the *left per-domain* of relation R, denoted $R\prec$, is defined by the equation

(29) $\quad R\prec \;=\; R/\!\!/R \circ R{<} \quad .$

□

The left and right per-domains are "pers" where "per" is an abbreviation of "partial equivalence relation".

Definition 30 (Partial Equivalence Relation (per)) A relation is a *partial equivalence relation* iff it is symmetric and transitive. That is, R is a partial equivalence relation iff

$$R = R^{\cup} \;\wedge\; R \circ R \subseteq R \quad .$$

Henceforth we abbreviate partial equivalence relation to *per*.
□

That $R\prec$ and $R\succ$ are indeed pers is a direct consequence of the symmetry and transitivity of $R\backslash\!\backslash R$.

The left and right per-domains are called "domains" because, like the coreflexive domains, we have the properties: for all pers P,

(31) $\quad R = R \circ P \;\equiv\; R\succ = R\succ \circ P$

and

(32) $\quad R = P \circ R \;\equiv\; R\prec = P \circ R\prec \quad .$

As with the coreflexive domains, we also have:

(33) $\quad R\prec \circ R \;=\; R \;=\; R \circ R\succ \quad .$

(The second of these equalities is an immediate consequence of (26) and the properties of (coreflexive-) domains; the first is symmetric.)

Indeed, $R{\prec}$ and $R{\succ}$ are the “least” pers that satisfy the equalities (33). (See [Voe99] for details of the ordering relation on pers.)

In order to prove additional properties, it is useful to record the left and right domains of the relation $R\backslash\!\backslash R \circ R{>}$.

Lemma 34 For all R,

$$(R\backslash\!\backslash R \circ R{>}){>} \;=\; R{>} \;=\; (R{>} \circ R\backslash\!\backslash R){<} \quad ,$$

$$(R\backslash\!\backslash R \circ R{>}){<} \;=\; R{>} \;=\; (R{>} \circ R\backslash\!\backslash R){>} \quad ,$$

$$R\backslash\!\backslash R \circ R{>} \;=\; R{>} \circ R\backslash\!\backslash R \circ R{>} \;=\; R{>} \circ R\backslash\!\backslash R \quad .$$

□

Lemma 34 has the consequence that $R{\succ}$ can be defined equivalently by the equation

(35) $\quad R{\succ} \;=\; R\backslash\!\backslash R \circ R{>}$

and, moreover,

(36) $\quad (R{\succ}){<} \;=\; R{>} \;=\; (R{\succ}){>} \quad .$

Symmetrical properties hold of $R{\prec}$.

A property that we need later is

Lemma 37 For all relations R,

$$R{>} \circ R\backslash R = R{\succ} \circ R\backslash R \quad .$$

Proof By anti-symmetry of the subset relation:

$$\begin{array}{ll}
 & R\backslash R \circ R{\succ} \\
\subseteq & \{ \quad \text{by (24), (35) and monotonicity, } R{\succ} \subseteq R\backslash R \circ R{>} \quad \} \\
 & R\backslash R \circ R\backslash R \circ R{>} \\
\subseteq & \{ \quad \text{by cancellation, } R\backslash R \circ R\backslash R \subseteq R\backslash R \quad \} \\
 & R\backslash R \circ R{>} \\
\subseteq & \{ \quad I \subseteq R\backslash\!\backslash R\text{, so by (35) and montonicity, } R{>} \subseteq R{\succ} \quad \} \\
 & R\backslash R \circ R{\succ} \quad .
\end{array}$$

□

The following lemma extends [Rig48, Corollaire, p.134] from equivalence relations to pers.

Lemma 38 For all relations R, the following statements are all equivalent.

(i) R is a per (i.e. $R = R^{\cup} \wedge R \circ R \subseteq R$) ,

(ii) $R = R^{\cup} \circ R$,

(iii) $R = R\prec$,

(iv) $R = R\succ$.

□

For further properties of pers and per-domains, see [Voe99].

3.3 Functionality

In this section, we present a number of lesser-known properties of "functional" relations. A relation R of type $A \sim B$ is said to be *left-functional* iff $R \circ R^{\cup} = R<$. Equivalently, R is *left-functional* iff $R \circ R^{\cup} \subseteq I_A$. It is said to be *right-functional* iff $R^{\cup} \circ R = R>$ (equivalently, $R^{\cup} \circ R \subseteq I_B$). A relation R is said to be a *bijection* iff it is both left- and right-functional.

Rather than left- and right-functional, the more common terminology is "functional" and "injective" but publications differ on which of left- or right-functional is "functional" or "injective". We choose to abbreviate "left-functional" to *functional* and to use the term *injective* instead of right-functional. Typically, we use f and g to denote functional relations, and Greek letters to denote bijections (although the latter is not always the case). Other authors make the opposite choice.

The properties we present here stem from the observation that functionality can be defined via a Galois connection. Specifically, the relation f is (left-)functional iff, for all relations R and S (of appropriate type),

$$(39) \qquad f \circ R \subseteq S \quad \equiv \quad f> \circ R \subseteq f^{\cup} \circ S \ .$$

It is a simple exercise to show that (39) is equivalent to the property $f \circ f^{\cup} \subseteq I$. (Although (39) doesn't immediately fit the standard definition of a Galois connection, it can be turned into standard form by restricting the range of the dummy R to relations that satisfy $f> \circ R = R$, i.e. relations R such that $R< \subseteq f>$.)

The converse-dual of (39) is also used frequently: g is functional iff, for all R and S,

(40) $\quad R \circ g^{\cup} \subseteq S \quad \equiv \quad R \circ g{>} \subseteq S \circ g \quad .$

Comparing the Galois connections defining the over and under operators with the Galois connection defining functionality (see (39)) suggests a formal relationship between "division" by a functional relation and composition with the relation's converse. The precise form of this relationship is given by the following lemma.

Lemma 41 For all R and all functional relations f,

$$f{>} \circ f \backslash R \;=\; f^{\cup} \circ R \quad .$$

Proof We use the anti-symmetry of the subset relation. First,

$$
\begin{array}{ll}
 & f^{\cup} \circ R \;\subseteq\; f{>} \circ f \backslash R \\
= & \{ \quad \text{domains} \quad \} \\
 & f{>} \circ f^{\cup} \circ R \;\subseteq\; f{>} \circ f \backslash R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & f^{\cup} \circ R \;\subseteq\; f \backslash R \\
= & \{ \quad \text{factors} \quad \} \\
 & f \circ f^{\cup} \circ R \;\subseteq\; R \\
\Leftarrow & \{ \quad \text{definition and monotonicity} \quad \} \\
 & f \text{ is functional} \quad .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & f{>} \circ f \backslash R \;\subseteq\; f^{\cup} \circ R \\
\Leftarrow & \{ \quad f{>} \subseteq f^{\cup} \circ f \,;\ \text{monotonicity and transitivity} \quad \} \\
 & f^{\cup} \circ f \circ f \backslash R \;\subseteq\; f^{\cup} \circ R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & f \circ f \backslash R \;\subseteq\; R \\
= & \{ \quad \text{cancellation} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

□

Two lemmas that will be needed later now follow. Lemma 42 allows the converse of a functional relation (i.e. an injective relation) to be cancelled, whilst lemma 43 expresses a distributivity property.

Lemma 42 For all R and all functional relations f,

$$f< \circ f^{\cup} \backslash (f^{\cup} \circ R) \quad = \quad f< \circ R \quad .$$

Proof

$$
\begin{array}{ll}
 & f< \circ f^{\cup} \backslash (f^{\cup} \circ R) \\
= & \{ \quad \text{assumption: } f \text{ is functional} \quad \} \\
 & f \circ f^{\cup} \circ f^{\cup} \backslash (f^{\cup} \circ R) \\
\subseteq & \{ \quad \text{cancellation} \quad \} \\
 & f \circ f^{\cup} \circ R \\
= & \{ \quad \text{assumption: } f \text{ is functional} \quad \} \\
 & f< \circ R \quad .
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & f< \circ R \subseteq f< \circ f^{\cup} \backslash (f^{\cup} \circ R) \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R \subseteq f^{\cup} \backslash (f^{\cup} \circ R) \\
= & \{ \quad \text{factors} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

The lemma follows by anti-symmetry of the subset relation.

□

Lemma 43 For all R and S and all functional relations f,

$$R \backslash (S \circ f) \circ f> \quad = \quad R \backslash S \circ f \quad .$$

Proof

$$
\begin{array}{ll}
 & R\backslash(S{\circ}f)\circ f{>} \;\subseteq\; R\backslash S\circ f \\
\Leftarrow & \{ \quad f{>} \subseteq f^{\cup}\circ f \text{ , monotonicity} \quad \} \\
 & R\backslash(S{\circ}f)\circ f^{\cup} \;\subseteq\; R\backslash S \\
= & \{ \quad \text{factors} \quad \} \\
 & R\circ R\backslash(S{\circ}f)\circ f^{\cup} \;\subseteq\; S \\
\Leftarrow & \{ \quad \text{cancellation} \quad \} \\
 & S\circ f\circ f^{\cup} \;\subseteq\; S \\
= & \{ \quad \text{assumption: } f \text{ is functional} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & R\backslash S\circ f \;\subseteq\; R\backslash(S{\circ}f)\circ f{>} \\
\Leftarrow & \{ \quad \text{monotonicity, } f = f\circ f{>} \quad \} \\
 & R\backslash S\circ f \;\subseteq\; R\backslash(S{\circ}f) \\
= & \{ \quad \text{factors and cancellation} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

The lemma follows by anti-symmetry of the subset relation.

□

The following lemma is crucial to fully understanding Riguet's "analogie frappante"; see lemma 221. (The lemma is complicated by the fact that it has five free variables. Simpler, possibly better known, instances can be obtained by instantiating one or more of f, g, U and W to the identity relation.)

Lemma 44 Suppose f and g are functional. Then, for all U, V and W,

$$
\begin{array}{ll}
 & f^{\cup}\circ(g{<}\circ U)\backslash V/(W\circ f{<})\circ g \\
= & f{>}\circ(g^{\cup}\circ U\circ f)\backslash(g^{\cup}\circ V\circ f)/(g^{\cup}\circ W\circ f)\circ g{>} \ .
\end{array}
$$

Proof Guided by the assumed functionality of f and g, we use the rule of indirect equality. Specifically, we have, for all R, U, V and W,

$$
\begin{array}{ll}
 & f{>}\circ R\circ g{>} \;\subseteq\; f^{\cup}\circ(g{<}\circ U)\backslash V/(W\circ f{<})\circ g \\
= & \{ \quad \text{assumption: } f \text{ and } g \text{ are functional, (39) and (40)} \quad \} \\
 & f\circ R\circ g^{\cup} \;\subseteq\; (g{<}\circ U)\backslash V/(W\circ f{<})
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{factors} \quad \} \\
& g{<} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f{<} \ \subseteq \ V \\
= & \{ \quad \text{assumption: } f \text{ and } g \text{ are functional} \\
& \quad\quad \text{i.e.} \quad f \circ f^{\cup} = f{<} \wedge g \circ g^{\cup} = g{<} \quad \} \\
& g \circ g^{\cup} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f \circ f^{\cup} \ \subseteq \ V \\
= & \{ \quad \text{assumption: } f \text{ and } g \text{ are functional, (39) and (40)} \quad \} \\
& g{>} \circ g^{\cup} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f \circ f{>} \ \subseteq \ g^{\cup} \circ V \circ f \\
= & \{ \quad \text{domains (four times)} \quad \} \\
& g^{\cup} \circ U \circ f \circ f{>} \circ R \circ g{>} \circ g^{\cup} \circ W \circ f \ \subseteq \ g^{\cup} \circ V \circ f \\
= & \{ \quad \text{factors} \quad \} \\
& f{>} \circ R \circ g{>} \ \subseteq \ (g^{\cup} \circ U \circ f) \backslash (g^{\cup} \circ V \circ f) / (g^{\cup} \circ W \circ f) \\
= & \{ \quad f{>} \text{ and } g{>} \text{ are coreflexives} \quad \} \\
& f{>} \circ R \circ g{>} \ \subseteq \ f{>} \circ (g^{\cup} \circ U \circ f) \backslash (g^{\cup} \circ V \circ f) / (g^{\cup} \circ W \circ f) \circ g{>}
\end{array}
$$

The lemma follows by instantiating R to the left and right sides of the claimed equation, simplifying using domain calculus, and then applying the reflexivity and anti-symmetry of the subset relation.

□

The final lemma in this section anticipates the discussion of per domains in section 5.

Lemma 45 Suppose relations R, f and g are such that

$$f \circ f^{\cup} \ = \ f{<} \ = \ R{<} \quad \wedge \quad g{<} \ = \ g \circ g^{\cup} \quad .$$

Then, for all S,

$$(46) \qquad g{>} \circ (f^{\cup} \circ R \circ g) \backslash (f^{\cup} \circ S) \ = \ g^{\cup} \circ R \backslash S \quad .$$

It follows that

$$(47) \qquad g{>} \circ (f^{\cup} \circ R \circ g) \backslash (f^{\cup} \circ R \circ g) \circ g{>} \ = \ g^{\cup} \circ R \backslash R \circ g \quad .$$

Proof The proof of (46) is as follows.

$$
\begin{array}{ll}
& g{>} \circ (f^{\cup} \circ R \circ g) \backslash (f^{\cup} \circ S) \\
= & \{ \quad \text{factors:} \quad \} \\
& g{>} \circ g \backslash ((f^{\cup} \circ R) \backslash (f^{\cup} \circ S))
\end{array}
$$

$=$ { lemma 41 with $f,R := g\,,(f^{\cup}\circ R)\backslash(f^{\cup}\circ S)$ }

$g^{\cup}\circ(f^{\cup}\circ R)\backslash(f^{\cup}\circ S)$

$=$ { factors }

$g^{\cup}\circ R\backslash(f^{\cup}\backslash(f^{\cup}\circ S))$

$=$ { $[\,R\backslash S = R\backslash(R{<}\circ S)\,]$ with $R,S := R\,,\,f^{\cup}\backslash(f^{\cup}\circ S)$

assumption: $f{<} = R{<}$ }

$g^{\cup}\circ R\backslash(f{<}\circ f^{\cup}\backslash(f^{\cup}\circ S))$

$=$ { lemma 42 with $f,R := f,S$ }

$g^{\cup}\circ R\backslash(f{<}\circ S)$

$=$ { assumption: $f{<} = R{<}$, $[\,R\backslash S = R\backslash(R{<}\circ S)\,]$ }

$g^{\cup}\circ R\backslash S$.

Now we prove (47).

$g{>}\circ(f^{\cup}\circ R\circ g)\backslash(f^{\cup}\circ R\circ g)\circ g{>}$

$=$ { (46) with $S := R\circ g$ }

$g^{\cup}\circ R\backslash(R\circ g)\circ g{>}$

$=$ { lemma 43 }

$g^{\cup}\circ R\backslash R\circ g$.

□

3.4 Difunctions

Formally, relation R is *difunctional* equivales

(48) $R\circ R^{\cup}\circ R \subseteq R$.

As for pers, there are several equivalent definitions of “difunctional”. We begin with the point-free definitions:

Theorem 49 For all R, the following statements are all equivalent.

(i) R is difunctional (i.e. $R\circ R^{\cup}\circ R \subseteq R$) ,

(ii) $R = R\circ R^{\cup}\circ R$,

(iii) $R{>}\circ R\backslash R = R^{\cup}\circ R$,

(iv) $R_{\succ} = R^{\cup} \circ R$,

(v) $R/R \circ R_{<} = R \circ R^{\cup}$,

(vi) $R_{\prec} = R \circ R^{\cup}$,

(vii) $R = R \cap (R\backslash R/R)^{\cup}$.

Proof For the equivalence of (i) and (ii), we first observe that, for all R,

$$R \subseteq R \circ R^{\cup} \circ R$$

since

$$
\begin{array}{ll}
 & R \subseteq R \circ R^{\cup} \circ R \\
\Leftarrow & \{ \quad R_{>} \subseteq R^{\cup} \circ R \text{ and monotonicity} \quad \} \\
 & R = R \circ R_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

That (i) and (ii) are equivalent thus follows from the anti-symmetry of the subset relation.

Next we establish the equivalence of (i) and (iii). Again, we begin by observing a property that holds for all R, namely

(50) $R^{\cup} \circ R \supseteq R_{>} \circ R\backslash R$.

The proof is as follows:

$$
\begin{array}{ll}
 & R^{\cup} \circ R \supseteq R_{>} \circ R\backslash R \\
= & \{ \quad \text{cancellation} \quad \} \\
 & R^{\cup} \circ R \circ R\backslash R \supseteq R_{>} \circ R\backslash R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R^{\cup} \circ R \supseteq R_{>} \\
\Leftarrow & \{ \quad \text{definition 15} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

We now prove that the opposite inclusion follows from (i).

$$R^{\cup}\circ R \;\subseteq\; R_{>}\circ R\backslash R$$

$$\Leftarrow \qquad \{ \quad R_{>}\circ R^{\cup} = R^{\cup} \text{ and monotonicity} \quad \}$$

$$R^{\cup}\circ R \;\subseteq\; R\backslash R$$

$$\Leftarrow \qquad \{ \quad \text{factors} \quad \}$$

$$R\circ R^{\cup}\circ R \;\subseteq\; R \quad .$$

Thus, by anti-symmetry, (iii) follows from (i). But

$$R_{>}\circ R\backslash R \;=\; R^{\cup}\circ R$$

$$\Rightarrow \qquad \{ \quad \text{Leibniz} \quad \}$$

$$R\circ R_{>}\circ R\backslash R \;=\; R\circ R^{\cup}\circ R$$

$$= \qquad \{ \quad \text{domains} \quad \}$$

$$R\circ R\backslash R \;=\; R\circ R^{\cup}\circ R$$

$$= \qquad \{ \quad \text{cancellation} \quad \}$$

$$R \;=\; R\circ R^{\cup}\circ R \quad .$$

That is, (iii) implies (ii) which, as we have already shown, is equivalent to (i). We conclude, by mutual implication, that (iii) and (i) are equivalent.

A similar proof establishes the equivalence of (i) and (iv). Once again we begin by observing a property that holds for all R, namely

$$(51) \qquad R^{\cup}\circ R \;\supseteq\; R_{\succ} \quad .$$

We have:

$$R^{\cup}\circ R$$

$$\supseteq \qquad \{ \quad (50) \quad \}$$

$$R_{>}\circ R\backslash R$$

$$\supseteq \qquad \{ \quad R\mathbin{\backslash\!\backslash}R = R\backslash R\cap(R\backslash R)^{\cup} \quad \}$$

$$R_{>}\circ R\mathbin{\backslash\!\backslash}R$$

$$= \qquad \{ \quad \text{definition: (28)} \quad \}$$

$$R_{\succ} \quad .$$

We now prove that the opposite inclusion follows from (i).

$$R^{\cup}\circ R \;\subseteq\; R_{\succ}$$

$$
\begin{array}{ll}
 & \\
= & \{ \quad \text{definition: (28)} \quad \} \\
 & R^{\cup}\circ R \;\subseteq\; R_{>}\circ R\mathbin{\backslash\!\!\backslash} R \\
\Leftarrow & \{ \quad R_{>}\circ R^{\cup} = R^{\cup} \text{ and monotonicity} \quad \} \\
 & R^{\cup}\circ R \;\subseteq\; R\mathbin{\backslash\!\!\backslash} R \\
= & \{ \quad R^{\cup}\circ R \text{ is symmetric, } R\mathbin{\backslash\!\!\backslash} R = R\backslash R\cap(R\backslash R)^{\cup} \quad \} \\
 & R^{\cup}\circ R \;\subseteq\; R\backslash R \\
\Leftarrow & \{ \quad \text{factors} \quad \} \\
 & R\circ R^{\cup}\circ R \;\subseteq\; R \;\; .
\end{array}
$$

Thus, by anti-symmetry, (iv) follows from (i). But

$$
\begin{array}{ll}
 & R_{\succ} \;=\; R^{\cup}\circ R \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & R\circ R_{\succ} \;=\; R\circ R^{\cup}\circ R \\
= & \{ \quad \text{per domains} \quad \} \\
 & R \;=\; R\circ R^{\cup}\circ R \;\; .
\end{array}
$$

The equivalence of (i), (v) and (vi) is symmetrical.

The proof that (v) is equivalent to (48) is straightforward:

$$
\begin{array}{ll}
 & R \;=\; R\cap(R\backslash R/R)^{\cup} \\
= & \{ \quad \text{definition of infimum} \quad \} \\
 & R\subseteq(R\backslash R/R)^{\cup} \\
= & \{ \quad \text{converse and factors} \quad \} \\
 & R\circ R^{\cup}\circ R \;\subseteq\; R \;\; .
\end{array}
$$

□

The equivalence of 49(i) and 49(ii) is well-known and due to Riguet [Rig48]; the equivalence of 49(i), (iv) and (vi) is due to Voermans [Voe99]. The equivalence of 49(i), (iii) and (v) is formally stronger: a consequence is that, if R is difunctional,

(52) $\quad R_{\succ} \;=\; R_{>}\circ R\backslash R \quad\wedge\quad R_{\prec} \;=\; R/R\circ R_{<} \;\;.$

(Cf. (28).) These formulae are exploited in section 12.4. Definition (48) is the most useful when it is required to establish that a particular relation is difunctional, whereas properties 49(ii)-(vii) are more useful when it is required to exploit the fact that a particular relation is difunctional.

The combination of theorem 49 (in particular 49(ii) and 49(iv) with lemma 38 allows one to prove that a per is a symmetric difunction. (We leave the easy calculation to the reader.) This property is sometimes used to specialise properties of difunctions to properties of pers.

3.5 Provisional Orderings

There are various well-known notions of ordering: preorder, partial and linear (aka total) ordering. For our purposes all of these are too strict. So, in this section, we introduce the notion of a "provisional ordering". The adjective "provisional" has been chosen because the notion "provides" just what we need.

The standard definition of an ordering is an anti-symmetric preorder whereby a preorder is required to be reflexive and transitive. It is the reflexivity requirement that is too strict for our purposes. So, with the intention of weakening the standard definition of a preorder to requiring reflexivity of a relation over some superset of its left and right domains, we propose the following definition.

Definition 53 Suppose T is a homogeneous relation. Then T is said to be a *provisional preorder* if

$$T{<} \subseteq T \quad \wedge \quad T{>} \subseteq T \quad \wedge \quad T{\circ}T \subseteq T \quad .$$

□

Fig. 2 depicts a provisional preorder on a set of eight elements as a directed graph. The blue squares should be ignored for the moment. (See the discussion following lemma 59.) Note that the relation depicted is not a preorder because it is not reflexive: the top-right node depicts an element that is not in the left or right domain of the relation.

An immediate consequence of the definition is that the left and right domains of a provisional preorder must be equal:

Lemma 54 If T is a provisional preorder then

$$T{<} = T{>} \quad .$$

Proof Suppose T is a provisional preorder. Then

$$
\begin{array}{ll}
 & T{>} \subseteq T{<} \\
= & \{ \quad \text{domains} \quad \} \\
 & (T{>}){<} \subseteq T{<} \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \}
\end{array}
$$

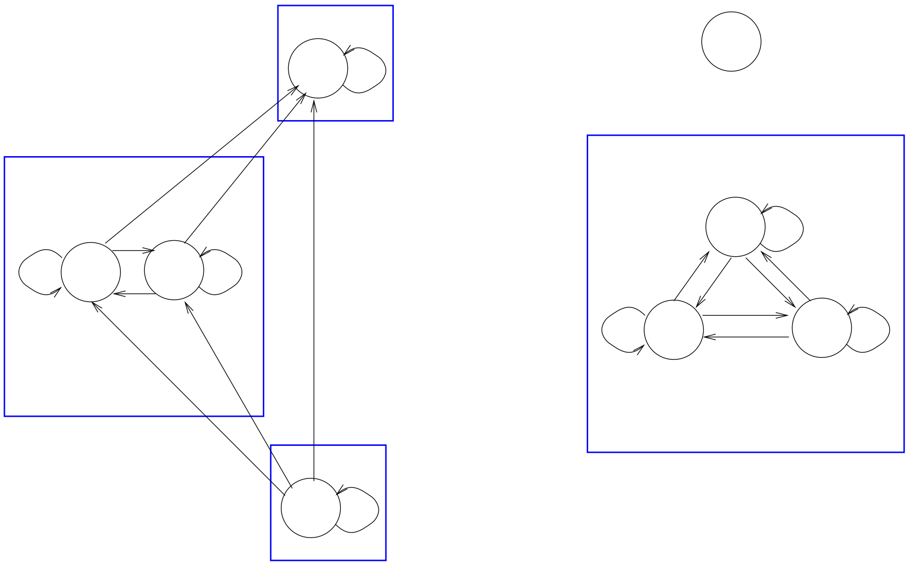


Figure 2: A Provisional Preorder

$$
\begin{array}{ll}
& T_> \subseteq T \\
= & \{ \quad \text{assumption: } T_> \subseteq T \quad \} \\
& \mathsf{true} \quad .
\end{array}
$$

That is, $T_> \subseteq T_<$. Dually, $T_< \subseteq T_>$. Thus, by anti-symmetry, $T_< = T_>$.
□

A trivial property that is nevertheless used frequently:

Lemma 55 T is a provisional preorder equivales $T^\cup$ is a provisional preorder.

Proof Immediate from the definition and properties of converse.
□

A preorder is a provisional preorder with left (equally right) domain equal to the identity relation. In other words, a preorder is a total provisional preorder. It is easy to show that, for any relation R, the relations $R\backslash R$ and R/R are preorders. It is also easy to show that R is a preorder if and only if $R = R\backslash R$ (or equivalently if and only if $R = R/R$). These properties generalise to provisional preorders.

Lemma 56 For all relations R, the relations $R_> \circ R\backslash R$ and $R/R \circ R_<$ are provisional preorders.

Proof The proof is very straightforward. First,

$$(R_> \circ R\backslash R)_<$$

$$
\begin{array}{ll}
= & \{ \quad I \subseteq R\backslash R\,,\ \text{so}\ \ (R\backslash R)_< = I\,;\ \text{domains} \quad \} \\
 & (R_>)_< \\
= & \{ \quad R_>\ \text{is a coreflexive} \quad \} \\
 & R_> \\
\subseteq & \{ \quad I \subseteq R\backslash R\,,\ \text{monotonicity} \quad \} \\
 & R_> \circ R\backslash R \ \ .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & (R_> \circ R\backslash R)_> \\
= & \{ \quad \text{domains} \quad \} \\
 & (R \circ R\backslash R)_> \\
= & \{ \quad \text{cancellation} \quad \} \\
 & R_> \\
\subseteq & \{ \quad I \subseteq R\backslash R\,,\ \text{monotonicity} \quad \} \\
 & R_> \circ R\backslash R \ \ .
\end{array}
$$

Third,

$$
\begin{array}{ll}
 & R_> \circ R\backslash R \circ R_> \circ R\backslash R \\
\subseteq & \{ \quad R_> \subseteq I\,,\ \text{monotonicity} \quad \} \\
 & R_> \circ R\backslash R \circ R\backslash R \\
\subseteq & \{ \quad R\backslash R \circ R\backslash R \subseteq R\backslash R \\
 & \quad\ \ \text{(easy use of definition of factors and cancellation)} \quad \} \\
 & R_> \circ R\backslash R \ \ .
\end{array}
$$

Comparing the above properties with definition 53, we have shown that $R_> \circ R\backslash R$ is a provisional preorder. The dual property, $R/R \circ R_<$ is a provisional preorder, is obtained by the instantiation $R := R^{\cup}$ and application of distributivity properties of converse.
□

Lemma 57 T is a provisional preorder equivales

$$
T \;=\; T_< \circ T\backslash T \;=\; T/T \circ T_> \;=\; T_< \circ T\backslash T/T \circ T_> \ \ .
$$

Proof Follows-from is a straightforward consequence of the fact that $T\backslash T$ is a preorder for arbitrary T.

Implication is also straightforward. Assume that T is a provisional preorder. The proof of the leftmost equality is by mutual inclusion. First

$$
\begin{array}{ll}
 & T \subseteq T_< \circ T\backslash T \\
\Leftarrow & \{ \quad T = T_< \circ T \text{ and monotonicity} \quad \} \\
 & T \subseteq T\backslash T \\
= & \{ \quad \text{factors} \quad \} \\
 & T \circ T \subseteq T \\
= & \{ \quad \text{assumption: } T \text{ is transitive} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

For the opposite inclusion we have

$$
\begin{array}{ll}
 & T_< \circ T\backslash T \subseteq T \\
\Leftarrow & \{ \quad \text{assumption: } T_< \subseteq T\text{, monotonicity} \quad \} \\
 & T \circ T\backslash T \subseteq T \\
= & \{ \quad \text{cancellation} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

Thus $T = T_< \circ T\backslash T$ by anti-symmetry. That $T = T/T \circ T_>$ follows from lemma 55 and the properties of converse. Finally,

$$
\begin{array}{ll}
 & T \\
= & \{ \quad T = T \circ T_> \text{ and } T = T_< \circ T\backslash T \text{ (proved above)} \quad \} \\
 & T_< \circ T\backslash T \circ T_> \\
= & \{ \quad T = T/T \circ T_> \text{ (proved above)} \quad \} \\
 & T_< \circ T\backslash(T/T \circ T_>) \circ T_> \\
= & \{ \quad [\ R\backslash(S \circ R_>) \circ R_> = R\backslash S \circ R_>\ ] \text{ with } R,S := T,T \quad \} \\
 & T_< \circ T\backslash T/T \circ T_> \quad .
\end{array}
$$

□

Lemma 57 is sometimes used in a form where the domains are replaced by per domains.

Lemma 58 Suppose T is a provisional preorder. Then

$$T \;=\; T_{\prec}\circ T\backslash T \;=\; T/T\circ T_{\succ} \;=\; T_{\prec}\circ T\backslash T/T\circ T_{\succ} \;\;.$$

Proof Immediate from lemma 57 and the per domain equations, for all R,

$$R \;=\; R_{\prec}\circ R \;=\; R_{\prec}\circ R_{<}\circ R \;=\; R\circ R_{\succ} \;=\; R\circ R_{>}\circ R_{\succ} \;\;.$$

For example,

$$T$$

$$= \quad \{ \quad [\, R \;=\; R_{\prec}\circ R \,] \text{ with } R{:=}T \quad \}$$

$$T_{\prec}\circ T$$

$$= \quad \{ \quad \text{lemma 57} \quad \}$$

$$T_{\prec}\circ T_{<}\circ T\backslash T$$

$$= \quad \{ \quad [\, R_{\prec}\circ R_{<} \;=\; R_{\prec} \,] \text{ with } R{:=}T \quad \}$$

$$T_{\prec}\circ T\backslash T \;\;.$$

□

Lemma 59 Suppose T is a provisional preorder. Then

$$T_{\prec} \;=\; T\cap T^{\cup} \;=\; T_{\succ} \;\;.$$

Hence $T\cap T^{\cup}$ is a per.

Proof We exploit lemma 57:

$$T_{\succ}$$

$$= \quad \{ \quad \text{definition: (28) and (24), lemma 34} \quad \}$$

$$T_{>}\circ (T\backslash T\cap (T\backslash T)^{\cup})\circ T_{>}$$

$$= \quad \{ \quad \text{distributivity } (T_{>} \text{ is coreflexive}) \quad \}$$

$$T_{>}\circ T\backslash T\circ T_{>} \;\;\cap\;\; (T^{\cup})_{<}\circ T^{\cup}/T^{\cup}\circ (T^{\cup})_{<}$$

$$= \quad \{ \quad \text{lemma 54} \quad \text{(twice, once with } T{:=}T^{\cup} \text{ using lemma 55)} \quad \}$$

$$T_{<}\circ T\backslash T\circ T_{>} \;\;\cap\;\; (T^{\cup})_{<}\circ T^{\cup}/T^{\cup}\circ (T^{\cup})_{>}$$

$$= \quad \{ \quad \text{lemma 57} \quad \}$$

$$T\circ T_{>} \;\cap\; (T^{\cup})_{<}\circ T^{\cup}$$

$$= \quad \{ \quad \text{domains} \quad \}$$

$$T\cap T^{\cup} \;\;.$$

The dual property $\mathsf{T}{\prec} = \mathsf{T}\cap\mathsf{T}^{\cup}$ is immediate from the properties of converse.
□

Referring back to fig. 2, the blue squares depict the equivalence classes of the symmetric closure of a provisional preorder. As remarked earlier, the depicted relation is not a preorder; correspondingly, the blue squares depict a truly *partial* equivalence relation.

We assume the reader is familiar with the notions of an ordering and a linear (or total) ordering. We now extend these notions to provisional orderings. (The at-most relation on the integers is both anti-symmetric and linear. The at-most relation restricted to some arbitrary subset of the integers is an example of a linear provisional ordering according to the definition below.)

Definition 60 Suppose T is a homogeneous relation of type $A{\sim}A$, for some A. Then T is said to be *provisionally anti-symmetric* if

$$\mathsf{T}\cap\mathsf{T}^{\cup} \subseteq \mathrm{I}_{\mathsf{A}} \quad .$$

Also, T is said to be a *provisional ordering* if T is provisionally anti-symmetric and T is a provisional preorder. Finally, T is said to be a *linear provisional ordering* if T is a provisional ordering and

$$\mathsf{T}\cup\mathsf{T}^{\cup} \;=\; (\mathsf{T}\cap\mathsf{T}^{\cup})\circ\top\!\!\top\circ(\mathsf{T}\cap\mathsf{T}^{\cup}) \quad .$$

□

Definition 60 weakens the equality in the standard notion of anti-symmetry to an inclusion. The standard definition of a partial ordering —an anti-symmetric preorder— is weakened accordingly (as mentioned earlier, in order to "provide" for our needs).

The following lemma anticipates the use of provisional preorders/orderings in examples presented later.

Lemma 61 Suppose T is a provisional ordering. Then

$$\mathsf{T}{<} \;=\; \mathsf{T}\cap\mathsf{T}^{\cup} \;=\; \mathsf{T}{>} \quad .$$

Proof For the first equality, we have

$$
\begin{array}{ll}
 & \mathsf{T}\cap\mathsf{T}^{\cup} \subseteq \mathsf{T}{<} \\
= & \{ \quad \mathrm{I} \text{ is unit of composition, definition of } \mathsf{T}{<} \quad \} \\
 & (\mathsf{T}\cap\mathsf{T}^{\cup})\circ\mathrm{I} \;\subseteq\; \mathrm{I}\cap\mathsf{T}\circ\top\!\!\top \\
= & \{ \quad \text{assumption: } \mathsf{T}\cap\mathsf{T}^{\cup} \subseteq \mathrm{I} \text{ ; infimum and monotonicity} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

Also,

$$\begin{aligned}
& \mathsf{T}_< \subseteq \mathsf{T} \cap \mathsf{T}^\cup \\
= \quad & \{ \quad \text{infimum} \quad \} \\
& \mathsf{T}_< \subseteq \mathsf{T} \ \wedge\ \mathsf{T}_< \subseteq \mathsf{T}^\cup \\
= \quad & \{ \quad \mathsf{T} \text{ is a provisional preorder, so } \mathsf{T}_< \subseteq \mathsf{T}\,;\ (\mathsf{T}_<)^\cup = \mathsf{T}_< \quad \} \\
& \mathsf{true} \quad .
\end{aligned}$$

The second equality is obtained by instantiating T to $\mathsf{T}^\cup$.

□

4 Squares and Rectangles

We now introduce the notions of a "rectangle" and a "square"; rectangles are typically heterogeneous whilst squares are, by definition, homogeneous relations. Squares are rectangles; properties of squares are typically obtained by specialising properties of rectangles. (For example, lemma 66 shows that the intersection of two rectangles is a rectangle by giving an explicit construction; the same construction applies to squares from which it is easily shown that the intersection of two squares is a square.)

Definition 62 (Rectangle and Square) A relation R is a *rectangle* iff $R = R\circ\top\!\!\top\circ R$. A relation R is a *square* iff R is a symmetric rectangle.

□

More generally, we have:

Lemma 63 For all relations R and S, $R\circ\top\!\!\top\circ S$ is a rectangle. It follows that $R\circ T\circ S$ is a rectangle if T is a rectangle.

Proof Because the proof is based on the cone rule, a case analysis is necessary. In the case that either R or S is the empty relation, the lemma clearly holds (because $R\circ\top\!\!\top\circ S$ is the empty relation, and the empty relation is a rectangle). Suppose now that both R and S are non-empty. Then

$$\begin{aligned}
& R\circ\top\!\!\top\circ S\circ\top\!\!\top\circ R\circ\top\!\!\top\circ S \\
= \quad & \{ \quad \text{cone rule: (4) (applied twice), assumption: } R \neq \perp\!\!\perp \text{ and } S \neq \perp\!\!\perp \quad \} \\
& R\circ\top\!\!\top\circ S \quad .
\end{aligned}$$

If T is a rectangle, $R\circ T\circ S = R\circ T\circ\top\!\!\top\circ T\circ S$; thus $R\circ T\circ S$ is a rectangle.

□

Lemma 64 A rectangle is a difunction and a square is a per.

Proof Suppose R is a rectangle. Then

$$
\begin{array}{ll}
 & R \circ R^{\cup} \circ R \\
= & \{ \quad \text{definition 62 (applied to outer terms)} \quad \} \\
 & R \circ \top\!\top \circ R \circ R^{\cup} \circ R \circ \top\!\top \circ R \\
\subseteq & \{ \quad \top\!\top \text{ is greatest relation, monotonicity} \quad \} \\
 & R \circ \top\!\top \circ R \\
= & \{ \quad \text{definition 62} \quad \} \\
 & R \ .
\end{array}
$$

That is, $R \circ R^{\cup} \circ R \subseteq R$. Thus, by definition, R is a difunction.

A similar calculation shows that a square is a per.

□

4.1 Inclusion and Intersection

Using colloquial terminology, the left and right domain of a rectangle are the "sides" of the rectangle. In general, a rectangle is defined by its two sides. More precisely:

Lemma 65 Suppose R and S are rectangles of the same type. Then

$$R \subseteq S \quad \equiv \quad R_< \subseteq S_< \ \wedge \ R_> \subseteq S_> \ .$$

It follows that

$$R = S \quad \equiv \quad R_< = S_< \ \wedge \ R_> = S_> \ .$$

Proof By mutual implication:

$$
\begin{array}{ll}
 & R \subseteq S \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R_< \subseteq S_< \ \wedge \ R_> \subseteq S_> \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R_< \circ \top\!\top \circ R_> \ \subseteq \ S_< \circ \top\!\top \circ S_> \\
= & \{ \quad \text{domains} \quad \} \\
 & R \circ \top\!\top \circ R \subseteq S \circ \top\!\top \circ S \\
= & \{ \quad \text{assumption: } R \text{ and } S \text{ are rectangles, definition 62} \quad \} \\
 & R \subseteq S \ .
\end{array}
$$

The second property follows straightforwardly from the anti-symmetry of the subset relation.
□

Lemma 66 The intersection of two rectangles is a rectangle. Specifically, for all rectangles R and S,

$$R\cap S \;=\; (R{<}\cap S{<})\circ\top\!\!\top\circ(R{>}\cap S{>}) \quad .$$

Proof We have, for all R, S, T and U,

$$
\begin{array}{ll}
 & R\circ\top\!\!\top\circ S \;\cap\; T\circ\top\!\!\top\circ U \\
= & \{ \quad \text{property of conditionals} \quad \} \\
 & R\circ\top\!\!\top \;\cap\; \top\!\!\top\circ S \;\cap\; T\circ\top\!\!\top \;\cap\; \top\!\!\top\circ U \\
= & \{ \quad \text{property of conditionals} \quad \} \\
 & (R\cap T)\circ\top\!\!\top \;\cap\; \top\!\!\top\circ(S\cap U) \\
= & \{ \quad \text{property of conditionals} \quad \} \\
 & (R\cap T)\circ\top\!\!\top\circ(S\cap U) \quad .
\end{array}
$$

(The properties of conditionals used above are not shown in this paper but easily proven. Hint: use the modularity rule (2).) Also, for all R and S, $R\circ\top\!\!\top\circ S = R{<}\circ\top\!\!\top\circ S{>}$. So

$$
\begin{array}{ll}
 & R\cap S \\
= & \{ \quad \text{assumption: } R \text{ and } S \text{ are rectangles} \quad \} \\
 & R\circ\top\!\!\top\circ R \cap S\circ\top\!\!\top\circ S \\
= & \{ \quad [\; R\circ\top\!\!\top\circ S = R{<}\circ\top\!\!\top\circ S{>} \;] \text{ with } R,S := R,R \text{ and } R,S := S,S \quad \} \\
 & R{<}\circ\top\!\!\top\circ R{>} \;\cap\; S{<}\circ\top\!\!\top\circ S{>} \\
= & \{ \quad \text{above with } R,S,T,U := R{<}, R{>}, S{<}, S{>} \quad \} \\
 & (R{<}\cap S{<})\circ\top\!\!\top\circ(R{>}\cap S{>}) \quad .
\end{array}
$$

□

5 Isomorphic Relations

Definition 67 Suppose R and S are two relations (not necessarily of the same type). Then we say that R and S are *isomorphic* and write $R\cong S$ iff

$$
\begin{array}{l}
\langle \exists \phi,\psi \\
\qquad : \quad \phi\circ\phi^{\cup} = R_{<} \;\wedge\; \phi^{\cup}\circ\phi = S_{<} \;\wedge\; \psi\circ\psi^{\cup} = R_{>} \;\wedge\; \psi^{\cup}\circ\psi = S_{>} \\
\qquad : \quad R = \phi\circ S\circ\psi^{\cup} \\
\rangle \quad .
\end{array}
$$

□

The relation between R and S in definition 67 can be strengthened to the conjunction

(68) $\quad R = \phi\circ S\circ\psi^{\cup} \quad\wedge\quad \phi^{\cup}\circ R\circ\psi = S \quad .$

Alternatively, the leftmost conjunct can be replaced by the rightmost conjunct. This is a consequence of the following lemma.

Lemma 69 For all ϕ, ψ, R and S,

$$
\begin{array}{ll}
 & (R = \phi\circ S\circ\psi^{\cup} \;\equiv\; \phi^{\cup}\circ R\circ\psi = S) \\
\Leftarrow & \phi\circ\phi^{\cup} = R_{<} \;\wedge\; \phi^{\cup}\circ\phi = S_{<} \;\wedge\; \psi\circ\psi^{\cup} = R_{>} \;\wedge\; \psi^{\cup}\circ\psi = S_{>} \quad .
\end{array}
$$

Proof The proof is by mutual implication.

$$
\begin{array}{ll}
 & R = \phi\circ S\circ\psi^{\cup} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & \phi^{\cup}\circ R\circ\psi \;=\; \phi^{\cup}\circ\phi\circ S\circ\psi^{\cup}\circ\psi \\
= & \{ \quad \text{assume: } \phi^{\cup}\circ\phi = S_{<} \text{ and } \psi^{\cup}\circ\psi = S_{>}\text{, domains} \quad \} \\
 & \phi^{\cup}\circ R\circ\psi \;=\; S \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & \phi\circ\phi^{\cup}\circ R\circ\psi\circ\psi^{\cup} \;=\; \phi\circ S\circ\psi^{\cup} \\
= & \{ \quad \text{ssume: } \phi\circ\phi^{\cup} = R_{<} \text{ and } \psi\circ\psi^{\cup} = R_{>}\text{, domains} \quad \} \\
 & R = \phi\circ S\circ\psi^{\cup} \quad .
\end{array}
$$

□

We often choose one or other of the conjuncts in (68), whichever being most convenient at the time.

Lemma 70 The relation $\cong$ is reflexive, transitive and symmetric. That is, $\cong$ is an equivalence relation.

Proof This is very straightforward. The details are left to the reader.
□

The task of proving that two relations are isomorphic involves constructing ϕ and ψ that satisfy the conditions of the existential quantification in definition 67; we call the constructed values *witnesses* to the isomorphism.

Note that the requirement on ϕ in definition 67 is that it is both functional and injective; thus it is required to "witness" a (1–1) correspondence between the points in the left domain of R and the points in the left domain of S. Similarly, the requirement on ψ is that it "witnesses" a (1–1) correspondence between the points in the right domain of R and the points in the right domain of S. Formally, $R{<}$ and $S{<}$ are isomorphic as "witnessed" by ϕ and $R{>}$ and $S{>}$ are isomorphic as "witnessed" by ψ:

Lemma 71 Suppose R and S are relations such that $R \cong S$. Then $R{<} \cong S{<}$ and $R{>} \cong S{>}$. Specifically, if ϕ and ψ witness the isomorphism $R \cong S$,

$$R{<} = \phi \circ S{<} \circ \phi^{\cup} \quad \wedge \quad R{>} = \psi \circ S{>} \circ \psi^{\cup} \quad .$$

Proof Suppose ϕ and ψ are such that

$$\phi \circ \phi^{\cup} = R{<} \;\wedge\; \phi^{\cup} \circ \phi = S{<} \;\wedge\; \psi \circ \psi^{\cup} = R{>} \;\wedge\; \psi^{\cup} \circ \psi = S{>} \quad .$$

Then

$$\begin{array}{ll}
 & R{<} \\
= & \{ \quad R{<} \text{ is a coreflexive} \quad \} \\
 & R{<} \circ R{<} \\
= & \{ \quad \text{assumption} \quad \} \\
 & \phi \circ \phi^{\cup} \circ \phi \circ \phi^{\cup} \\
= & \{ \quad \text{assumption} \quad \} \\
 & \phi \circ S{<} \circ \phi^{\cup} \quad .
\end{array}$$

That is $R{<} = \phi \circ S{<} \circ \phi^{\cup}$. Similarly, $R{>} = \psi \circ S{>} \circ \psi^{\cup}$. But also (because the domain operators are closure operators),

$$\phi \circ \phi^{\cup} = (R{<}){<} \wedge \phi^{\cup} \circ \phi = (S{<}){<} \wedge \psi \circ \psi^{\cup} = (R{>}){>} \wedge \psi^{\cup} \circ \psi = (S{>}){>} \quad .$$

Applying definition 67 with $R,S,\phi,\psi := R{<}, S{<}, \phi, \phi$ and $R,S,\phi,\psi := R{>}, S{>}, \psi, \psi$, the lemma is proved.
□

The property of the left and right domains stated in lemma 71 is also valid for the left and right per domains:

Lemma 72 Suppose R and S are relations such that $R \cong S$. Then $R_{\prec} \cong S_{\prec}$ and $R_{\succ} \cong S_{\succ}$. Specifically, if ϕ and ψ witness the isomorphism $R \cong S$,

$$R_{\prec} = \phi \circ S_{\prec} \circ \phi^{\cup} \quad \wedge \quad R_{\succ} = \psi \circ S_{\succ} \circ \psi^{\cup} \quad .$$

Proof Suppose ϕ and ψ witness the isomorphism $R \cong S$. Then

$$
\begin{array}{ll}
 & R_{>} \circ R \backslash R \circ R_{>} \\
= & \{ \quad \text{assumption: } \psi \circ \psi^{\cup} = R_{>} \quad \} \\
 & \psi \circ \psi^{\cup} \circ R \backslash R \circ \psi \circ \psi^{\cup} \\
= & \{ \quad (47) \text{ with } f,g := \phi,\psi \quad \} \\
 & \psi \circ \psi_{>} \circ (\phi^{\cup} \circ R \circ \psi) \backslash (\phi^{\cup} \circ R \circ \psi) \circ \psi_{>} \circ \psi^{\cup} \\
= & \{ \quad \text{domains, assumption } S = \phi^{\cup} \circ R \circ \psi \quad \} \\
 & \psi \circ S \backslash S \circ \psi^{\cup} \quad .
\end{array}
$$

So

$$
\begin{array}{ll}
 & R_{\succ} \\
= & \{ \quad \text{definition: (28)} \quad \} \\
 & R_{>} \circ R \backslash R \circ R_{>} \ \cap \ (R_{>} \circ R \backslash R \circ R_{>})^{\cup} \\
= & \{ \quad \text{above} \quad \} \\
 & \psi \circ S \backslash S \circ \psi^{\cup} \ \cap \ (\psi \circ S \backslash S \circ \psi^{\cup})^{\cup} \\
= & \{ \quad \text{assumption: } \psi_{>} = S_{>} \text{, domains} \quad \} \\
 & \psi \circ S_{>} \circ S \backslash S \circ S_{>} \circ \psi^{\cup} \ \cap \ (\psi \circ S_{>} \circ S \backslash S \circ S_{>} \circ \psi^{\cup})^{\cup} \\
= & \{ \quad \text{distributivity (} \psi \text{ is a bijection)} \quad \} \\
 & \psi \circ (S_{>} \circ S \backslash S \circ S_{>} \ \cap \ (S_{>} \circ S \backslash S \circ S_{>})^{\cup}) \circ \psi^{\cup} \\
= & \{ \quad \text{definition} \quad \} \\
 & \psi \circ S_{\succ} \circ \psi^{\cup} \quad .
\end{array}
$$

We have thus calculated that the the pair (ψ, ψ) is a candidate witness of the isomorphism $R_{\succ} \cong S_{\succ}$. It remains to check the domain requirements in definition 67. By assumption, $\psi \circ \psi^{\cup} = R_{>}$ and $\psi^{\cup} \circ \psi = S_{>}$. Moreover, for arbitrary relation R, $(R_{\succ})_{>} = (R_{\succ})_{<} = R_{>}$; so $\psi \circ \psi^{\cup} = (R_{\succ})_{>}$ and $\psi^{\cup} \circ \psi = (S_{\succ})_{>}$. Applying definition 67 with $R,S,\phi,\psi := R_{\succ}, S_{\succ}, \psi, \psi$, we have proved that $R_{\succ} \cong S_{\succ}$.

The proof that $R_{\prec} \cong S_{\prec}$ is symmetrical.

A quite different proof of lemma 72 is as follows. (It is always reasssuring to have different proofs.)

Alternative proof

**** Ed. Ik ben niet overtuigd dat dit bewijs beter is want er zitten twee konijntjes in: eerst (ψ,ψ) als witness kiezen en tweede de transitivity stap (zie beneden). Het nadeel van mijn bewijs is dat we eigenschappen zoals (47) moeten opnemen in het stuk. Voorlopig laat ik allebei blijven staan. ****

Suppose ϕ and ψ witness the isomorphism $R \cong S$. We show that the pair (ψ,ψ) witnesses the isomorphism $R_{\succ} \cong S_{\succ}$. As above, $\psi \circ \psi^{\cup} = R_{>}$, $\psi^{\cup} \circ \psi = S_{>}$, $\psi \circ \psi^{\cup} = (R_{\succ})_{>}$ and $\psi^{\cup} \circ \psi = (S_{\succ})_{>}$. So it remains to show that $R_{\succ} = \psi \circ S_{\succ} \circ \psi^{\cup}$. Now

$$
\begin{array}{ll}
 & R_{\succ} \;=\; \psi \circ S_{\succ} \circ \psi^{\cup} \\
\Leftarrow & \{ \quad \text{transitivity} \quad \} \\
 & R_{\succ} \;=\; R_{\succ} \circ \psi \circ S_{\succ} \circ \psi^{\cup} \;=\; \psi \circ S_{\succ} \circ \psi^{\cup} \quad .
\end{array}
$$

The calculation thus splits into two steps: the proof of the leftmost equality and the proof of the rightmost equality. The leftmost equality proceeds as follows.

$$
\begin{array}{ll}
 & R_{\succ} \;=\; R_{\succ} \circ \psi \circ S_{\succ} \circ \psi^{\cup} \\
= & \{ \quad (31),\ \psi \circ S_{\succ} \circ \psi^{\cup} \text{ is a per (see below)} \quad \} \\
 & R \;=\; R \circ \psi \circ S_{\succ} \circ \psi^{\cup} \quad .
\end{array}
$$

Continuing with the right hand side:

$$
\begin{array}{ll}
 & R \circ \psi \circ S_{\succ} \circ \psi^{\cup} \\
= & \{ \quad R = \phi \circ S \circ \psi^{\cup} \quad \} \\
 & \phi \circ S \circ \psi^{\cup} \circ \psi \circ S_{\succ} \circ \psi^{\cup} \\
= & \{ \quad \psi^{\cup} \circ \psi = S_{>} \text{, domains: (18) and (33)} \quad \} \\
 & \phi \circ S \circ \psi^{\cup} \\
= & \{ \quad \text{see lemma 70} \quad \} \\
 & R \quad .
\end{array}
$$

Combining the two calculations, we have established that

$$
R_{\succ} \;=\; R_{\succ} \circ \psi \circ S_{\succ} \circ \psi^{\cup} \quad .
$$

Now, for the rightmost equality, we have:

$$
\begin{array}{ll}
& R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \;=\; \psi \circ S{\succ} \circ \psi^{\cup} \\
= & \{ \quad (R{\succ}){<} = R{>} \text{, domains} \quad \} \\
& R{>} \circ R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \;=\; \psi \circ S{\succ} \circ \psi^{\cup} \\
= & \{ \quad R{>} = \psi \circ \psi^{\cup} \quad \} \\
& \psi \circ \psi^{\cup} \circ R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \;=\; \psi \circ S{\succ} \circ \psi^{\cup} \\
\Leftarrow & \{ \quad \text{Leibniz} \quad \} \\
& \psi^{\cup} \circ R{\succ} \circ \psi \circ S{\succ} \;=\; S{\succ} \\
= & \{ \quad \text{converse (noting that } R{\succ} \text{ and } S{\succ} \text{ are symmetric)} \quad \} \\
& S{\succ} \circ \psi^{\cup} \circ R{\succ} \circ \psi \;=\; S{\succ} \\
= & \{ \quad \text{(31), } \psi^{\cup} \circ R{\succ} \circ \psi \text{ is a per (see below)} \quad \} \\
& S \circ \psi^{\cup} \circ R{\succ} \circ \psi \;=\; S \\
= & \{ \quad \text{as above, with } R,S,\psi := S\,,R\,,\psi^{\cup} \quad \} \\
& \mathsf{true} \quad .
\end{array}
$$

Note that the usage of (31) relies on the fact that both $\psi \circ S{\succ} \circ \psi^{\cup}$ and $\psi^{\cup} \circ R{\succ} \circ \psi$ are pers. The straightforward proof is omitted.
□

Lemma 73 A relation R is isomorphic to a coreflexive iff R is a bijection.

Proof The proof is by mutual implication. Suppose first that R is a bijection. That is,

$$R \circ R^{\cup} = R{<} \;\;\wedge\;\; R^{\cup} \circ R = R{>} \quad .$$

We prove that R is isomorphic to $R{<}$. (Symmetrically, R is isomorphic to $R{>}$.) For the witnesses we take $R{<}$ and R. Instantiating definition 67, we have to verify that

$$R{<} \circ (R{<})^{\cup} = R{<} \;\;\wedge\;\; (R{<})^{\cup} \circ R{<} = R{<} \;\;\wedge\;\; R \circ R^{\cup} = (R{<}){>} \;\;\wedge\;\; R^{\cup} \circ R = R{>}$$

and

$$R{<} = R{<} \circ R \circ R^{\cup} \quad .$$

The verification is a straightforward application of properties of the left domain operator.

Now suppose that coreflexive p is isomorphic to R. Suppose the witnesses are ϕ and ψ. That is,

(74) $\quad \phi \circ \phi^{\cup} \;=\; p \;\;\wedge\;\; \phi^{\cup} \circ \phi = R{<} \;\;\wedge\;\; \psi^{\cup} \circ \psi = R{>}$

and

(75) $\quad p = \phi \circ R \circ \psi^{\cup}$.

Then

$$
\begin{array}{cl}
 & R_{<} \\
= & \{ \quad \phi^{\cup} \circ \phi = R_{<} = R_{<} \circ R_{<} \quad \} \\
 & \phi^{\cup} \circ \phi \circ \phi^{\cup} \circ \phi \\
= & \{ \quad \phi \circ \phi^{\cup} = p = p \circ p^{\cup} \quad \} \\
 & \phi^{\cup} \circ p \circ p^{\cup} \circ \phi \\
= & \{ \quad (75) \quad \} \\
 & \phi^{\cup} \circ \phi \circ R \circ \psi^{\cup} \circ (\phi \circ R \circ \psi^{\cup})^{\cup} \circ \phi \\
= & \{ \quad \text{converse} \quad \} \\
 & \phi^{\cup} \circ \phi \circ R \circ \psi^{\cup} \circ \psi \circ R^{\cup} \circ \phi^{\cup} \circ \phi \\
= & \{ \quad (74) \quad \} \\
 & R_{<} \circ R \circ R_{>} \circ R^{\cup} \circ R_{<} \\
= & \{ \quad \text{domains} \quad \} \\
 & R \circ R^{\cup} \quad .
\end{array}
$$

We conclude that $R_{<} = R \circ R^{\cup}$. Symmetrically, $R_{>} = R^{\cup} \circ R$. That is, R is a bijection.
$\square$

Theorem 76 Suppose P is a per. Then,

$$P_{<} = P \quad \Leftarrow \quad P_{<} \cong P \quad .$$

In particular, for all R,

$$R_{<} = R_{\prec} \quad \Leftarrow \quad R_{<} \cong R_{\prec} \quad .$$

Symmetrically, for all R,

$$R_{>} = R_{\succ} \quad \Leftarrow \quad R_{>} \cong R_{\succ} \quad .$$

Proof This is an instance of lemma 73. Specifically, assuming that $P_{<} \cong P$, we may apply the instantiation $p,R := P_{<}, P$ in lemma 73 to deduce that P is a bijection. That is, $P \circ P^{\cup} = P_{<}$. But P is a per (i.e. $P = P \circ P^{\cup}$). So we conclude that

$$P = P_{<} \quad .$$

That, for all R, $R< = R\prec$ if $R< \cong R\prec$ now follows by making the instantiation $P := R\prec$ and using the fact that $(R\prec)< = R<$. The symmetric property of the right domain operators follows by making the instantiation $P := R\succ$ and using the fact that $(R\succ)< = R>$.
□

6 Indexes and Core Relations

This section introduces the notions of "index" and "core" of a relation. An "index" is a special case of a "core" of a relation but, in general, it is more useful. The properties of both notions are explored in depth.

6.1 Indexes

Recall fig. 1. We said that the middle and rightmost figures depict "core relations". The property that is common to both is captured by the following definition.

Definition 77 (Core Relation) A relation R is a *core relation* iff $R< = R\prec$ and $R> = R\succ$.
□

The rightmost figure of fig. 1 is what we call an "index" of the relation depicted by the leftmost figure. The definition of an "index" of a relation is as follows.

Definition 78 (Index) An *index* of a relation R is a relation J that has the following properties:

(a) $J \subseteq R$,

(b) $R\prec \circ J \circ R\succ = R$,

(c) $J< \circ R\prec \circ J< = J<$,

(d) $J> \circ R\succ \circ J> = J>$.

□

Note particularly requirement 78(a). A consequence of this requirement is that an index of a relation has the same type as the relation. This means that the relation depicted by the middle figure of fig. 1 is *not* an index of the relation depicted by the leftmost figure because the relations have different types.

An obvious property is that a core relation is an index of itself:

Theorem 79 Suppose R is a core relation. Then R is an index of R.

Proof Straightforward application of definitions 77 and 78 together with the properties of (coreflexive and per) domains.
□

In general, the existence of an index of an arbitrary relation is *not* derivable in systems that axiomatise point-free relation algebra. In section 7.2 we add a limited form of the axiom of choice that guarantees the existence of indexes of arbitrary pers; we also show that this then guarantees the existence of indexes for arbitrary relations. For the moment, we establish a number of properties of indexes assuming they exist. For example, we show that all indexes of a given relation are isomorphic: see theorem 89.

Lemma 80 If J is an index of the relation R then

$$J_{\prec} \subseteq R_{\prec} \quad \wedge \quad J_{\succ} \subseteq R_{\succ} \quad .$$

It follows that

$$J_{<} = J_{\prec} \quad \wedge \quad J_{>} = J_{\succ} \quad .$$

That is, an index is a core relation.

Proof We first prove that $J_{\prec} \subseteq R_{\prec}$.

$$R_{\prec}$$

$$= \qquad \{ \qquad \text{definition} \qquad \}$$

$$R /\!\!/ R \circ R_{<}$$

$$\supseteq \qquad \{ \qquad \text{78(a) and monotonicity} \qquad \}$$

$$R /\!\!/ R \circ J_{<}$$

$$\supseteq \qquad \{ \qquad \text{see below} \qquad \}$$

$$J_{\prec} \quad .$$

The last step in the above calculation proceeds as follows.

$$J_{\prec} \;\subseteq\; R /\!\!/ R \circ J_{<}$$

$$\Leftarrow \qquad \{ \qquad (J_{\prec})_{>} = J_{<} \ \text{(so } J_{\prec} \;=\; J_{\prec} \circ J_{<}\text{) and } J_{<} \circ J_{<} = J_{<}$$

$$\text{monotonicity} \qquad \}$$

$$J_{\prec} \;\subseteq\; R /\!\!/ R$$

$$= \qquad \{ \qquad \text{definition of } R /\!\!/ R \qquad \}$$

$$J_{\prec} \;\subseteq\; R/R \cap (R/R)^{\cup}$$

$=\quad\{\quad J{\prec} = (J{\prec})^{\cup}\quad\}$

$\quad J{\prec} \subseteq R/R$

$=\quad\{\quad$ shunting $\quad\}$

$\quad J{\prec}\circ R \subseteq R$.

We continue with the lefthand side of the above inclusion.

$\quad J{\prec}\circ R$

$=\quad\{\quad$ 78(b) $\quad\}$

$\quad J{\prec}\circ R{\prec}\circ J\circ R{\succ}$

$=\quad\{\quad (J{\prec}){>} = J{<}$ and domains $\quad\}$

$\quad J{\prec}\circ J{<}\circ R{\prec}\circ J{<}\circ J\circ R{\succ}$

$=\quad\{\quad$ 78(c) $\quad\}$

$\quad J{\prec}\circ J{<}\circ J\circ R{\succ}$

$=\quad\{\quad$ (corefexive and per) domains $\quad\}$

$\quad J\circ R{\succ}$

$\subseteq\quad\{\quad$ 78(a) $\quad\}$

$\quad R\circ R{\succ}$

$=\quad\{\quad$ per domains $\quad\}$

$\quad R$.

We conclude that $J{\prec} \subseteq R{\prec}$. The equation $J{\prec} = J{<}$ uses anti-symmetry.

$\quad J{\prec}$

$\supseteq\quad\{\quad$ per domains $\quad\}$

$\quad J{<}$

$=\quad\{\quad$ 78(c) $\quad\}$

$\quad J{<}\circ R{\prec}\circ J{<}$

$\supseteq\quad\{\quad J{\prec} \subseteq R{\prec}$ (see above), composition of coreflexives is idempotent $\quad\}$

$\quad J{<}$.

The other two properties are symmetrical.

□

An immediate corollary of lemma 80 is the following theorem.

Theorem 81 If J is an index (of some relation) then J is an index of J.

Proof Suppose J is an index of R. Then we have to prove the properties 78(a), (b), (c) and (d) with $R{:=}J$. These are the properties:

(e) $J \subseteq J$,

(f) $J_{\prec} \circ J \circ J_{\succ} \;=\; J$,

(g) $J_{<} \circ J_{\prec} \circ J_{<} \;=\; J_{<}$,

(h) $J_{>} \circ J_{\succ} \circ J_{>} \;=\; J_{>}$.

Properties (e) and (f) are true of all relations J. Properties (g) and (h) follow from lemma 80 and the fact that composition of coreflexives is idempotent.
□

The indexes of a relation are uniquely defined by their left and right domains. See corollary 83, which is an immediate consequence of the following lemma.

Lemma 82 Suppose J is an index of the relation R. Then

$$J \;=\; J_{<} \circ R \circ J_{>} \;\; .$$

Proof

$$
\begin{array}{ll}
 & J \\
= & \{ \quad \text{domains} \quad \} \\
 & J_{<} \circ J \circ J_{>} \\
= & \{ \quad \text{78(c) and (d)} \quad \} \\
 & J_{<} \circ R_{\prec} \circ J_{<} \circ J \circ J_{>} \circ R_{\succ} \circ J_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & J_{<} \circ R_{\prec} \circ J \circ R_{\succ} \circ J_{>} \\
= & \{ \quad \text{78(b)} \quad \} \\
 & J_{<} \circ R \circ J_{>} \;\; .
\end{array}
$$

□

Corollary 83 Suppose J and K are both indexes of the relation R. Then

$$J = K \quad \equiv \quad J_{<} = K_{<} \;\wedge\; J_{>} = K_{>} \;\; .$$

Proof Implication is an immediate consequence of Leibniz's rule. For the "if" part, we assume that $J_< = K_<$ and $J_> = K_>$. Then

$$
\begin{array}{ll}
 & J \\
= & \{ \quad J \text{ is an index of } R\text{, lemma 82} \quad \} \\
 & J_< \circ R \circ J_> \\
= & \{ \quad \text{assumption: } J_< = K_< \ \wedge\ J_> = K_> \quad \} \\
 & K_< \circ R \circ K_> \\
= & \{ \quad K \text{ is an index of } R\text{, lemma 82 with } J := K \quad \} \\
 & K \ .
\end{array}
$$

□

The following lemma becomes relevant when we study indexes of difunctions. (See section 7.1.)

Lemma 84 Suppose J is an index of R. Then

$$R \circ J^{\cup} \circ R \;=\; R \circ R^{\cup} \circ R \ .$$

Proof

$$
\begin{array}{ll}
 & R \circ J^{\cup} \circ R \\
= & \{ \quad \text{per domains} \quad \} \\
 & R \circ R_{\succ} \circ J^{\cup} \circ R_{\prec} \circ R \\
= & \{ \quad \text{78(b) and converse} \quad \} \\
 & R \circ R^{\cup} \circ R \ .
\end{array}
$$

□

We now formulate a couple of lemmas that lead to lemma 87 which, in turn, leads to theorem 88.

Lemma 85 Suppose J is an index of R. Then $R_{\prec} \circ J_< \circ R_{\prec}$ and $R_{\succ} \circ J_> \circ R_{\succ}$ are pers.

Proof We prove that

$$R_{\prec} \circ J_< \circ R_{\prec} \;=\; R_{\prec} \circ J_< \circ R_{\prec} \circ (R_{\prec} \circ J_< \circ R_{\prec})^{\cup} \ .$$

We have:

$$R{\prec}\circ J{<}\circ R{\prec}\circ(R{\prec}\circ J{<}\circ R{\prec})^{\cup}$$

$= \quad \{ \quad R{\prec}$ is a per, $J{<}$ is a coreflexive, converse $\}$

$$R{\prec}\circ J{<}\circ R{\prec}\circ J{<}\circ R{\prec}$$

$= \quad \{ \quad 78(c) \quad \}$

$$R{\prec}\circ J{<}\circ R{\prec} \quad .$$

□

Lemma 86 Suppose J is an index of R. Then

$$(R{\prec}\circ J{<}\circ R{\prec}){<} \;=\; R{<} \quad .$$

Symmetrically,

$$(R{\succ}\circ J{>}\circ R{\succ}){>} \;=\; R{>} \quad .$$

Proof

$$(R{\prec}\circ J{<}\circ R{\prec}){<}$$

$= \quad \{ \quad$ domains, $(R{\prec}){<} \;=\; R{<} \quad \}$

$$(R{\prec}\circ J{<}\circ R{<}){<}$$

$= \quad \{ \quad$ by 78(a), $J{<}\subseteq R{<}$, domains $\}$

$$(R{\prec}\circ J){<}$$

$= \quad \{ \quad$ by 78(a), $J{>}\subseteq R{>}$, domains $\}$

$$(R{\prec}\circ J\circ R{>}){<}$$

$= \quad \{ \quad$ domains, $(R{\succ}){<} \;=\; R{>} \quad \}$

$$(R{\prec}\circ J\circ R{\succ}){<}$$

$= \quad \{ \quad 78(b) \quad \}$

$$R{<} \quad .$$

□

Lemma 87 Suppose J is an index of R. Then

(a) $R{\prec}\circ J{<}\circ R{\prec} \;=\; R{\prec}$,

(b) $R{\succ}\circ J{>}\circ R{\succ} \;=\; R{\succ}$.

Proof

$$
\begin{array}{ll}
& R_{\prec} \\
= & \{ \quad R_{\prec} \text{ is a per } \} \\
& R_{\prec} \circ R_{\prec} \circ R_{\prec} \\
\supseteq & \{ \quad R_{\prec} \supseteq R_{<} \quad \} \\
& R_{\prec} \circ R_{<} \circ R_{\prec} \\
\supseteq & \{ \quad J \text{ is an index of } R\,;\ \text{definition 78(a) and monotonicity } \} \\
& R_{\prec} \circ J_{<} \circ R_{\prec} \\
= & \{ \quad R_{\prec} \text{ is a per } \} \\
& R_{\prec} \circ J_{<} \circ R_{\prec} \circ R_{\prec} \\
\supseteq & \{ \quad \text{lemma 85: } R_{\prec} \circ J_{<} \circ R_{\prec} \text{ is a per } \} \\
& (R_{\prec} \circ J_{<} \circ R_{\prec})_{<} \circ R_{\prec} \\
= & \{ \quad \text{lemma 86 } \} \\
& R_{<} \circ R_{\prec} \\
= & \{ \quad (R_{\prec})_{<} = R_{<} \quad \} \\
& R_{\prec} \ .
\end{array}
$$

By anti-symmetry of the subset relation we have proved (a). Property (b) is symmetrical.

□

Theorem 88 Suppose J is an index of R. Then $J_{<}$ is an index of $R_{\prec}$ and $J_{>}$ is an index of $R_{\succ}$.

Proof We prove that $J_{<}$ is an index of $R_{\prec}$. That $J_{>}$ is an index of $R_{\succ}$ is symmetrical.
Instantiating definition 78 with $R,J := R_{\prec}, J_{<}$, our task is to prove the four properties:

(a) $J_{<} \subseteq R_{\prec}$,

(b) $(R_{\prec})_{\prec} \circ (J_{<})_{<} \circ (R_{\succ})_{\prec} = R_{\prec}$,

(c) $(J_{<})_{<} \circ (R_{\prec})_{\prec} \circ (J_{<})_{<} = (J_{<})_{<}$,

(d) $(J_{<})_{>} \circ (R_{\prec})_{\succ} \circ (J_{<})_{>} = (J_{<})_{>}$.

The proof of property (a) is straightforward:

$$
\begin{array}{ll}
 & J_{<} \subseteq R_{\prec} \\
\Leftarrow & \{ \quad R_{<} \subseteq R_{\prec} \text{ , transitivity} \quad \} \\
 & J_{<} \subseteq R_{<} \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & J \subseteq R \\
= & \{ \quad J \text{ is an index of } R \text{ , 78(a)} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

Property (b) simplifies using the fact that $(R_{\prec})_{\prec} = R_{\prec}$, $(R_{\succ})_{\prec} = R_{\succ}$ and $J_{<} = (J_{<})_{<}$ to:

(b') $R_{\prec} \circ J_{<} \circ R_{\succ} \;=\; R_{\prec}$,

This is the first of the two properties proved in lemma 87. Using the fact that $(R_{\prec})_{\prec} = R_{\prec}$ and $J_{<} = (J_{<})_{<}$, property (c) is the same as property (c) of definition 78; similarly, using the fact that $R_{\prec} = (R_{\prec})_{\succ}$, and $J_{<} = (J_{<})_{>}$, property (d) is also the same as property (c) of definition 78.
□

We show later that the converse of theorem 88 is a prescription for constructing an index of an arbitrary relation. See theorem 107.

Theorem 89 If R and S are isomorphic relations then indexes of R and S are also isomorphic. In particular, indexes of a relation R are isomorphic.

Proof Suppose ϕ and ψ witness the isomorphism $R \cong S$ and J is an index of R and K is an index of S . We verify that λ and ρ defined by

$$\lambda = J_{<} \circ R_{\prec} \circ \phi \circ S_{\prec} \circ K_{<} \quad \wedge \quad \rho = J_{>} \circ R_{\succ} \circ \psi \circ S_{\succ} \circ K_{>}$$

witness the isomorphism $J \cong K$.

The task is to verify that

$$J_{<} = \lambda \circ \lambda^{\cup} \quad \wedge \quad \lambda^{\cup} \circ \lambda = K_{<} \quad \wedge \quad \rho \circ \rho^{\cup} = J_{>} \quad \wedge \quad \rho^{\cup} \circ \rho = K_{>}$$

and

$$J \;=\; \lambda \circ K \circ \rho^{\cup} \quad .$$

The four domain properties are all essentially the same so we only verify the first conjunct:

$$
\begin{array}{ll}
& \lambda \circ \lambda^{\cup} \\
= & \{ \quad \text{definition, converse} \quad \} \\
& J_{<} \circ R_{\prec} \circ \phi \circ S_{\prec} \circ K_{<} \circ S_{\prec} \circ \phi^{\cup} \circ R_{\prec} \circ J_{<} \\
= & \{ \quad K \text{ is an index of } S \text{, lemma 85 with } J,R := K,S \quad \} \\
& J_{<} \circ R_{\prec} \circ \phi \circ S_{\prec} \circ \phi^{\cup} \circ R_{\prec} \circ J_{<} \\
= & \{ \quad \text{theorem 89} \quad \} \\
& J_{<} \circ R_{\prec} \circ R_{\prec} \circ R_{\prec} \circ J_{<} \\
= & \{ \quad R_{\prec} \text{ is a per, } J \text{ is an index of } R \text{, definition 78(c)} \quad \} \\
& J_{<} \quad .
\end{array}
$$

Finally,

$$
\begin{array}{ll}
& \lambda \circ K \circ \rho^{\cup} \\
= & \{ \quad \text{definition, converse} \quad \} \\
& J_{<} \circ R_{\prec} \circ \phi \circ S_{\prec} \circ K_{<} \circ K \circ K_{>} \circ S_{\succ} \circ \psi^{\cup} \circ R_{\succ} \circ J_{>} \\
= & \{ \quad \text{domains} \quad \} \\
& J_{<} \circ R_{\prec} \circ \phi \circ S_{\prec} \circ K \circ S_{\succ} \circ \psi^{\cup} \circ R_{\succ} \circ J_{>} \\
= & \{ \quad K \text{ is an index of } S \text{, definition 78(b)} \quad \} \\
& J_{<} \circ R_{\prec} \circ \phi \circ S \circ \psi^{\cup} \circ R_{\succ} \circ J_{>} \\
= & \{ \quad R = \phi \circ S \circ \psi^{\cup} \quad \} \\
& J_{<} \circ R_{\prec} \circ R \circ R_{\succ} \circ J_{>} \\
= & \{ \quad \text{per domains} \quad \} \\
& J_{<} \circ R \circ J_{>} \\
= & \{ \quad J \text{ is an index of } R \text{, definition 78(b)} \quad \} \\
& J \quad .
\end{array}
$$

That the indexes of a relation R are isomorphic follows because R is isomorphic to itself (with witnesses $R_{<}$ and $R_{>}$), i.e. the isomorphism relation is reflexive.
□

The construction of the witnesses λ and ρ looks very much like the proverbial rabbit out of a hat! In fact, they were calculated using the type judgements formulated in Voermans' thesis [Voe99]. We hope at a later date to exploit Voermans' calculus in order to make the process of constructing witnesses much more methodical.

6.2 Core Relations

Indexes are a special case of what we call “core” relations. (Recall definition 77.) This section is about the properties of a “core” of a given relation R.

Definition 90 (Core) Suppose R is an arbitrary relation and suppose C is a relation such that

$$C \;=\; \lambda \circ R \circ \rho^{\cup}$$

for some relations λ and ρ satisfying

$$R{\prec} \;=\; \lambda^{\cup} \circ \lambda \quad \wedge \quad \lambda{<} = \lambda \circ \lambda^{\cup} \quad \wedge \quad R{\succ} \;=\; \rho^{\cup} \circ \rho \quad \wedge \quad \rho{<} = \rho \circ \rho^{\cup} \quad .$$

Then C is said to be a *core of* R *as witnessed by* λ *and* ρ.
□

The existence of a core of a given relation R has a constructive element: it is necessary to construct the “witnesses” λ and ρ. In general, given a per P, a functional relation f with the property that P equals $f^{\cup} \circ f$ is called a “splitting” of P. Constructing a core of relation R thus involves “splitting ” the pers $R{\prec}$ and $R{\succ}$ into functional relations λ and ρ. As with indexes, the existence of cores is not derivable in point-free relation algebra. However, just as for indexes, all cores of a given relation are isomorphic in the sense of definition 67: see theorem 93. See section 8 for further discussion of the construction of cores of pers.

Immediately obvious is that an index of a relation is a core of the relation:

Theorem 91 Suppose R is an arbitrary relation and suppose J is an index of R. Then J is a core of R as witnessed by $J{<} \circ R{\prec}$ and $J{>} \circ R{\succ}$.

Proof First,

$$\begin{array}{ll}
 & J \\
= & \{ \quad \text{lemma 82} \quad \} \\
 & J{<} \circ R \circ J{>} \\
= & \{ \quad \text{per domains} \quad \} \\
 & J{<} \circ R{\prec} \circ R \circ R{\succ} \circ J{>} \\
= & \{ \quad \text{converse, domains} \quad \} \\
 & (J{<} \circ R{\prec}) \circ R \circ (J{>} \circ R{\succ})^{\cup} \quad .
\end{array}$$

This establishes the required property of C in definition 90, with $C{:=}J$. (The parentheses in the last line of the calculation indicate the definitions of the splittings λ and ρ.) Second,

$$
\begin{array}{ll}
 & (J_< \circ R_\prec)^\cup \circ J_< \circ R_\prec \\
= & \{ \quad \text{converse, } (R_\prec)^\cup = R_\succ \text{ and } (J_<)^\cup \circ J_< = J_< \quad \} \\
 & R_\prec \circ J_< \circ R_\succ \\
= & \{ \quad \text{lemma 87} \quad \} \\
 & R_\prec \quad .
\end{array}
$$

Third,

$$
\begin{array}{ll}
 & J_< \circ R_\prec \circ (J_< \circ R_\prec)^\cup \\
= & \{ \quad \text{converse, } (J_<)^\cup = J_< \text{ and } R_\prec \circ (R_\prec)^\cup = R_\prec \quad \} \\
 & J_< \circ R_\prec \circ J_< \\
= & \{ \quad J \text{ is an index of } R\text{, definition 78(c)} \quad \} \\
 & J_< \\
= & \{ \quad \text{theorem 88; in particular, } J_< \subseteq R_< \quad \} \\
 & (J_< \circ R_<)_< \\
= & \{ \quad (R_\prec)_< = R_<\text{, domains} \quad \} \\
 & (J_< \circ R_\prec)_< \quad .
\end{array}
$$

This establishes the required properties of λ in definition 90 (with $\lambda := J_< \circ R_\prec$). The properties of ρ in definition 90 (with $\rho := J_> \circ R_\succ$) are established similarly.
□

Fig. 3 illustrates theorem 91 applied to the relation introduced in fig. 1. The index J is depicted by the green edges in the lower bipartite graph. The decomposition of the relation in the definition of a core is illustrated by the row of bipartite graphs at the top; the relations depicted are, in order, $\lambda^\cup$, λ, R, ρ and $\rho^\cup$. The composition of the middle three figures is the index J.

A number of properties of indexes are derived from the fact that indexes are cores. The remainder of this section catalogues such properties.

The name "core" in definition 90 anticipates theorem 96 where we show that the relation C is a core relation as defined by definition 77. Some preliminary lemmas are needed first.

Lemma 92 Suppose R, C, λ and ρ are as in definition 90. Then

$$R \;=\; \lambda^\cup \circ C \circ \rho \quad .$$

Proof

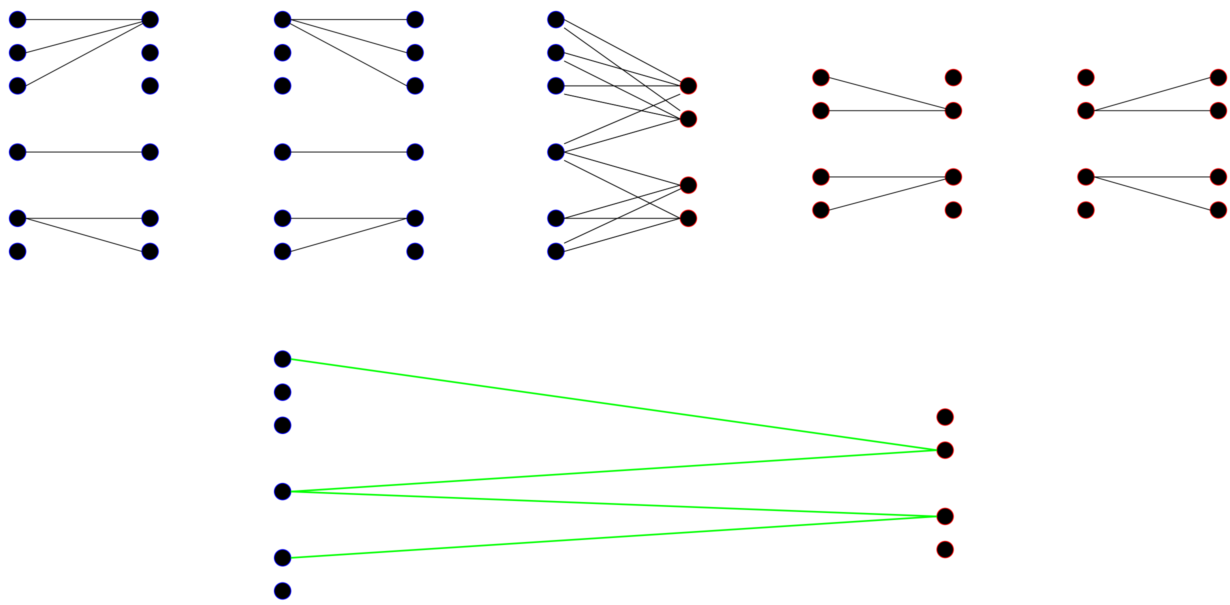


Figure 3: Decomposition of a Relation into a Core and Witnesses

$$
\begin{aligned}
& R \\
= \quad & \{ \quad \text{per domains: (33)} \quad \} \\
& R{\prec} \circ R \circ R{\succ} \\
= \quad & \{ \quad R{\prec} = \lambda^{\cup} \circ \lambda \ \text{ and } \ R{\succ} = \rho^{\cup} \circ \rho \quad \} \\
& \lambda^{\cup} \circ \lambda \circ R \circ \rho^{\cup} \circ \rho \\
= \quad & \{ \quad \text{definition 90} \quad \} \\
& \lambda^{\cup} \circ C \circ \rho \quad .
\end{aligned}
$$

□

Lemma 92 has the corollary that cores of a given relation are isomorphic:

Theorem 93 Suppose S_0 and S_1 are both cores of R. Then $S_0 \cong S_1$.

Proof Suppose, for $i = 0$ and $i = 1$, $S_i = \lambda_i \circ R \circ \rho_i^{\cup}$ where $R{\prec} = \lambda_i^{\cup} \circ \lambda_i$ and $R{\succ} = \rho_i^{\cup} \circ \rho_i$. (That is, S_0 and S_1 are both cores of R.) Then

$$
\begin{aligned}
& S_0 \\
= \quad & \{ \quad \text{assumption} \quad \} \\
& \lambda_0 \circ R \circ \rho_0^{\cup} \\
= \quad & \{ \quad \text{lemma 92} \quad \} \\
& \lambda_0 \circ \lambda_1^{\cup} \circ S_1 \circ \rho_1 \circ \rho_0^{\cup} \quad .
\end{aligned}
$$

Applying definition 67 with $f{,}g := \lambda_0 \circ \lambda_1^{\cup}$, $\rho_1 \circ \rho_0^{\cup}$ in combination with theorem 116, we conclude that $S_0 \cong S_1$.
□

For later use, we calculate the left and right domains of the core of a relation.

Lemma 94 Suppose R , λ , ρ and C are as in definition 90. Then

$$R_< = \lambda_> \;\wedge\; C_< = \lambda_< \;\wedge\; R_> = \rho_> \;\wedge\; C_> = \rho_< \quad .$$

Proof We prove the middle two equations. First,

$$
\begin{array}{ll}
& R_> \\
= & \{ \quad (36) \quad \} \\
& (R_{\succ})_< \\
= & \{ \quad \text{definition 90} \quad \} \\
& (\rho^{\cup} \circ \rho)_< \\
= & \{ \quad \text{domains} \quad \} \\
& \rho_> \quad .
\end{array}
$$

The dual equation, $R_< = \lambda_>$, is proved similarly. Second,

$$
\begin{array}{ll}
& C_< \\
= & \{ \quad \text{definition 90} \quad \} \\
& (\lambda \circ R \circ \rho^{\cup})_< \\
= & \{ \quad R_> = \rho_> \ \text{(just proved)} \quad \} \\
& (\lambda \circ R \circ R_>)_< \\
= & \{ \quad \text{domains} \quad \} \\
& (\lambda \circ R_<)_< \\
= & \{ \quad R_< = \lambda_> \ \text{(see above)} \quad \} \\
& \lambda_< \quad .
\end{array}
$$

The final equation is also proved similarly.
□

Lemma 95 Suppose R , λ , ρ and C are as in definition 90. Suppose also that J is an index of R . Then $C \cong J$ as witnessed by $\lambda \circ J_<$ and $\rho \circ J_>$.

Proof We construct the witnesses as follows.

$$
\begin{array}{ll}
 & C \\
= & \{ \quad \text{definition 90} \quad \} \\
 & \lambda \circ R \circ \rho^{\cup} \\
= & \{ \quad J \text{ is an index of } R\text{, definition 78(b)} \quad \} \\
 & \lambda \circ R_{\prec} \circ J \circ R_{\succ} \circ \rho^{\cup} \\
= & \{ \quad \text{definition 90} \quad \} \\
 & \lambda \circ \lambda^{\cup} \circ \lambda \circ J \circ \rho^{\cup} \circ \rho \circ \rho^{\cup} \\
= & \{ \quad \lambda \text{ and } \rho \text{ are functional,} \\
 & \qquad \text{so } \lambda_{<} = \lambda \circ \lambda^{\cup} \text{ and } \rho_{<} = \rho \circ \rho^{\cup} \quad \} \\
 & \lambda \circ J \circ \rho^{\cup} \\
= & \{ \quad \text{domains} \quad \} \\
 & \lambda \circ J_{<} \circ J \circ (\rho \circ J_{>})^{\cup} \quad .
\end{array}
$$

Comparing the last line with the definition of an isomorphism of relations (definition 67 with the instantiation $R,S,\phi,\psi := C\ ,\ J\ ,\ \lambda \circ J_{<}\ ,\ \rho \circ J_{>}$), we postulate $\lambda \circ J_{<}$ and $\rho \circ J_{>}$ as witnesses to the isomorphism.

It remains to show that $\lambda \circ J_{<}$ and $\rho \circ J_{>}$ are bijections on the appropriate domains. First,

$$
\begin{array}{ll}
 & (\rho \circ J_{>})^{\cup} \circ \rho \circ J_{>} \\
= & \{ \quad \text{converse} \quad \} \\
 & J_{>} \circ \rho^{\cup} \circ \rho \circ J_{>} \\
= & \{ \quad \text{definition 90} \quad \} \\
 & J_{>} \circ R_{\succ} \circ J_{>} \\
= & \{ \quad J \text{ is an index of } R\text{, definition 78(d)} \quad \} \\
 & J_{>} \quad .
\end{array}
$$

Symmetrically,

$$
(\lambda \circ J_{<})^{\cup} \circ \lambda \circ J_{<} \;=\; J_{<} \quad .
$$

Finally,

$$
\begin{array}{ll}
 & (\rho \circ J_{>})_{<} \\
= & \{ \quad \rho \text{ is functional, and } \rho^{\cup} \circ \rho = R_{\succ}\text{,}
\end{array}
$$

$$
\begin{array}{ll}
 & \qquad \text{i.e. } \rho = \rho \circ \rho^{\cup} \circ \rho = \rho \circ R_{\succ} \quad \} \\
 & (\rho \circ R_{\succ} \circ J_{>})_{<} \\
= & \quad \{ \qquad J_{>} \subseteq R_{>} \text{ and } R_{>} = (R_{\succ})_{>} \quad \} \\
 & (\rho \circ R_{\succ} \circ J_{>} \circ (R_{\succ})_{>})_{<} \\
= & \quad \{ \qquad \text{domains, } R_{\succ} = (R_{\succ})^{\cup} \quad \} \\
 & (\rho \circ R_{\succ} \circ J_{>} \circ R_{\succ})_{<} \\
= & \quad \{ \qquad \text{domains} \quad \} \\
 & (\rho \circ (R_{\succ} \circ J_{>} \circ R_{\succ})_{<})_{<} \\
= & \quad \{ \qquad \text{lemmas 85 and 86(b)} \quad \} \\
 & (\rho \circ R_{>})_{<} \\
= & \quad \{ \qquad \text{(36) and domains} \quad \} \\
 & (\rho \circ R_{\succ})_{<} \\
= & \quad \{ \qquad \rho = \rho \circ R_{\succ} \text{ (see first step)} \quad \} \\
 & \rho_{<} \\
= & \quad \{ \qquad \text{lemma 94} \quad \} \\
 & C_{>} \quad .
\end{array}
$$

Symmetrically, $(\lambda \circ J_{<})_{<} = C_{<}$.

Putting all the calculations together, we conclude that $\lambda \circ J_{<}$ and $\rho \circ J_{>}$ are bijections; the left domain of $\lambda \circ J_{<}$ is $C_{<}$ and its right domain is $J_{<}$; the left domain of $\rho \circ J_{>}$ is $C_{>}$ and its right domain is $J_{>}$.

□

We now prove the theorem alluded to by the nomenclature of definition 90, namely any core of a given relation R is a core relation in the sense of definition 77.

Theorem 96 Suppose C is a core of R. Then, if R has an index,

(97) $\quad C_{\succ} = C_{>}$, and

(98) $\quad C_{\prec} = C_{<}$.

That is, if R has an index, any core C of R is a core relation. (See definition 77.)

Proof Assume that J is an index of R. The proof is a combination of several preceding lemmas and theorems.

$$C_{\prec} \;=\; C_{<}$$

$$\Leftarrow \qquad \{ \qquad \text{theorem 76} \qquad \}$$

$$C_{\prec} \;\cong\; C_{<}$$

$$\Leftarrow \qquad \{ \qquad \text{Leibniz} \qquad \}$$

$$J_{\prec} = J_{<} \;\wedge\; C_{\prec} \cong J_{\prec} \;\wedge\; J_{<} \cong C_{<}$$

$$\Leftarrow \qquad \{ \qquad \text{index } J \text{ is a core relation (lemma 80)} \qquad \}$$

$$C_{\prec} \cong J_{\prec} \;\wedge\; J_{<} \cong C_{<}$$

$$\Leftarrow \qquad \{ \qquad \text{lemmas 72 and 71} \qquad \}$$

$$C \cong J$$

$$= \qquad \{ \qquad \text{lemma 95} \qquad \}$$

$$\text{true} \quad .$$

□

Note Theorem 96 assumes that relation R has an index J. Likewise, a corollary of lemma 95 is that, assuming relation R has an index, all cores of R are isomorphic. As mentioned earlier, it can be proven that all cores of R are isomorphic without the assumption that R has an index. Similarly, theorem 96 can be proved without this assumption but the proof is quite long and complex. See [Bac21] for full details.

We argue later that this assumption has no practical significance: in section 7.3 we show that every relation R has an index if both its per domains have an index. This means that, for a given R, it is necessary to calculate indices of $R_{\prec}$ and $R_{\succ}$; however, in practice, this is not an issue. **End of Note**

7 Indexes of Difunctions and Pers

7.1 Indexes of Difunctions

We now specialise the notion of index to difunctions.

Lemma 99 Suppose J is an index of relation R and J is difunctional. Then R is difunctional.

Proof

$$R \circ R^{\cup} \circ R$$

$$= \qquad \{ \qquad J \text{ is an index of } R\text{, lemma 84} \qquad \}$$

$$R \circ J^{\cup} \circ R$$

$$
\begin{array}{ll}
= & \{ \quad J \text{ is an index of } R\,,\ 78(\mathrm{b}) \quad \} \\
 & R\!\prec \circ J \circ R\!\succ \circ J^{\cup} \circ R\!\prec \circ J \circ R\!\succ \\
= & \{ \quad \text{domains} \quad \} \\
 & R\!\prec \circ J \circ J\!> \circ R\!\succ \circ J\!> \circ J^{\cup} \circ J\!< \circ R\!\prec \circ J\!< \circ J \circ R\!\succ \\
= & \{ \quad J \text{ is an index of } R\,,\ 78(\mathrm{d}) \text{ and } (\mathrm{c}) \quad \} \\
 & R\!\prec \circ J \circ J\!> \circ J^{\cup} \circ J\!< \circ J \circ R\!\succ \\
= & \{ \quad \text{domains and } J \text{ is difunctional (i.e. } J = J \circ J^{\cup} \circ J\,) \quad \} \\
 & R\!\prec \circ J \circ R\!\succ \\
= & \{ \quad 78(\mathrm{b}) \quad \} \\
 & R\ .
\end{array}
$$

□

The property that R is a difunction is equivalent to $R\!\prec \,=\, R \circ R^{\cup}$ (and symmetrically to $R\!\succ \,=\, R^{\cup} \circ R$). Also, since $R = R \circ R^{\cup} \circ R$, the right side of lemma 84 simplifies to R. In this way, the definition of an index of a difunction can be restated as follows.

Definition 100 (Difunction Index) An index of a difunction R is a relation J that has the following properties:

(a) $J \subseteq R$,

(b) $R \circ J^{\cup} \circ R \,=\, R$.

(c) $J\!< \circ R \circ R^{\cup} \circ J\!< \,=\, J\!<$,

(d) $J\!> \circ R^{\cup} \circ R \circ J\!> \,=\, J\!>$.

□

Lemma 101 An index J of a difunction R is a bijection between $J\!<$ and $J\!>$.

Proof

$$
\begin{array}{ll}
 & J\!< \\
= & \{ \quad 100(\mathrm{c}) \quad \} \\
 & J\!< \circ R^{\cup} \circ R \circ J\!< \\
\supseteq & \{ \quad 100(\mathrm{a}) \quad \} \\
 & J\!< \circ J^{\cup} \circ J \circ J\!<
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{domains} \quad \} \\
& J^{\cup} \circ J \\
\supseteq & \{ \quad \text{domains} \quad \} \\
& J{<} \ .
\end{array}
$$

Thus, by anti-symmetry,

$$J{<} \ = \ J^{\cup} \circ J \ .$$

Symmetrically, $J{>} \ = \ J \circ J^{\cup}$. That is, J is a bijection.
□

Corollary 102 formulates a method to determine whether a relation is a difunction: compute an index of the relation and then determine whether it is a difunction. By 78(a), the second step in this process will be no less efficient than determining difunctionality directly and, in many cases, may be substantially more efficient. (There is, however, no guarantee of improved efficiency since the inequality in 78(a) may be an equality.)

Corollary 102 Suppose J is an index of relation R. Then R is a difunction iff J is a difunction.

Proof Lemma 99 establishes “if”. Lemma 101 establishes “only if” (since a bijection is a difunction).
□

7.2 Indexes of Pers

That every difunction has an index is a desirable property but it is not provable in standard axiomatic formulations of relation algebra. Rather than postulate its truth, we shall postulate that all pers have an index, and then show that a consequence of the postulate is that all difunctions have an index.

A relation R is a per iff $R = R{\prec} = R{\succ}$. Using this property, the definition of index can be simplified for pers. Specifically, an index J of per R has the following properties. (Cf. definition 78.)

(a) $J \subseteq R$,

(b) $R \circ J \circ R = R$,

(c) $J{<} \circ R \circ J{<} \ = \ J{<}$,

(d) $J{>} \circ R \circ J{>} \ = \ J{>}$, Lemmas 103 and 104 prepare the way for definition 105.

Lemma 103 If a per has an index, then it has an index that is a coreflexive.

Proof Suppose R is a per and J is an index of R. The lemma is proved if we show that $J<$ is an index of R. We thus have to prove that

(e) $J< \subseteq R$,

(f) $R \circ J< \circ R = R$,

(g) $(J<)< \circ R \circ (J<)< = (J<)<$,

(h) $(J>)> \circ R \circ (J>)> = (J>)>$,

assuming the properties (a), (b), (c) and (d) above.

Of the four properties, only (f) is non-trivial. (Properties (g) and (h) follow because $J< = (J<)<$ and $J> = (J>)>$. Property (e) follows because, since R is a per, $R< \subseteq R$.)

Property (f) is proved as follows.

$$R \circ J< \circ R$$

$=$ { by lemma 101, $J \circ J^{\cup} = J<$ }

$$R \circ J \circ J^{\cup} \circ R$$

$=$ { domains }

$$R \circ J \circ J> \circ J^{\cup} \circ R$$

$=$ { (d) }

$$R \circ J \circ J> \circ R \circ J> \circ J^{\cup} \circ R$$

$=$ { domains }

$$R \circ J \circ R \circ J^{\cup} \circ R$$

$=$ { (b) }

$$R \circ J^{\cup} \circ R$$

$=$ { R is a per, so $R = R^{\cup}$; converse }

$$(R \circ J \circ R)^{\cup}$$

$=$ { R is a per, so $R = R^{\cup}$; (b) and converse }

$$R \ .$$

□

Lemma 104 For all pers R, if R has an index then there is a relation J such that

(a) $J \subseteq R_{<}$,

(b) $J \circ R \circ J = J$,

(c) $R \circ J \circ R = R$.

Conversely, for all pers R, if relation J satisfies the properties (a), (b) and (c) above, then J is an index of R.

Proof First, suppose R is a per that has an index. By lemma 103, R has a coreflexive index. Let J be such a coreflexive index of R. We must show that properties (a), (b) and (c) hold. We have

$$
\begin{array}{ll}
 & J \subseteq R_{<} \\
\Leftarrow & \{ \quad 78(a) \text{ and monotonicity} \quad \} \\
 & J = J_{<} \\
= & \{ \quad J \text{ is a coreflexive} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

This proves (a). Now for (b):

$$
\begin{array}{ll}
 & J \circ R \circ J \\
= & \{ \quad J \text{ is a coreflexive, so } J = J_{<} , \\
 & \quad\;\; R \text{ is a per, so } R = R_{\prec} \quad \} \\
 & J_{<} \circ R_{\prec} \circ J_{<} \\
= & \{ \quad 78(c) \quad \} \\
 & J_{<} \\
= & \{ \quad J \text{ is a coreflexive, so } J = J_{<} \quad \} \\
 & J \quad .
\end{array}
$$

Finally, (c):

$$
\begin{array}{ll}
 & R \circ J \circ R \\
= & \{ \quad R \text{ is a per, so } R = R_{\prec} \quad \} \\
 & R_{\prec} \circ J \circ R_{\prec} \\
= & \{ \quad 78(b) \quad \} \\
 & R \quad .
\end{array}
$$

For the converse, assume R is a per and J satisifies the properties (a), (b) and (c) above. We have to check the four properties listed in definition 78. First, 78(a):

$$
\begin{array}{ll}
 & J \\
\subseteq & \{ \quad \text{assumption: (a) above} \quad \} \\
 & R_{<} \\
\subseteq & \{ \quad R \text{ is per} \quad \} \\
 & R \quad .
\end{array}
$$

The properties 78(b), (c) and (d) follow because $J = J_{<} = J_{>}$ and $R = R_{\prec} = R_{\succ}$.
□

As a consequence of lemma 104, we postulate the following definition of an index of a per.

Definition 105 (Index of a Per) Suppose P is a per. Then a (*coreflexive*) *index* of P is a relation J such that

(a) $J \subseteq P_{<}$,

(b) $J \circ P \circ J = J$,

(c) $P \circ J \circ P = P$.

□

We also postulate that every per has a coreflexive index. We call this the *axiom of choice*.

Axiom 106 (Axiom of Choice) Every per has a coreflexive index.
□

7.3 From Pers To Relations

It is a desirable property that every relation has an index. However, as mentioned earlier, this can't be proved in standard relation algebra. It can be proved if we assume that every per has an index. The construction is suggested by theorem 88.

Theorem 107 Suppose J and K are (coreflexive) indices of $R_{\prec}$ and $R_{\succ}$, respectively. Then $J \circ R \circ K$ is an index of R.

Proof For convenience, we list the properties of J and K. These are obtained by instantiating definition 105 with $J,R := J, R_{\prec}$ and $J,R := K, R_{\succ}$. (Domain properties have been used to simplify (a) and (d).)

(a) $J \subseteq R_<$,

(b) $J \circ R_\prec \circ J \;=\; J$,

(c) $R_\prec \circ J \circ R_\prec \;=\; R_\prec$,

(d) $K \subseteq R_>$,

(e) $K \circ R_\succ \circ K \;=\; K$,

(f) $R_\succ \circ K \circ R_\succ \;=\; R_\succ$.

We have to prove the four properties 78(a)-(d) with the instantiation $J{,}R := J{\circ}R{\circ}K\,,R$. By (a), $J = J^\cup = J_< = J_>$. Similarly for K. The proof obligations are thus:

(g) $J{\circ}R{\circ}K \subseteq R$,

(h) $R_\prec \circ J \circ R \circ K \circ R_\succ \;=\; R$.

(i) $(J{\circ}R{\circ}K)_< \circ R_\prec \circ (J{\circ}R{\circ}K)_< \;=\; (J{\circ}R{\circ}K)_<$,

(j) $(J{\circ}R{\circ}K)_> \circ R_\succ \circ (J{\circ}R{\circ}K)_> \;=\; (J{\circ}R{\circ}K)_>$,

Property (g) is an easy combination of (a) and (d). For (h) we have:

$$
\begin{array}{ll}
 & R_\prec \circ J \circ R \circ K \circ R_\succ \\
= & \{ \quad \text{per domains} \quad \} \\
 & R_\prec \circ J \circ R_\prec \circ R \circ R_\succ \circ K \circ R_\succ \\
= & \{ \quad \text{(b) and (f)} \quad \} \\
 & R_\prec \circ R \circ R_\succ \\
= & \{ \quad \text{per domains} \quad \} \\
 & R \;\; .
\end{array}
$$

For (i), we have

$$
\begin{array}{ll}
 & (J{\circ}R{\circ}K)_> \circ R_\succ \circ (J{\circ}R{\circ}K)_> \\
= & \{ \quad (J{\circ}R{\circ}K)_> \;\subseteq\; K_> \;=\; K\,, \\
 & \quad\;\; \text{composition of coreflexives is intersection} \quad \} \\
 & (J{\circ}R{\circ}K)_> \circ K \circ R_\succ \circ K \circ (J{\circ}R{\circ}K)_> \\
= & \{ \quad \text{(e)} \quad \}
\end{array}
$$

$$
\begin{array}{lll}
 & (J{\circ}R{\circ}K){>} \circ K \circ (J{\circ}R{\circ}K){>} & \\
= & \{ \quad (J{\circ}R{\circ}K){>} \subseteq K{>} = K & \\
 & \quad\;\; \text{composition of coreflexives is intersection} & \} \\
 & (J{\circ}R{\circ}K){>} \;\; . &
\end{array}
$$

The proof is (j) is symmetrical.
□

Theorem 107 shows how to construct an index of a relation R from indexes J and K of its left and right per domains. In combination with lemma 82 and corollary 83, the construction is unique. Specifically, the steps are, first to choose from each equivalence class of $R{\prec}$ and each equivalence class of $R{\succ}$ a single representative. The collection of such representatives defines the coreflexives J and K. Then the index is defined to be $J{\circ}R{\circ}K$.

8 Characterisations of Pers and Difunctions

This section is about characterising pers and difunctions in terms of functional relations. Although the characterisations are well known, they are not derivable in point-free relation algebra. We show that they are derivable using our axiom of choice.

8.1 Characterisation of Pers

A well-known property is that a relation R is a per iff

(108) $\langle \exists f : f \circ f^{\cup} = f{<} : R = f^{\cup} \circ f \rangle$.

This property is said to be a *characteristic* property of pers. Perhaps surprisingly, it is *not* derivable in systems that axiomatise point-free relation algebra. Freyd and Ščedrov [Fv90, 1.281] call the functional f witnessing the existential quantification a "splitting[2]" of R. Typically, the existence of "splittings" is either postulated as an axiom (eg. Winter [Win04]) or by adding axioms formulating relations as a so-called "power allegory" [Fv90, 2.422], or by adding the so-called "all-or-nothing" axiom [Bac21]. (See section 9.6 for discussion of "all or nothing".) The existence of "splittings" is a consequence of our axiom of choice:

Theorem 109 If per P has a coreflexive index J , then

$$P = (J{\circ}P)^{\cup} \circ (J{\circ}P) \quad \wedge \quad J = (J{\circ}P) \circ (J{\circ}P)^{\cup} \;\; .$$

[2]Freyd and Ščedrov define a "splitting" in the more general context of a category rather than an allegory; the notion is applicable to "idempotents" which are also more general than pers.

Thus, assuming the axiom of choice, for all relations R,

$$\mathrm{per}.R \;\equiv\; \langle \exists f : f\circ f^{\cup} = f< : R = f^{\cup}\circ f\rangle \;.$$

Proof The proof is very straightforward. We have

$$
\begin{array}{ll}
 & (J\circ P)^{\cup}\circ(J\circ P) \\
= & \{ \quad \text{distributivity} \quad \} \\
 & P^{\cup}\circ J\circ J\circ P \\
= & \{ \quad J \text{ is coreflexive, so } J\circ J = J\,;\; P = P^{\cup} \quad \} \\
 & P\circ J\circ P \\
= & \{ \quad J \text{ is an index of } P\text{, definition 105(c)} \quad \} \\
 & P
\end{array}
$$

and

$$
\begin{array}{ll}
 & (J\circ P)\circ(J\circ P)^{\cup} \\
= & \{ \quad \text{distributivity} \quad \} \\
 & J\circ P\circ P^{\cup}\circ J \\
= & \{ \quad P \text{ is a per, so by lemma 38(ii), } P = P^{\cup}\circ P \quad \} \\
 & J\circ P\circ J \\
= & \{ \quad J \text{ is an index of } P\text{, definition 105(b)} \quad \} \\
 & J \;.
\end{array}
$$

This proves the first property. It also establishes that (assuming the axiom of choice), for all R,

$$\mathrm{per}.R \;\Rightarrow\; \langle \exists f : f\circ f^{\cup} = f< : R = f^{\cup}\circ f\rangle \;.$$

(The witness is $J\circ R$.) The converse is obvious because, for all f such that $f\circ f^{\cup} = f<$,

$$
\begin{array}{ll}
 & f^{\cup}\circ f\circ(f^{\cup}\circ f)^{\cup} \\
= & \{ \quad \text{converse} \quad \} \\
 & f^{\cup}\circ f\circ f^{\cup}\circ f \\
= & \{ \quad \text{assumption: } f\circ f^{\cup} = f< \quad \} \\
 & f^{\cup}\circ f<\circ f
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{domains} \quad \} \\
 & f^{\cup} \circ f \ .
\end{array}
$$

That is, by lemma 38(ii),

$$\langle \forall f : f \circ f^{\cup} = f< : \mathsf{per}.(f^{\cup} \circ f) \rangle$$

and hence

$$\mathsf{per}.R \ \Leftarrow \ \langle \exists f : f \circ f^{\cup} = f< : R = f^{\cup} \circ f \rangle \ .$$

The equivalence follows by mutual implication.
□

8.2 Characterisation of Difunctions

A second so-called "characteristic" property is that a relation R is a difunctional iff

$$\langle \exists f,g \ : \ f \circ f^{\cup} = f< = g \circ g^{\cup} = g< \ : \ R = f^{\cup} \circ g \rangle \ .$$

Like the characteristic property of pers, it is not derivable in systems that axiomatise point-free relation algebra. However, it is a corollary of theorem 109 as we now show. The basis for the construction is the construction of a per from a difunctional relation:

Lemma 110 For all relations R, $R \circ R^{\cup}$ is a per if R is difunctional.

Proof Suppose R is difunctional. We exploit lemma 38 :

$$
\begin{array}{ll}
 & R \circ R^{\cup} \ \text{is a per} \\
= & \{ \quad \text{lemma 38(ii) with } R := R \circ R^{\cup} \text{ and converse} \quad \} \\
 & R \circ R^{\cup} \ = \ R \circ R^{\cup} \circ R \circ R^{\cup} \\
\Leftarrow & \{ \quad \text{Leibniz} \quad \} \\
 & R \ = \ R \circ R^{\cup} \circ R \\
= & \{ \quad \text{theorem 49} \quad \} \\
 & R \ \text{is difunctional.}
\end{array}
$$

□

Theorem 111 Assuming the axiom of choice (axiom 106), for all relations R,

$$\mathsf{difunction}.R \ \equiv \ \langle \exists f,g : f \circ f^{\cup} = f< = g \circ g^{\cup} = g< : R = f^{\cup} \circ g \rangle \ .$$

Proof The proof is by mutual implication. First assume that $R = f^{\cup} \circ g$ where

$$f \circ f^{\cup} \;=\; f< \;=\; g \circ g^{\cup} \;=\; g< \quad .$$

Then

$$\begin{array}{ll}
 & R \circ R^{\cup} \circ R \\
= & \{ \quad \text{assumption: } R = f^{\cup} \circ g \text{ and converse} \quad \} \\
 & f^{\cup} \circ g \circ g^{\cup} \circ f \circ f^{\cup} \circ g \\
= & \{ \quad f \circ f^{\cup} \;=\; f< \;=\; g \circ g^{\cup} \;=\; g< \quad \} \\
 & f^{\cup} \circ g< \circ g< \circ g \\
= & \{ \quad \text{domains} \quad \} \\
 & f^{\cup} \circ g \\
= & \{ \quad \text{assumption: } R = f^{\cup} \circ g \quad \} \\
 & R \quad .
\end{array}$$

Applying lemma 38, we conclude that R is difunctional.

Suppose now that R is difunctional. (We owe the following construction to Winter [Win04].) Exploiting lemma 110 combined with theorem 109,

(112) $\langle \exists f \,:\, f \circ f^{\cup} \;=\; f< \,:\, R \circ R^{\cup} = f^{\cup} \circ f \rangle$.

Suppose therefore that $f \circ f^{\cup} \;=\; f<$ and $R \circ R^{\cup} = f^{\cup} \circ f$. Define the relation g by

(113) $g \;=\; f \circ R$.

Then

$$\begin{array}{ll}
 & g \circ g^{\cup} \\
= & \{ \quad (113) \text{ and converse} \quad \} \\
 & f \circ R \circ R^{\cup} \circ f^{\cup} \\
= & \{ \quad (112) \quad \} \\
 & f \circ f^{\cup} \circ f \circ f^{\cup} \\
= & \{ \quad (112) \quad \} \\
 & f< \circ f< \\
= & \{ \quad f< \text{ is a coreflexive} \quad \} \\
 & f< \quad .
\end{array}$$

It follows that $g< = g \circ g^{\cup}$. Thus

(114) $\quad f \circ f^{\cup} \;=\; f< = g< = g \circ g^{\cup} \quad .$

Moreover,

$$
\begin{array}{ll}
 & f^{\cup} \circ g \\
= & \{ \qquad (113) \quad \} \\
 & f^{\cup} \circ f \circ R \\
= & \{ \qquad R \circ R^{\cup} = f^{\cup} \circ f \quad \} \\
 & R \circ R^{\cup} \circ R \\
= & \{ \qquad R \text{ is difunctional: theorem 49} \quad \} \\
 & R \quad .
\end{array}
$$

Combined with (114), we have thus shown that

(115) $\quad \langle \exists f,g \;\; : \;\; f \circ f^{\cup} \;=\; f< \;=\; g \circ g^{\cup} \;=\; g< \;\; : \;\; R = f^{\cup} \circ g \rangle \quad .$

□

8.3 Unicity of Characterisations

The characterisation of a per in the form $f^{\cup} \circ f$ where f is a functional relation is not unique. (There are typically many representatives one can choose for each equivalence class; so there are very many distinct indexes of a per.) The characterisation is sometimes described as being "essentially" unique or sometimes as unique "up to isomorphism". This is made precise by theorem 116:

Theorem 116 Suppose R is a per and suppose f and g are functional relations such that $R \;=\; f^{\cup} \circ f \;=\; g^{\cup} \circ g$. Then $f \cong g$.

Proof We have

$$
\begin{array}{ll}
 & f \circ g^{\cup} \circ (f \circ g^{\cup})^{\cup} \\
= & \{ \qquad \text{converse} \quad \} \\
 & f \circ g^{\cup} \circ g \circ f^{\cup} \\
= & \{ \qquad \text{assumption: } f^{\cup} \circ f \;=\; g^{\cup} \circ g \quad \} \\
 & f \circ f^{\cup} \circ f \circ f^{\cup} \\
= & \{ \qquad \text{assumption: } f \text{ is functional, i.e. } f \circ f^{\cup} = f< \quad \} \\
 & f< \quad .
\end{array}
$$

That is,

(117) $f \circ g^{\cup} \circ (f \circ g^{\cup})^{\cup} \;=\; f{<}$.

Similarly,

(118) $(f \circ g^{\cup})^{\cup} \circ f \circ g^{\cup} \;=\; g{<}$.

Also,

$$
\begin{array}{ll}
 & g{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & (g^{\cup} \circ g){>} \\
= & \{ \quad \text{assumption: } f^{\cup} \circ f \;=\; g^{\cup} \circ g \quad \} \\
 & (f^{\cup} \circ f){>} \\
= & \{ \quad \text{domains} \quad \} \\
 & f{>} \;\; .
\end{array}
$$

That is,

(119) $f{>} \;=\; g{>}$.

Hence,

$$
\begin{array}{ll}
 & f \\
= & \{ \quad \text{domains} \quad \} \\
 & f{<} \circ f \\
= & \{ \quad (117) \quad \} \\
 & f \circ g^{\cup} \circ (f \circ g^{\cup})^{\cup} \circ f \\
= & \{ \quad \text{properties of converse} \quad \} \\
 & f \circ g^{\cup} \circ g \circ f^{\cup} \circ f \\
= & \{ \quad \text{assumption: } f^{\cup} \circ f \;=\; g^{\cup} \circ g \quad \} \\
 & f \circ g^{\cup} \circ g \circ g^{\cup} \circ g \\
= & \{ \quad \text{assumption: } g \text{ is functional, i.e. } g \circ g^{\cup} = g{<} \quad \} \\
 & f \circ g^{\cup} \circ g \;\; .
\end{array}
$$

Applying definition 67 with $R,S,\phi,\psi \;:=\; f\,,\,g\,,\,f\circ g^{\cup}\,,\,g{>}$, we conclude that $f\cong g$. (Properties (117) and (118) are the required properties of ϕ; property (119) together with straightforward properties of the right-domain operator establish the required properties of ψ.)

□

It is important to note that theorem 116 assumes that there is at least one characterisation of per R by a functional relation; it thus establishes that there is *at most one* such characterisation ("up to isomorphism").

Uniqueness "up to isomorphism" is a common phenomenon. The characterisation of difunctional relations is another example:

Theorem 120 Suppose f and g are relations such that

$$f\circ f^{\cup} \;=\; f{<} \;=\; g\circ g^{\cup} \;=\; g{<} \quad .$$

Suppose also that h and k are relations such that

$$h\circ h^{\cup} \;=\; h{<} \;=\; k\circ k^{\cup} \;=\; k{<} \quad .$$

Suppose further that

$$f^{\cup}\circ g \;=\; h^{\cup}\circ k \quad .$$

Then

$$f\cong h \;\;\wedge\;\; g\cong k \quad .$$

Proof Our task is to construct witnesses ϕ and ψ satisfying definition 67 (with $R,S{:=}f,h$ and $R,S{:=}g,k$). Define ϕ by $\phi = f\circ h^{\cup}$. We prove that

$$(121)\quad \phi\circ\phi^{\cup} \;=\; f{<} \;\;\wedge\;\; \phi^{\cup}\circ\phi \;=\; h{<} \quad .$$

(In words, ϕ is a bijection with left domain the common left domain of f and g, and right domain the common left domain of h and k.) The proof is as follows.

$$\begin{array}{ll}
 & \phi\circ\phi^{\cup} \\
= & \{ \quad \text{definition, converse} \quad \} \\
 & f\circ h^{\cup}\circ h\circ f^{\cup} \\
= & \{ \quad \text{assumption: } h{<} \;=\; k\circ k^{\cup} \quad \} \\
 & f\circ h^{\cup}\circ k\circ k^{\cup}\circ h\circ f^{\cup} \\
= & \{ \quad \text{assumption: } f^{\cup}\circ g \;=\; h^{\cup}\circ k \quad \} \\
 & f\circ f^{\cup}\circ g\circ g^{\cup}\circ f\circ f^{\cup} \\
= & \{ \quad \text{assumption: } f\circ f^{\cup} \;=\; f{<} \;=\; g\circ g^{\cup} \quad \} \\
 & f{<}
\end{array}$$

and

$$
\begin{array}{ll}
& \phi^{\cup}\circ\phi \\
= & \{ \quad \text{definition, converse} \quad \} \\
& h\circ f^{\cup}\circ f\circ h^{\cup} \\
= & \{ \quad \text{assumption: } f{<} \;=\; g\circ g^{\cup} \quad \} \\
& h\circ f^{\cup}\circ g\circ g^{\cup}\circ f\circ h^{\cup} \\
= & \{ \quad \text{assumption: } f^{\cup}\circ g \;=\; h^{\cup}\circ k \text{ (used twice)} \quad \} \\
& h\circ h^{\cup}\circ k\circ k^{\cup}\circ h\circ h^{\cup} \\
= & \{ \quad \text{assumption: } h\circ h^{\cup} \;=\; h{<} \;=\; k\circ k^{\cup} \quad \} \\
& h{<} \quad .
\end{array}
$$

We now prove that $f=\phi\circ h$.

$$
\begin{array}{ll}
& \phi\circ h \\
= & \{ \quad \text{definition} \quad \} \\
& f\circ h^{\cup}\circ h \\
= & \{ \quad \text{assumption: } h{<} \;=\; k\circ k^{\cup} \quad \} \\
& f\circ h^{\cup}\circ k\circ k^{\cup}\circ h \\
= & \{ \quad \text{assumption: } f^{\cup}\circ g \;=\; h^{\cup}\circ k \text{ (used twice)} \quad \} \\
& f\circ f^{\cup}\circ g\circ g^{\cup}\circ f \\
= & \{ \quad \text{assumption: } f\circ f^{\cup} \;=\; f{<} \;=\; g\circ g^{\cup} \quad \} \\
& f \quad .
\end{array}
$$

It follows that

(122) $\quad f \;=\; \phi\circ h\circ h{>} \quad \wedge \quad h{>} \;=\; f{>} \quad .$

The combination of (121) and (122) (together with straightforward properties of $h{>}$) establishes that ϕ and $h{>}$ witness the isomorphism $f\cong h$. The property $g\cong k$ is proved similarly.

□

Part II
Pointwise Reasoning

9 Enabling Pointwise Reasoning

In this section, our goal is to capture the notion that a relation is a set with elements pairs of points.

In traditional pointwise reasoning about relations, a basic assumption is that a type is a set that forms a complete, universally distributive lattice under the subset ordering; the type of a (binary) relation is a set of pairs. The set of relations of a given type thus forms a powerset of a set of pairs.

In section 9.1, we recall a general theorem on the structure of powersets. Briefly, theorem 125 states that a set is isomorphic to the powerset of its "atoms" iff it is "saturated". The section defines these concepts; the concepts then form the backbone of later sections where we specialise the theorem to relations.

One (of several) mechanisms for introducing pointwise reasoning within the framework of point-free relation algebra involves the introduction of the so-called "all-or-nothing rule" which was postulated as an axiom by Glück [Glü17]. This rule is combined with completeness and "extensionality" axioms which state that, for each type A, the coreflexives of type A form a complete, saturated lattice. This was the approach taken in [BDGv22, Bac22] where pointwise reasoning was used to formulate and prove properties of graphs. Theorem 148 establishes that the all-or-nothing rule is a consequence of our axiom of choice (axiom 106: every per has an index). Together with the "extensionality" axiom, this enables the application of theorem 125 to establish that the type $A{\sim}B$ is isomorphic to the powerset $2^{A\times B}$ (the set of subsets of the cartesian product $A\times B$). See theorems 148 and 149 in section 9.6.

Section 9.2 introduces "points" and states the extensionality axiom that we assume. A number of sections are then necessary in order to establish theorem 149. Section 9.3 introduces "particles" and "pairs"; it is then shown that particles are points whilst section 9.4 shows that —assuming the axiom of choice— points are particles. (For this reason, the terminology "particle" is temporary.) Section 9.5 shows that proper atoms (of a given type) are "pairs". These are the ingredients for deriving the "all-or-nothing" rule in section 9.6. Section 9.6 also shows that the point-free definition of a "pair" in section 9.3 does correspond to what one normally understand to be a pair of points. The section concludes with theorem 149.

9.1 Powersets

As mentioned above, this section defines “atoms” and “saturated” in the context of a partially ordered set. We then state a fundamental theorem relating these concepts to powersets.

The definition of an atom is the following.

Definition 123 (Atom and Atomicity) Suppose $\mathcal{A}$ is a set partially ordered by the relation $\sqsubseteq$. Then, the element p is an *atom* iff

$$\langle \forall q \; :: \; q \sqsubseteq p \;\; \equiv \;\; q = p \; \vee \; q = \perp\!\!\perp \rangle \quad .$$

Note that $\perp\!\!\perp$ is an atom according to this definition. If p is an atom that is different from $\perp\!\!\perp$ we say that it is a *proper* atom. A lattice is said to be *atomic* if

$$\langle \forall q \; :: \; q \neq \perp\!\!\perp \;\; \equiv \;\; \langle \exists a : \mathsf{atom}.a \wedge a \neq \perp\!\!\perp : a \sqsubseteq q \rangle \rangle \quad .$$

In words, a lattice is atomic if every proper element includes a proper atom.
□

The definition of saturated is as follows.

Definition 124 (Saturated) A complete lattice (ordered by $\sqsubseteq$) is *saturated* iff

$$\langle \forall p \; :: \; p \;\; = \;\; \langle \sqcup a : \mathsf{atom}.a \; \wedge \; a \sqsubseteq p : a \rangle \rangle \quad .$$

□

The set of subsets of a type is a powerset iff the lattice is saturated, as formulated in the following theorem.

Theorem 125 Suppose $\mathcal{A}$ is a complete, universally distributive lattice. Then the following statements are equivalent.

(a) $\mathcal{A}$ is saturated,

(b) $\mathcal{A}$ is atomic and complemented,

(c) $\mathcal{A}$ is isomorphic to the powerset of its atoms.

□

(See [ABH$^+$92, theorem 6.43] for the proof of theorem 125.)

We use theorem 125 in two ways. Firstly, for all types A, we simply postulate that the set of coreflexives of type A is isomorphic to a powerset under the $\subseteq$ ordering: the atoms are the “points” introduced in section 9.2. Second, we use this postulate together with our axiom of choice to show that, for all types A and B, the type $A{\sim}B$ of (heterogeneous) relations is also isomorphic to a powerset under the $\subseteq$ ordering: the atoms are “pairs” introduced in section 9.3. The proof that “pairs” are indeed atoms is the subject of section 9.5. A prelude to this is theorem 139, proved in sections 9.3 and 9.4, is that “points” are a special case of “pairs”.

9.2 Points

We begin by postulating that each type A is a set of "points". We also postulate that the set of coreflexives of type A forms a complete, universally distributive lattice under the subset ordering. Finally, we postulate that the lattice is saturated. With theorem 125 in mind, we define "points" to be the proper atoms of the lattice:

Definition 126 (Point) A homogeneous relation a of type A is a *point* iff it has the following three properties.

(a) $a \neq \perp\!\!\perp$,

(b) $a \subseteq I$, and

(c) $\langle \forall b : b \neq \perp\!\!\perp \wedge b \subseteq a : b = a \rangle$.

In words, a point is proper, coreflexive and an atom.
□

If A is a type, we use a, a' etc. to denote "points" of type A. Similarly for "points" of type B. "Points" represent elements of the appropriate type.

For points a and a' of the same type,

(127) $a = a' \ \vee \ a \circ a' = \perp\!\!\perp$.

The proof is straightforward. Suppose a and a' are points. Then

$$
\begin{array}{ll}
 & a = a \circ a' \\
\Leftarrow & \{ \quad a \text{ is an atom, definition 123} \quad \} \\
 & a \circ a' \neq \perp\!\!\perp \ \wedge \ a \circ a' \subseteq a \\
\Leftarrow & \{ \quad a' \subseteq I \quad \} \\
 & a \circ a' \neq \perp\!\!\perp \ .
\end{array}
$$

Interchanging a and a',

$$a' = a \circ a' \Leftarrow a' \circ a \neq \perp\!\!\perp \ .$$

But, since composition of coreflexives is symmetric, $a \circ a' = a' \circ a$. We conclude that

$$a = a \circ a' = a' \ \Leftarrow \ a \circ a' \neq \perp\!\!\perp \ .$$

This is equivalent to (127).

In point-free relation algebra, subsets of a type are modelled by coreflexives of that type. In order to model the property that the coreflexives of a given type form a lattice that is isomorphic to the set of subsets of the type we need to add to our axiom system a *saturation* property, viz.:

Definition 128 (Saturation) Suppose A is a type. The lattice of coreflexives of type A is said to be *saturated* iff

(129) $\langle\forall p :: p \subseteq I_A \equiv p = \langle\cup a : \mathsf{point}.a \wedge a \subseteq p : a\rangle\rangle$.

□

The axiom that we call "extensionality" is then:

Axiom 130 (Extensionality) For each type A, the points of type A form a complete, universally distributive, saturated lattice under the subset ordering.
□

Applying theorem 125, a consequence of axiom 130 is that the coreflexives of type A form a lattice that is isomorphic to the powerset 2^A. In this sense, the coreflexives in point-free relation algebra represent sets of points in traditional pointwise formulations of relation algebra.

We now want to show how to formulate the property that the set of relations of type $A{\sim}B$ is isomorphic to the powerset $2^{A\times B}$, i.e. relations in point-free relation algebra represent pairs (a,b) of points a and b of type A and B, respectively.

9.3 Pairs and Particles

We now turn our attention to the lattice of relations of a given type. We begin with a point-free definition of a "pair". In subsection 9.6, we show that definition 131 does indeed capture the notion of a "pair of points" whereby the points are the "particles" also introduced in the definition.

Definition 131 (Pair) A relation Z is a *pair* iff it has the following properties:

(a) $Z \neq \perp\!\!\perp$,

(b) $Z = Z{\circ}\top\!\!\top{\circ}Z$,

(c) $Z{<} = Z \circ Z^{\cup}$,

(d) $Z{>} = Z^{\cup} \circ Z$.

We call a relation a *particle* if it is a pair and it is symmetric.
□

In words, a pair Z is a non-empty "rectangle" (properties 131(a) and 131(b)) that is a "bijection" on its left domain and right domains (properties 131(c) and 131(d)).

(Definition 131 was introduced in [Voe99] but using the terminology "singleton" instead of "pair", and "singleton square" instead of "particle".)

Our goal is to prove that the points are exactly the particles. This section is about showing that a particle is a point. See corollary 136.

One task is to show that particles are atoms. The more general property, which we need in later sections, is that pairs are atoms.

Lemma 132 A pair is an atom.

Proof Suppose Z is a pair and suppose Y is such that $Y \subseteq Z$. By the definition of atom, definition 123, we must show that $Y = \perp\!\!\perp \vee Y = Z$. Equivalently, assuming $Y \neq \perp\!\!\perp$, we must show that $Y = Z$. This is done as follows.

$$
\begin{array}{ll}
 & Y \\
= & \{ \quad \text{assumption: } Y \subseteq Z\text{. So, by monotonicity, } Y_< \subseteq Z_< \text{ and } Y_> \subseteq Z_>; \\
 & \quad\;\; \text{domains} \quad \} \\
 & Z_< \circ Y \circ Z_> \\
= & \{ \quad Z \text{ is a pair, so } Z_< = Z \circ Z^\cup = (Z \circ \top\!\!\top \circ Z) \circ Z^\cup \\
 & \quad\;\; \text{similarly for } Z_> \quad \} \\
 & Z \circ \top\!\!\top \circ Z \circ Z^\cup \circ Y \circ Z^\cup \circ Z \circ \top\!\!\top \circ Z \\
= & \{ \quad \text{domains} \quad \} \\
 & Z \circ \top\!\!\top \circ Z_< \circ Y \circ Z_> \circ \top\!\!\top \circ Z \\
= & \{ \quad Z_< \circ Y \circ Z_> = Y \text{ (see first step above)} \quad \} \\
 & Z \circ \top\!\!\top \circ Y \circ \top\!\!\top \circ Z \\
= & \{ \quad \text{assumption: } Y \neq \perp\!\!\perp\text{, cone rule: (4)} \quad \} \\
 & Z \circ \top\!\!\top \circ Z \\
= & \{ \quad Z \text{ is a pair} \quad \} \\
 & Z \;\; .
\end{array}
$$

□

Since a particle is, by definition, a pair, we have:

Corollary 133 A particle is an atom.
□

Lemma 134 A particle is coreflexive.

Proof Suppose Z is square and a pair. Then

$$
\begin{array}{ll}
 & \mathsf{Z} \\
= & \{ \quad \text{assumption: } \mathsf{Z} \text{ is a pair, so } \mathsf{Z}=\mathsf{Z}\circ\top\!\!\top\circ\mathsf{Z}\,; \\
 & \quad [\ \top\!\!\top\circ\mathsf{Z} \ = \ \top\!\!\top\circ\mathsf{Z}_{<}\circ\mathsf{Z} \ = \ \top\!\!\top\circ\mathsf{Z}^{\cup}\circ\mathsf{Z}\ ] \quad \} \\
 & \mathsf{Z}\circ\top\!\!\top\circ\mathsf{Z}^{\cup}\circ\mathsf{Z} \\
= & \{ \quad \text{assumption: } \mathsf{Z} \text{ is a square, so } \mathsf{Z} = \mathsf{Z}\circ\top\!\!\top\circ\mathsf{Z}^{\cup} = \mathsf{Z}^{\cup} \quad \} \\
 & \mathsf{Z}^{\cup}\circ\mathsf{Z} \\
= & \{ \quad \text{assumption: } \mathsf{Z} \text{ is a pair, so } \mathsf{Z}_{>} = \mathsf{Z}^{\cup}\circ\mathsf{Z} \quad \} \\
 & \mathsf{Z}_{>}\ .
\end{array}
$$

That is, Z equals $\mathsf{Z}_{>}$ which is coreflexive.
□

Corollary 135 (Particle) A relation Z is a *particle* iff it has the following three properties.

(a) $\mathsf{Z}\neq\perp\!\!\perp$,

(b) $\mathsf{Z}\subseteq\mathrm{I}$, and

(c) $\mathsf{Z} = \mathsf{Z}\circ\top\!\!\top\circ\mathsf{Z}$.

In words, a particle is a proper, coreflexive rectangle.

Proof "Only-if" is the combination of the definition of a particle and lemma 134. "If" is a straightforward consequence of the properties of domains and coreflexives.
□

Corollary 136 A particle is proper, coreflexive and an atom. That is, a particle is a point.

Proof This is a combination of lemmas 132 and 134.
□

9.4 Points are Particles

We now prove the converse of corollary 136. We use the assumption that every per has a coreflexive index: the axiom of choice (axiom 106).

Lemma 137 Assuming axiom 106, a point is a particle.

Proof Suppose that a is a point. Comparing the definition of a point, definition 126, with the defining properties of a particle, corollary 136, it suffices to prove that $a = a\circ\top\!\top\circ a$. Clearly $a\circ\top\!\top\circ a$ is a per. (The simple proof uses the fact that $a = a^{\cup}$, because a is coreflexive, and $\top\!\top\circ a\circ\top\!\top = \top\!\top$ because $a \neq \bot\!\bot$.) So, by the axiom of choice, $a\circ\top\!\top\circ a$ has an index J, say. We show that J is a particle and $J = a$.

To show that J is a particle, we must establish the three properties listed in corollary 135 with the instantiation $Z := J$. Part (a) is proved as follows.

$$
\begin{array}{ll}
 & J = \bot\!\bot \\
\Rightarrow & \{ \quad \bot\!\bot \text{ is zero of composition} \quad \} \\
 & a\circ\top\!\top\circ a\circ J\circ a\circ\top\!\top\circ a \;=\; \bot\!\bot \\
= & \{ \quad J \text{ is an index of per } a\circ\top\!\top\circ a\text{, definition 105(c)} \quad \} \\
 & a\circ\top\!\top\circ a \;=\; \bot\!\bot \\
\Rightarrow & \{ \quad a\circ a\circ a \subseteq a\circ\top\!\top\circ a \text{ and } a\circ a\circ a = a \text{ (because } a \subseteq I\text{)} \quad \} \\
 & a \subseteq \bot\!\bot \\
= & \{ \quad [\, R \subseteq \bot\!\bot \;\equiv\; R = \bot\!\bot \,] \text{ with } R := a \quad \} \\
 & a = \bot\!\bot \\
= & \{ \quad \text{assumption: } a \text{ is proper, i.e. } a \neq \bot\!\bot \quad \} \\
 & \mathsf{false} \quad .
\end{array}
$$

We conclude that $J \neq \bot\!\bot$. The next step is to show that $J = a$.

$$
\begin{array}{ll}
 & J = a \\
\Leftarrow & \{ \quad \text{assumption: } a \text{ is an atom} \quad \} \\
 & J = \bot\!\bot \;\vee\; J \subseteq a \\
= & \{ \quad J \neq \bot\!\bot \text{ (see above)} \quad \} \\
 & J \subseteq a \\
= & \{ \quad \text{assumption: } a \subseteq I\text{, so } a = (a\circ\top\!\top\circ a)_{<} \quad \} \\
 & J \subseteq (a\circ\top\!\top\circ a)_{<} \\
= & \{ \quad \text{assumption: } J \text{ is an index of } a\circ\top\!\top\circ a \\
 & \quad\quad \text{definition 105(a)} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

Property (b) of corollary 135 immediately follows because a is coreflexive. We now show that $J = J\circ\top\!\top\circ J$.

$$
\begin{array}{ll}
 & J\circ\top\!\!\top\circ J \\
= & \{ \quad J = a \ \text{(proved above) and } \ a \subseteq I \quad \} \\
 & J\circ a\circ\top\!\!\top\circ a\circ J \\
= & \{ \quad \text{assumption: } J \text{ is an index of } a\circ\top\!\!\top\circ a \\
 & \quad\ \ \text{definition 105(c) with } P := a\circ\top\!\!\top\circ a \quad \} \\
 & J \ .
\end{array}
$$

We conclude that $J = a = J\circ\top\!\!\top\circ J$. Thus $a = a\circ\top\!\!\top\circ a$ as required.
□

Relations of the form $R\circ b\circ S$, where b is a point, play an important role later when we consider "polar coverings". Such relations are always rectangles:

Lemma 138 If R has type $A{\sim}B$, S has type $B{\sim}C$, and b is a point of type B , the relation $R\circ b\circ S$ is a rectangle.

Proof Immediate consequence of lemma 63 since, by lemma 137, b is a rectangle if b is a point.
□

Combining corollary 136 with lemma 137, we conclude:

Theorem 139 A relation is a point iff it is a particle.
□

9.5 Proper Atoms are Pairs

The goal of this section is to show that a proper atom is a pair. Aiming to exploit the equivalence of points and particles, we begin with lemmas on the left and right domains of a proper atom.

Lemma 140 Suppose R is a proper atom. Then $R_<$ and $R_>$ are proper atoms[3].

Proof First, that $R_<$ and $R_>$ are both proper is immediate from (22).

To show that $R_<$ is an atom we have to show that, for all p ,

$$p \subseteq R_< \wedge p \neq \perp\!\!\perp \ \equiv \ p = R_< \ .$$

We do this by mutual implication. First, the follows-from:

[3]Note: strictly we should detail the lattice under consideration here. However, it is easy to show that a coreflexive being an atom in the lattice of coreflexives is equivalent to its being an atom in the lattice of relations. This justifies the omission.

$$
\begin{array}{ll}
& p \subseteq R{<} \ \wedge\ p \neq \perp\!\!\perp \ \Leftarrow\ p = R{<} \\
= & \{ \quad \text{predicate calculus} \quad \} \\
& (p \subseteq R{<} \ \Leftarrow\ p = R{<}) \ \wedge\ (p \neq \perp\!\!\perp \ \Leftarrow\ p = R{<}) \\
\Leftarrow & \{ \quad \text{left conjunct: anti-symmetry, right conjunct: Leibniz} \quad \} \\
& \mathsf{true} \ \wedge\ R{<} \neq \perp\!\!\perp \\
\Leftarrow & \{ \quad R{<} \text{ is proper (see above)} \quad \} \\
& \mathsf{true} \ .
\end{array}
$$

Now we prove the converse. Assume $p \subseteq R{<}$ and $p \neq \perp\!\!\perp$. Then

$$
\begin{array}{ll}
& p = R{<} \\
= & \{ \quad \text{anti-symmetry and assumption: } p \subseteq R{<} \quad \} \\
& R{<} \subseteq p \\
\Leftarrow & \{ \quad \text{assumption: } p \subseteq R{<} \text{ and } R{<} \subseteq I \text{ , so } p = p{<} ;\ (p \circ R){<} \subseteq p{<} \quad \} \\
& R{<} \ =\ (p \circ R){<} \\
\Leftarrow & \{ \quad \text{Leibniz} \quad \} \\
& R = p \circ R \\
= & \{ \quad p \circ R \neq \perp\!\!\perp \text{ (see below for proof)} \\
& \qquad R \text{ is an atom, definition 123 (appropriately instantiated)} \quad \} \\
& p \circ R \subseteq R \\
= & \{ \quad \text{assumption: } p \subseteq R{<} \text{ and } R{<} \subseteq I \text{ , monotonicity} \quad \} \\
& \mathsf{true} \ .
\end{array}
$$

In order to verify the penultimate step in the above calculation, we show that $p \circ R = \perp\!\!\perp \Rightarrow \mathsf{false}$ under the assumption that $p \subseteq R{<}$ and $p \neq \perp\!\!\perp$.

$$
\begin{array}{ll}
& p \circ R = \perp\!\!\perp \\
= & \{ \quad \text{cone rule: (4)} \quad \} \\
& \top\!\!\top \circ p \circ R \circ \top\!\!\top \ =\ \perp\!\!\perp \\
= & \{ \quad \text{domains: (dual of) theorem 23(a)} \quad \} \\
& \top\!\!\top \circ p \circ R{<} \circ \top\!\!\top \ =\ \perp\!\!\perp \\
\Rightarrow & \{ \quad \text{assumption: } p \subseteq R{<} \text{, composition of coreflexives is intersection} \quad \} \\
& \top\!\!\top \circ p \circ \top\!\!\top \ =\ \perp\!\!\perp
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{assumption: } p \neq \bot\!\!\bot \text{, cone rule: (4)} \quad \} \\
 & \mathsf{false} \quad .
\end{array}
$$

□

Corollary 141 If R is a proper atom, $R<$ and $R>$ are particles.

Proof By lemma 140 and definition 126 of a point, if R is a proper atom, $R<$ and $R>$ are points. Thus, by lemma 137, $R<$ and $R>$ are particles.

□

We now aim to verify properties 131(b), (c) and (d) of a pair, with Z instantiated to proper atom R. Property 131(b) is the following lemma.

Lemma 142 A proper atom is a rectangle.

Proof Suppose R is a proper atom. Then

$$
\begin{array}{ll}
 & R \circ \top\!\!\top \circ R \\
= & \{ \quad \text{domains} \quad \} \\
 & R< \circ \top\!\!\top \circ R> \\
= & \{ \quad R \neq \bot\!\!\bot \text{, cone rule: (4)} \quad \} \\
 & R< \circ \top\!\!\top \circ R \circ \top\!\!\top \circ R> \\
= & \{ \quad \text{domains} \quad \} \\
 & R< \circ \top\!\!\top \circ R< \circ R \circ R> \circ \top\!\!\top \circ R> \\
= & \{ \quad \text{by corollary 141, } R< \text{ and } R> \text{ are particles} \\
 & \quad\quad \text{corollary 135(c) with } Z{:=}R< \text{ and } Z{:=}R> \quad \} \\
 & R< \circ R \circ R> \\
= & \{ \quad \text{domains} \quad \} \\
 & R \quad .
\end{array}
$$

That is, $R \circ \top\!\!\top \circ R = R$. Thus, by definition, R is a rectangle.

□

Properties 131(c) and (d) require a proper atom to be a bijection. Aiming to apply lemma 101, we introduce an obvious property of rectangles.

Lemma 143 A rectangle is a difunction.

Proof Suppose R is a rectangle. Then

$$
\begin{array}{ll}
 & R\circ R^{\cup}\circ R \subseteq R \\
= & \{ \quad R \text{ is a rectangle, so } R = R\circ\top\!\top\circ R \quad \} \\
 & R\circ R^{\cup}\circ R \subseteq R\circ\top\!\top\circ R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R^{\cup} \subseteq \top\!\top \\
= & \{ \quad [R\subseteq\top\!\top] \text{ with } R:=R^{\cup} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

□

Now we have all the ingredients for our goal.

Lemma 144 Suppose R is a proper atom. Then, assuming axiom 106, R is a pair.

Proof Suppose R is a proper atom. We have to verify properties 131(b), (c) and (d) (with $Z:=R$) of a pair.

Property 131(b) is lemma 142. Properties 131(c) and (d) assert that R is a bijection. To prove this, let J be an index of R. (This is where axiom 106 is assumed.) Then

$$
\begin{array}{ll}
 & J = R \\
= & \{ \quad R \text{ is an atom} \quad \} \\
 & J \neq \bot\!\bot \wedge J \subseteq R \\
= & \{ \quad J \text{ is an index of } R\text{, definition 78} \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

That is, $J=R$. But R is a rectangle and thus a difunction. So, applying lemma 101, J —and thus R— is a bijection, as required.

□

To conclude this section and sections 9.3 and 9.4, we have:

Theorem 145 Assuming axiom 106, for all types A and B, and all relations R of type $A{\sim}B$, R is a proper atom iff R is a pair.

Proof This is a combination of lemmas 132 and 144.

□

9.6 Pairs of Points and the All-or-Nothing Rule

The final step is to show that we can derive the “all-or-nothing” rule.

Lemma 146 If Z is a pair then $Z_<$ and $Z_>$ are particles.

Proof Suppose Z is a pair. We begin by showing that its left and right domains are also pairs.

Properties 131(a), (c) and (d) —with $Z{:=}Z_<$ and $Z{:=}Z_>$— are properties of the domain operators . This leaves 131(b). For the instance $Z{:=}Z_<$, we have:

$$
\begin{array}{ll}
 & Z_< \circ \top\!\!\top \circ Z_< \\
= & \{ \quad \text{domains (specifically} \\
 & \quad [\ Z_< \circ \top\!\!\top = Z \circ \top\!\!\top\ ] \text{ and } [\ \top\!\!\top \circ Z^{\cup} = \top\!\!\top \circ Z_< = \top\!\!\top \circ Z \circ Z^{\cup}\ ]) \quad \} \\
 & Z \circ \top\!\!\top \circ Z \circ Z^{\cup} \\
= & \{ \quad \text{assumption: } Z \text{ is a pair, so } Z \circ \top\!\!\top \circ Z = Z \quad \} \\
 & Z \circ Z^{\cup} \\
= & \{ \quad \text{assumption: } Z \text{ is a pair, so } Z \circ Z^{\cup} = Z_< \quad \} \\
 & Z_< \ .
\end{array}
$$

The proof that $Z_>$ is a pair is symmetrical.

It now follows immediately that $Z_<$ and $Z_>$ are squares: a square is a symmetric rectangle, and both are rectangles (see above); also, both are coreflexives, and coreflexives are symmetric.

□

The following theorem is [Voe99, lemma 4.41(d)].

Theorem 147 For all Z,

$$\text{pair}.Z \ \equiv\ \langle \exists\, a,b : \text{point}.a \wedge \text{point}.b : Z = a \circ \top\!\!\top \circ b \rangle \ .$$

Proof By mutual implication. First,

$$
\begin{array}{ll}
 & \text{pair}.Z \\
\Rightarrow & \{ \quad \text{lemma 146;} \\
 & \quad \text{definition 131(b) and } [\ Z \circ \top\!\!\top \circ Z = Z_< \circ \top\!\!\top \circ Z_>\ ] \quad \} \\
 & \text{particle}.Z_< \ \wedge\ \text{particle}.Z_> \ \wedge\ Z = Z_< \circ \top\!\!\top \circ Z_> \\
\Rightarrow & \{ \quad \text{corollary 136} \quad \} \\
 & \text{point}.Z_< \ \wedge\ \text{point}.Z_> \ \wedge\ Z = Z_< \circ \top\!\!\top \circ Z_> \\
\Rightarrow & \{ \quad a,b := Z_<\,, Z_> \quad \} \\
 & \langle \exists\, a,b : \text{point}.a \wedge \text{point}.b : Z = a \circ \top\!\!\top \circ b \rangle \ .
\end{array}
$$

For the converse, assume that a and b are points. We have to prove that $a{\circ}\top\!\!\top{\circ}b$ is a pair. Applying definition 131, this means checking four properties:

(a) $a{\circ}\top\!\!\top{\circ}b \neq \bot\!\!\bot$,

(b) $a{\circ}\top\!\!\top{\circ}b = a{\circ}\top\!\!\top{\circ}b{\circ}\top\!\!\top{\circ}a{\circ}\top\!\!\top{\circ}b$,

(c) $(a{\circ}\top\!\!\top{\circ}b)_< = (a{\circ}\top\!\!\top{\circ}b) \circ (a{\circ}\top\!\!\top{\circ}b)^\cup$,

(d) $(a{\circ}\top\!\!\top{\circ}b)_> = (a{\circ}\top\!\!\top{\circ}b)^\cup \circ (a{\circ}\top\!\!\top{\circ}b)$.

Properties (a) and (b) are instances of the cone rule together with the assumption that a and b are proper. We prove (c) as follows.

$(a{\circ}\top\!\!\top{\circ}b) \circ (a{\circ}\top\!\!\top{\circ}b)^\cup$

$=$ { converse }

$a \circ \top\!\!\top \circ b \circ b^\cup \circ \top\!\!\top \circ a$

$=$ { assumption: b is a point, cone rule: (4) }

$a{\circ}\top\!\!\top{\circ}a$

$=$ { assumption: a is a point;

so, by corollary 137, a is a pair;

definition 131(b) with $Z{:=}a$ }

a

$=$ { $a{\circ}\top\!\!\top{\circ}b$ is a non-empty rectangle }

$(a{\circ}\top\!\!\top{\circ}b)_<$.

Property (d) is proved symmetrically.

□

We conclude with the theorem that Glück's "all-or-nothing" axiom [Glü17] is a consequence of our axiom of choice.

Theorem 148 (All or Nothing)

$$\langle \forall a,b,R \;:\; \mathsf{point}.a \wedge \mathsf{point}.b \;:\; a{\circ}R{\circ}b = \bot\!\!\bot \vee a{\circ}R{\circ}b = a{\circ}\top\!\!\top{\circ}b \rangle \quad .$$

Proof Suppose a and b are points. By theorem 147, $a{\circ}\top\!\!\top{\circ}b$ is a pair. So, by lemma 132, $a{\circ}\top\!\!\top{\circ}b$ is an atom. Applying the definition of an atom, we have, for all R,

$$\begin{array}{ll} & \mathsf{true} \\ = & \{ \quad \text{monotonicity, } R \subseteq \top\!\top \quad \} \\ & a \circ R \circ b \subseteq a \circ \top\!\top \circ b \\ = & \{ \quad a \circ \top\!\top \circ b \text{ is an atom, definition 123} \quad \} \\ & a \circ R \circ b = \bot\!\bot \vee a \circ R \circ b = a \circ \top\!\top \circ b \quad . \end{array}$$

□

The significance of the all-or-nothing rule is that, together with theorem 125, it follows that the lattice of relations of type $A \sim B$ is isomorphic to the powerset $2^{A \times B}$.

Theorem 149 Suppose, for types A and B, the lattices of coreflexives of types A and B are both complete, universally distributive and extensional. Then the lattice of relations of type $A \sim B$ is saturated; the atoms are elements of the form $a \circ \top\!\top \circ b$ where a and b are atoms of the poset of coreflexives (of types A and B, respectively). It follows that, if the lattice of relations of type $A \sim B$ is complete and universally distributive, it is isomorphic to the powerset of the set of elements of the form $a \circ \top\!\top \circ b$ where a and b are points of types A and B, respectively.

Proof By theorems 147 and 145, $a \circ \top\!\top \circ b$ is an atom . It suffices to prove that the lattice of relations of type $A \sim B$ is saturated. This is easy: for all R of type $A \sim B$,

$$\begin{array}{ll} & R \\ = & \{ \quad I \text{ is unit of composition,} \\ & \quad \text{lattices of coreflexives of types } A \text{ and } B \text{ are extensional} \quad \} \\ & \langle \cup a : \mathsf{point}.a : a \rangle \circ R \circ \langle \cup b : \mathsf{point}.b : b \rangle \\ = & \{ \quad \text{distributivity of composition over } \cup \quad \} \\ & \langle \cup\, a,b : \mathsf{point}.a \wedge \mathsf{point}.b : a \circ R \circ b \rangle \\ = & \{ \quad \text{all-or-nothing rule: theorem 148, } \bot\!\bot \text{ is zero of supremum} \quad \} \\ & \langle \cup\, a,b : \mathsf{point}.a \wedge \mathsf{point}.b \wedge a \circ R \circ b \neq \bot\!\bot : a \circ \top\!\top \circ b \rangle \quad . \end{array}$$

That the lattice of relations is a powerset follows from theorem 125. By theorem 147, every pair is a relation of the form $a \circ \top\!\top \circ b$; also, by lemma 132, $a \circ \top\!\top \circ b$ is an atom.
□

Henceforth, we assume that, for each type A, the lattice of coreflexives of type A is complete, universally distributive and saturated (in other words, we postulate axiom 130). That is, recalling theorem 125, we assume that the coreflexives of a given type form a powerset. We also assume that, for each pair of types A and B, the lattice of relations of type $A{\sim}B$ is complete and universally distributive. Theorem 149 then states that —with the additional postulate of our axiom of choice (axiom 106)— , for each pair of types A and B, the lattice of relations of type $A{\sim}B$ is a powerset with atoms of the form $a{\circ}\top\!\!\top{\circ}b$ where a and b are points of type A and B, respectively. Standard properties of powersets —the properties of set union, intersection and complementation— will be assumed, sometimes without specific mention and sometimes with the hint "set theory".

Summarising theorem 149, the *saturation* property is that

$$(150)\quad \langle\forall R :: R = \langle\cup a,b : a{\circ}\top\!\!\top{\circ}b \subseteq R : a{\circ}\top\!\!\top{\circ}b\rangle\rangle\ .$$

Combining theorem 149 with theorem 125, we get the *irreducibility* property: if $\mathcal{R}$ is a function with range relations of type $A{\sim}B$ and source K, then, for all points a and b of appropriate type,

$$(151)\quad a{\circ}\top\!\!\top{\circ}b \subseteq \cup\mathcal{R} \equiv \langle\exists k : k{\in}K : a{\circ}\top\!\!\top{\circ}b \subseteq \mathcal{R}.k\rangle\ .$$

Theorem 149 assumes that the lattices of coreflexives (of appropriate type) are extensional. Conversely, if we assume that, for all types A and B, the lattice of relations of type $A{\sim}B$ is extensional then so is the lattice of coreflexives of type A, for all A. This is theorem 154. First, we need a lemma.

Lemma 152 The identity relation I_A of type A satisfies, for all points a and a' of type A,

$$(153)\quad a{\circ}\top\!\!\top{\circ}a' \subseteq I_A \equiv a = a'\ .$$

Proof The proof is by mutual implication. First,

$$\begin{array}{ll}
 & a = a' \\
\Rightarrow & \{\quad \text{Leibniz}\quad\} \\
 & a{\circ}\top\!\!\top{\circ}a' = a{\circ}\top\!\!\top{\circ}a \\
\Rightarrow & \{\quad \text{a point is a particle (lemma 137)} \\
 & \quad\;\; \text{131(b) (with } Z{:=}a\text{)}\quad\} \\
 & a{\circ}\top\!\!\top{\circ}a' = a \\
\Rightarrow & \{\quad \text{definition a point (definition 126)}\quad\} \\
 & a{\circ}\top\!\!\top{\circ}a' \subseteq I_A\ .
\end{array}$$

For the converse, we first prove that, for arbitrary points a and a', $a{\circ}\top\!\!\top{\circ}a' \neq \bot\!\!\bot$.

$$
\begin{array}{ll}
 & a \circ \top\!\top \circ a' \neq \bot\!\bot \\
= & \{ \quad \text{cone rule: (4) with } R := a \circ \top\!\top \circ a' \quad \} \\
 & \top\!\top \circ a \circ \top\!\top \circ a' \circ \top\!\top \;=\; \top\!\top \\
= & \{ \quad \text{cone rule: (4) (with } R := a \text{ and } R := a'\text{), } a \neq \bot\!\bot \text{ and } a' \neq \bot\!\bot \quad \} \\
 & \mathsf{true} \quad .
\end{array}
$$

So

$$
\begin{array}{ll}
 & a \circ \top\!\top \circ a' \subseteq I_A \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & a \circ a \circ \top\!\top \circ a' \circ a' \subseteq a \circ I_A \circ a' \\
= & \{ \quad a \circ a = a \,,\; a' \circ a' = a' \,,\; I_A \text{ is identity of composition} \quad \} \\
 & a \circ \top\!\top \circ a' \subseteq a \circ a' \\
= & \{ \quad \text{composition of coreflexives is intersection} \quad \} \\
 & a \circ \top\!\top \circ a' \subseteq a \;\wedge\; a \circ \top\!\top \circ a' \subseteq a' \\
= & \{ \quad a \circ \top\!\top \circ a' \neq \bot\!\bot \text{ (proved above), } a \text{ and } a' \text{ are atoms} \quad \} \\
 & a \circ \top\!\top \circ a' = a \;\wedge\; a \circ \top\!\top \circ a' = a' \\
\Rightarrow & \{ \quad \text{transitivity} \quad \} \\
 & a = a' \quad .
\end{array}
$$

□

Theorem 154 Suppose, for all types A and B, the lattice of relations of type $A{\sim}B$ is extensional, whereby the atoms are elements of the form $a \circ \top\!\top \circ b$ where a and b are atoms of the poset of coreflexives (of types A and B, respectively). Then, for all A, the lattice of coreflexives of type A is extensional.

Proof By assumption, for all A, the lattice of relations of type $A{\sim}A$ is complete and universally distributive. It follows straightforwardly that the lattice of relations of type $A{\sim}A$ bounded above by any fixed relation is also complete and universally distributive. In particular, the coreflexives (which are bounded above by I_A) form a complete and universally distributive lattice. It suffices thus to prove that the lattice of coreflexives of type A is saturated. That is, we have to prove that, for all coreflexives p of type A,

$$
p \;=\; \langle \cup a : a \subseteq p : a \rangle
$$

where dummy a ranges over points of type A. This we do as follows.

$$\begin{array}{ll}
& \langle\cup a : a \subseteq p : a\rangle \\
= & \{ \quad (153) \text{ with } a,a' := a,a \quad \} \\
& \langle\cup a \;:\; a\circ\top\!\!\top\circ a \subseteq p \;:\; a\circ\top\!\!\top\circ a\rangle \\
= & \{ \quad \text{one-point rule} \quad \} \\
& \langle\cup a,b \;:\; a=b \wedge a\circ\top\!\!\top\circ b \subseteq p \;:\; a\circ\top\!\!\top\circ b\rangle \\
= & \{ \quad p \text{ is coreflexive, i.e. } p \subseteq I_A \\
& \quad\quad a\circ\top\!\!\top\circ b \subseteq I_A \;\Rightarrow \{(153) \text{ with } a,a' := a,b\} \;\; a=b \quad \} \\
& \langle\cup a,b : a\circ\top\!\!\top\circ b \subseteq p : a\circ\top\!\!\top\circ b\rangle \\
= & \{ \quad \text{assumption: lattice } A{\sim}A \text{ is saturated} \quad \} \\
& p \;\; .
\end{array}$$

□

Combining theorems 149 and 154, we get:

Corollary 155 Suppose, for all types A and B, the lattice of relations of type $A{\sim}B$ is complete and universally distributive. Then for all types A and B, the lattice of relations of type $A{\sim}B$ is extensional iff for all types A, the lattice of coreflexives of type A is extensional.

□

Although the saturation property allows us to identify atoms of the form $a\circ\top\!\!\top\circ b$ with elements (a,b) of the set $A{\times}B$, it does not establish that the operators of relation algebra (converse, composition etc.) correspond to their standard set-theoretic interpretations. This is straightforward. For example, for composition we have, for all R and S,

$$\begin{array}{ll}
& R\circ S \\
= & \{ \quad \text{saturation: (150)} \quad \} \\
& \langle\cup a,b : a\circ\top\!\!\top\circ b \subseteq R : a\circ\top\!\!\top\circ b\rangle \circ \langle\cup b',c : b'\circ\top\!\!\top\circ c \subseteq S : b'\circ\top\!\!\top\circ c\rangle \\
= & \{ \quad \text{distributivity} \quad \} \\
& \langle\cup a,b,b',c \;:\; a\circ\top\!\!\top\circ b \subseteq R \wedge b'\circ\top\!\!\top\circ c \subseteq S \;:\; a\circ\top\!\!\top\circ b\circ b'\circ\top\!\!\top\circ c\rangle \\
= & \{ \quad b \text{ and } b' \text{ are points, so } b\circ b' \neq \bot\!\!\bot \equiv b'=b \\
& \quad\quad \text{case analysis on } b'=b \vee b' \neq b \text{, one-point rule} \quad \} \\
& \langle\cup a,b,c \;:\; a\circ\top\!\!\top\circ b \subseteq R \;\wedge\; b\circ\top\!\!\top\circ c \subseteq S \;:\; a\circ\top\!\!\top\circ b\circ b\circ\top\!\!\top\circ c\rangle \\
= & \{ \quad b \text{ ranges over points, so } b\circ b = b \neq \bot\!\!\bot \text{, cone rule: (4)} \quad \}
\end{array}$$

$$
\begin{array}{ll}
 & \langle \cup\, a,b,c \ : \ a\circ\top\!\!\top\circ b \subseteq R \ \wedge \ b\circ\top\!\!\top\circ c \subseteq S \ : \ a\circ\top\!\!\top\circ c\rangle \\
= & \qquad \{ \qquad \text{range disjunction} \qquad \} \\
 & \langle \cup\, a,c \ : \ \langle \exists b \ :: \ a\circ\top\!\!\top\circ b \subseteq R \wedge b\circ\top\!\!\top\circ c \subseteq S\rangle \ : \ a\circ\top\!\!\top\circ c\rangle \quad .
\end{array}
$$

Comparing the first and last lines of this calculation (and interpreting $a\circ\top\!\!\top\circ b \subseteq R$ as $(a,b)\in R$ and $b\circ\top\!\!\top\circ c \subseteq S$ as $(b,c)\in S$) we recognise the standard set-theoretic definition of $R\circ S$.

The important step to note in the above calculation is the use of the distributivity of composition over union. The validity of such universal distributivity — both from the left and from the right— is a consequence of the Galois connections (5) and (6) defining factors. A similar step needed in the calculation for converse relies on the fact that converse is the upper and lower adjoint of itself.

We conclude this section with a brief comparison of extensionality as formulated here with the notion of extensionality formulated by Voermans [Voe99].

Voermans [Voe99, section 4.5] postulated that the lattice of binary relations of a given type is saturated by relations of the form $X\circ\top\!\!\top\circ Y$ where X and Y are *particles*. Relations of this form are then shown to model pairs (x,y) in standard set-theoretic presentations of relation algebra. Here, we have postulated that each type A forms a lattice that is saturated by *points*: see axiom 130; this postulate is combined with our axiom of choice: all pers have an index. Then pairs in standard set-theoretic presentations of relation algebra are modelled by relations of the form $a\circ\top\!\!\top\circ b$, where a and b are points. Because particles are points (corollary 136), the saturation property postulated by Voermans is formally stronger than axiom 130. As a consequence, it becomes slightly harder to establish that, for example, the composition of two relations does indeed correspond to the set-theoretic notion of composition. (See [Voe99, section 4.5] for details of what is involved.) More importantly, the combination of axioms 106 and 130 facilitates a better separation of concerns: axiom 106 provides a powerful extension of point-free reasoning, whilst axiom 130 fills the gap where pointwise reasoning is unavoidable.

10 Pointwise Interpretations

We have now shown that, with the addition of axioms on the completeness and universal distributivity of the relations of a given type together with the axiom of choice, axiom 106, the type $A{\sim}B$ (for each type A and B) is isomorphic to the powerset $2^{A\times B}$. The proper atoms are events of the form $a\circ\top\!\!\top\circ b$ where a and b are points; such an event models the pair (a,b) in conventional pointwise formulations of relation algebra. Specfically, the property

$$a\circ\top\!\!\top\circ b \subseteq R$$

models the property $(a, b) \in R$ in conventional formulations, whilst

$$a \circ R \circ b = \perp\!\!\perp$$

models the converse property $(a, b) \notin R$.

A major benefit of enabling pointwise reasoning in this way is that we can derive pointwise interpretations of the operators in the calculus in a precise and concise fashion. This section is about the pointwise interpretations of some of the less familiar operators. The properties presented are needed in later sections.

Lemma 156 gives pointwise interpretations of the factor operators.

Lemma 156 For all relations R of type $A{\sim}C$ and S of type $B{\sim}C$ (for some A, B and C) and all points a and b,

$$a \circ \top\!\!\top \circ b \subseteq R/S \quad \equiv \quad (b \circ S)_> \subseteq (a \circ R)_> \quad .$$

Dually, for all relations R of type $C{\sim}A$ and S of type $C{\sim}B$, and all points a and b,

$$a \circ \top\!\!\top \circ b \subseteq R \backslash S \quad \equiv \quad (R \circ a)_< \subseteq (S \circ b)_< \quad .$$

Proof By mutual implication:

$$
\begin{array}{ll}
 & a \circ \top\!\!\top \circ b \subseteq R/S \\
= & \{ \quad \text{definition of factor} \quad \} \\
 & a \circ \top\!\!\top \circ b \circ S \subseteq R \\
\Rightarrow & \{ \quad a \text{ and } b \text{ are points, monotonicity and domains} \quad \} \\
 & (b \circ S)_> \subseteq (a \circ R)_> \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & a \circ \top\!\!\top \circ (b \circ S)_> \subseteq a \circ \top\!\!\top \circ (a \circ R)_> \\
= & \{ \quad \text{domains} \quad \} \\
 & a \circ \top\!\!\top \circ b \circ S \subseteq a \circ \top\!\!\top \circ a \circ R \\
= & \{ \quad a \text{ is a point (so } a \circ \top\!\!\top \circ a = a\text{)} \quad \} \\
 & a \circ \top\!\!\top \circ b \circ S \subseteq a \circ R \\
\Rightarrow & \{ \quad a \text{ is a coreflexive} \quad \} \\
 & a \circ \top\!\!\top \circ b \circ S \subseteq R \\
= & \{ \quad \text{definition of factor} \quad \} \\
 & a \circ \top\!\!\top \circ b \subseteq R/S \quad .
\end{array}
$$

The second equivalence is proved similarly.

$$
\begin{array}{ll}
& a \circ \top\!\!\top \circ b \subseteq R \backslash S \\
= & \{ \quad \text{definition of factor} \quad \} \\
& R \circ a \circ \top\!\!\top \circ b \subseteq S \\
\Rightarrow & \{ \quad \text{monotonicity and coreflexives} \quad \} \\
& (R \circ a)_< \subseteq (S \circ b)_< \\
\Rightarrow & \{ \quad \text{(as in above calculation)} \quad \} \\
& a \circ \top\!\!\top \circ b \subseteq R \backslash S \quad .
\end{array}
$$

□

For relations R and S with the same source, the relation $R/S \cap (S/R)^\cup$ is the "symmetric left division" of R and S. Dually, for relations R and S with the same target, the relation $R\backslash S \cap (S\backslash R)^\cup$ is their "symmetric right division". The following corollary of lemma 156 gives a pointwise interpretation of these "division" operators.

Corollary 157 For all relations R and S with the same source, and all points a and b (of appropriate type),

$$a \circ \top\!\!\top \circ b \subseteq R/S \cap (S/R)^\cup \quad \equiv \quad (a \circ R)_> = (b \circ S)_> \quad .$$

Dually, for all relations R and S with the same target, and all points a and b (of appropriate type),

$$a \circ \top\!\!\top \circ b \subseteq R\backslash S \cap (S\backslash R)^\cup \quad \equiv \quad (R \circ a)_< = (S \circ b)_< \quad .$$

Proof Straightforward application of lemma 156 and anti-symmetry:

$$
\begin{array}{ll}
& a \circ \top\!\!\top \circ b \subseteq R/S \cap (S/R)^\cup \\
= & \{ \quad \text{infima and converse} \quad \} \\
& a \circ \top\!\!\top \circ b \subseteq R/S \ \wedge \ b \circ \top\!\!\top \circ a \subseteq S/R \\
= & \{ \quad \text{lemma 156} \quad \} \\
& (b \circ S)_> \subseteq (a \circ R)_> \ \wedge \ (a \circ R)_> \subseteq (b \circ S)_> \\
= & \{ \quad \text{anti-symmetry} \quad \} \\
& (a \circ R)_> = (b \circ S)_> \quad .
\end{array}
$$

□

The pointwise interpretations of the left and right per domains are given by the following lemma.

Lemma 158 For all relations R of type $A{\sim}B$ and all points a and a' of type A,

$$a{\circ}\top\!\!\top{\circ}a' \subseteq R{\prec} \quad\equiv\quad a \subseteq R{<} \;\;\wedge\;\; (a{\circ}R){>} = (a'{\circ}R){>} \;\;\wedge\;\; a' \subseteq R{<} \quad .$$

Dually, for all relations R of type $A{\sim}B$ and all points b and b' of type B,

$$b{\circ}\top\!\!\top{\circ}b' \subseteq R{\succ} \quad\equiv\quad b \subseteq R{>} \;\;\wedge\;\; (R{\circ}b){<} = (R{\circ}b'){<} \;\;\wedge\;\; b' \subseteq R{>} \quad .$$

Proof Assume that b and b' are points. Then

$$
\begin{array}{cl}
 & b{\circ}\top\!\!\top{\circ}b' \subseteq R{\succ} \\
= & \{ \quad \text{definition (28) and lemma 34} \quad \} \\
 & b{\circ}\top\!\!\top{\circ}b' \subseteq R{>} \circ R\backslash\!\!\backslash R \circ R{>} \\
= & \{ \quad \text{domains (using mutual implication)} \quad \} \\
 & b \subseteq R{>} \;\;\wedge\;\; b{\circ}\top\!\!\top{\circ}b' \subseteq R\backslash\!\!\backslash R \;\;\wedge\;\; b' \subseteq R{>} \\
= & \{ \quad \text{corollary 157, with } R,S := R,R \quad \} \\
 & b \subseteq R{>} \;\;\wedge\;\; (R{\circ}b){<} = (R{\circ}b'){<} \;\;\wedge\;\; b' \subseteq R{>} \quad .
\end{array}
$$

The dual property follows from the distributivity properties of converse.
□

Part III
Applications

11 Coverings

This section is motivated by Riguet's study of so-called "relations de Ferrers" [Rig51] (which we call "staircase relations" [Bac21]). A central element in Riguet's study was a theorem characterising such relations as the "réunion" of "rectangles" that have a very special property. We abstract the notion of a "polar covering" of a relation and we prove the theorem that *every* relation has a polar covering. See definition 163 and theorem 166. In anticipation of section 12, we also define the notion of a "non-redundant" polar covering. For finite relations, it is straightforward to show that a non-redundant polar covering can always be constructed from a given polar covering of the relation. The algorithm may, however, not be practical; moreover, there are infinite relations that do not have a non-redundant polar covering. (The less-than relation on real numbers is an example.)

11.1 Completely Disjoint Rectangles

Definition 159 (Indexed Bag/Set) Suppose $\mathcal{R}$ is a function with source K. Then $\mathcal{R}$ is said to be a *bag indexed by* K. The values $\mathcal{R}.\mathsf{k}$, where k ranges over K, are said to be the *elements* of $\mathcal{R}$. In the case that $\mathcal{R}$ is injective, it is said to be an *indexed set*.
□

The distinction between "bag" and "set" in definition 159 emphasises the fact that the same element may occur repeatedly in an indexed bag whereas each element occurs exactly once in an indexed set. That is, an indexed set $\mathcal{R}$ has the property that, for all j and k in K,

$$\mathcal{R}.\mathsf{j} = \mathcal{R}.\mathsf{k} \quad \equiv \quad \mathsf{j} = \mathsf{k} \quad .$$

We normally apply definition 159 to bags/sets of rectangles. Specifically, suppose A, B and K are types and $\mathcal{R}$ is a function with source K and target rectangles of type $A{\sim}\mathsf{B}$. Then $\mathcal{R}$ is said to be an *indexed bag of rectangles*; it is an indexed *set* of rectangles if it is injective.

Two relations R and S are *disjoint* if $\mathsf{R}{\cap}\mathsf{S} = \perp\!\!\!\perp$. One can show that, for all rectangles R and S,

$$\mathsf{R}{\cap}\mathsf{S} = \perp\!\!\!\perp \quad \equiv \quad \mathsf{R}{<}{\cap}\,\mathsf{S}{<} = \perp\!\!\!\perp \;\vee\; \mathsf{R}{>}{\cap}\,\mathsf{S}{>} = \perp\!\!\!\perp \quad .$$

(This is a consequence of lemma 66.) The definition of "completely" disjoint strengthens the disjunction to a conjunction. Note that we don't use continued equality because the symbol " $\perp\!\!\perp$ " is overloaded.

Definition 160 (Completely Disjoint) Two rectangles R and S are said to be *completely disjoint* iff

$$R{<} \cap S{<} = \perp\!\!\perp \quad \wedge \quad R{>} \cap S{>} = \perp\!\!\perp \quad .$$

Suppose $\mathcal{R}$ is an indexed bag of rectangles. Then $\mathcal{R}$ is said to be a *completely disjoint bag of rectangles* iff, for all j and k in the index set of $\mathcal{R}$,

$$\mathcal{R}.j \neq \mathcal{R}.k \quad \equiv \quad (\mathcal{R}.j){<} \cap (\mathcal{R}.k){<} = \perp\!\!\perp \wedge (\mathcal{R}.j){>} \cap (\mathcal{R}.k){>} = \perp\!\!\perp \quad .$$

$\mathcal{R}$ is said to be a completely disjoint *set* of rectangles iff in addition it is injective. That is, $\mathcal{R}$ is a *completely disjoint set of rectangles* iff, for all j and k in the index set of $\mathcal{R}$,

$$j \neq k \quad \equiv \quad (\mathcal{R}.j){<} \cap (\mathcal{R}.k){<} = \perp\!\!\perp \wedge (\mathcal{R}.j){>} \cap (\mathcal{R}.k){>} = \perp\!\!\perp \quad .$$

□

We give several constructions of bags/sets of rectangles. When we do so, the verification that the bags/sets are completely disjoint is achieved by mutual implication. The "if" part is established by proving its contrapositive. That is, the proof obligation becomes to show that, for all indices j and k,

$$\mathcal{R}.j = \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j){<} \cap (\mathcal{R}.k){<} \neq \perp\!\!\perp \wedge (\mathcal{R}.j){>} \cap (\mathcal{R}.k){>} \neq \perp\!\!\perp$$

which simplifies to, for all j,

$$\mathcal{R}.j \neq \perp\!\!\perp \quad .$$

(The same simplification is valid whether the construction yields a bag or a set.) Thus the first step is to show that the construction yields non-empty elements. The "only-if" part is to show that, for all indices j and k,

$$\mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j){<} \cap (\mathcal{R}.k){<} = \perp\!\!\perp \wedge (\mathcal{R}.j){>} \cap (\mathcal{R}.k){>} = \perp\!\!\perp \quad .$$

For this part, the following lemma is exploited.

Lemma 161 For all relations R and S,

$$R{<} \cap S{<} \;=\; \perp\!\!\perp \quad \equiv \quad R^{\cup} \circ S \;=\; \perp\!\!\perp \quad .$$

Symmetrically,

$$R{>} \cap S{>} \;=\; \perp\!\!\perp \quad \equiv \quad R \circ S^{\cup} \;=\; \perp\!\!\perp \quad .$$

Proof First note that

$$R< \cap S< \;=\; \bot\!\!\bot \quad\equiv\quad R< \circ S< \;=\; \bot\!\!\bot$$

since the intersection of coreflexives is the same as their composition. Then

$$
\begin{array}{ll}
 & R< \circ S< \;=\; \bot\!\!\bot \\
\Rightarrow & \{ \quad \bot\!\!\bot \text{ is zero of composition} \quad \} \\
 & R^{\cup} \circ R< \circ S< \circ S \;=\; \bot\!\!\bot \\
= & \{ \quad \text{domains: (18)} \quad \} \\
 & R^{\cup} \circ S \;=\; \bot\!\!\bot \\
\Rightarrow & \{ \quad \bot\!\!\bot \text{ is zero of composition} \quad \} \\
 & R \circ R^{\cup} \circ S \circ S^{\cup} \;=\; \bot\!\!\bot \\
\Rightarrow & \{ \quad \text{monotonicity, } [\, R = \bot\!\!\bot \equiv R \subseteq \bot\!\!\bot \,] \text{ (applied twice)} \quad \} \\
 & (I \cap R \circ R^{\cup}) \circ (I \cap S \circ S^{\cup}) = \bot\!\!\bot \\
= & \{ \quad \text{domains: definition 15} \quad \} \\
 & R< \circ S< \;=\; \bot\!\!\bot \quad .
\end{array}
$$

The lemma follows by mutual implication.

□

The foregoing discussion is formalised in the following lemma.

Lemma 162 Suppose $\mathcal{R}$ is an indexed bag of rectangles. Then $\mathcal{R}$ is completely disjoint iff

$$
\begin{array}{ll}
 & \langle \forall j :: \mathcal{R}.j \neq \bot\!\!\bot \rangle \\
\wedge & \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \;=\; \bot\!\!\bot \;\wedge\; \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \bot\!\!\bot \rangle \quad .
\end{array}
$$

Also, $\mathcal{R}$ is completely disjoint and injective —i.e. an indexed set— iff

$$
\begin{array}{ll}
 & \langle \forall j :: \mathcal{R}.j \neq \bot\!\!\bot \rangle \\
\wedge & \langle \forall j,k \;::\; j \neq k \;\Rightarrow\; (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \;=\; \bot\!\!\bot \;\wedge\; \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \bot\!\!\bot \rangle \quad .
\end{array}
$$

Proof

$$
\begin{array}{ll}
 & \mathcal{R} \text{ is completely disjoint} \\
= & \{ \quad \text{definition 160} \quad \} \\
 & \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\equiv\; (\mathcal{R}.j)< \cap (\mathcal{R}.k)< = \bot\!\!\bot \;\wedge\; (\mathcal{R}.j)> \cap (\mathcal{R}.k)> = \bot\!\!\bot \rangle
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{mutual implication} \quad \} \\
& \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Leftarrow (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \bot\!\!\bot \wedge (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \bot\!\!\bot \rangle \\
\wedge & \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Rightarrow (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \bot\!\!\bot \wedge (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \bot\!\!\bot \rangle \\
= & \{ \quad \text{contrapositive; lemma 161} \quad \} \\
& \langle \forall j,k :: \mathcal{R}.j = \mathcal{R}.k \Rightarrow (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< \neq \bot\!\!\bot \vee (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> \neq \bot\!\!\bot \rangle \\
\wedge & \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Rightarrow \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \bot\!\!\bot \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \bot\!\!\bot \rangle \\
= & \{ \quad \text{Leibniz, reflexivity of equality, idempotence of intersection} \quad \} \\
& \langle \forall j :: (\mathcal{R}.j)_< \neq \bot\!\!\bot \vee (\mathcal{R}.j)_> \neq \bot\!\!\bot \rangle \\
\wedge & \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Rightarrow \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \bot\!\!\bot \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \bot\!\!\bot \rangle \\
= & \{ \quad \text{domains} \\
& \qquad (\; [\, (R_< = \bot\!\!\bot) = (R = \bot\!\!\bot) = (R_> = \bot\!\!\bot) \,] \text{ with } R := \mathcal{R}.j \,)) \quad \} \\
& \langle \forall j :: \mathcal{R}.j \neq \bot\!\!\bot \rangle \\
\wedge & \langle \forall j,k :: \mathcal{R}.j \neq \mathcal{R}.k \Rightarrow \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \bot\!\!\bot \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \bot\!\!\bot \rangle \; .
\end{array}
$$

Injectivity of $\mathcal{R}$ is the property that $\langle \forall j,k :: \mathcal{R}.j = \mathcal{R}.k \equiv j = k \rangle$. The characterisation of completely disjoint and injective thus follows by the use of Leibniz's rule.
□

11.2 Polar Coverings

Definition 163 (Polar Covering) Suppose $\mathcal{R}$ is an indexed bag of rectangles. (See definition 159.) Then $\mathcal{R}$ is said to be *polar* if, for all elements U and V of $\mathcal{R}$,

$$U_< \subseteq V_< \equiv U_> \supseteq V_> \; .$$

Also, $\mathcal{R}$ is said to be *linear* if, for all elements U and V of $\mathcal{R}$,

$$U_< \subseteq V_< \quad \vee \quad V_< \subseteq U_< \; .$$

(Equivalently,

$$U_> \subseteq V_> \quad \vee \quad V_> \subseteq U_> \; .)$$

A relation R is *covered* by $\mathcal{R}$ if $R = \cup\mathcal{R}$. The covering $\mathcal{R}$ is *non-redundant* if there is a total function $\mathcal{D}$ from indices of $\mathcal{R}$ to a set of completely disjoint subrectangles of $\cup\mathcal{R}$ that "defines" the elements of $\mathcal{R}$. To be precise, the covering $\mathcal{R}$ is *non-redundant* if there is a function $\mathcal{D}$ with the same source as $\mathcal{R}$ such that

$$\begin{array}{ll} & \langle\forall k :: \mathrm{rectangle}.(\mathcal{D}.k) \wedge \mathcal{D}.k \subseteq \mathcal{R}.k\rangle \\ \wedge & \langle\forall j,k :: \mathcal{D}.j \neq \mathcal{D}.k \equiv (\mathcal{D}.j)_< \cap (\mathcal{D}.k)_< = \perp\!\!\perp \wedge (\mathcal{D}.j)_> \cap (\mathcal{D}.k)_> = \perp\!\!\perp\rangle \\ \wedge & \langle\forall j,k :: \mathcal{D}.j = \mathcal{D}.k \equiv \mathcal{R}.j = \mathcal{R}.k\rangle \ . \end{array}$$

In such a case, we call the indexed bag $\mathcal{D}$ a *definiens* of $\mathcal{R}$.
□

Lemma 164 Suppose $\mathcal{R}$ is an indexed bag of rectangles and suppose $\mathcal{R}$ is polar. Then, for all elements U and V of $\mathcal{R}$,

$$U = V \equiv U_< = V_< \ .$$

Proof

$$\begin{array}{ll} & U = V \\ = & \{ \quad U \text{ and } V \text{ are rectangles: lemma 65} \quad \} \\ & U_< = V_< \wedge U_> = V_> \\ = & \{ \quad \text{anti-symmetry} \quad \} \\ & U_< \subseteq V_< \wedge U_< \supseteq V_< \wedge U_> \subseteq V_> \wedge U_> \supseteq V_> \\ = & \{ \quad \mathcal{R} \text{ is polar: definition 163} \quad \} \\ & U_< \subseteq V_< \wedge U_> \subseteq V_> \\ = & \{ \quad \text{anti-symmetry} \quad \} \\ & U_< = V_< \ . \end{array}$$

□

Definition 165 Suppose $\mathcal{R}$ is a polar covering of relation R. The *polar ordering* of the elements of $\mathcal{R}$, denoted henceforth by the symbol $\sqsubseteq$, is defined by, for all indices j and k of $\mathcal{R}$,

$$\mathcal{R}.j \sqsubseteq \mathcal{R}.k \equiv (\mathcal{R}.j)_< \subseteq (\mathcal{R}.k)_< \ .$$

Equivalently,

$$\mathcal{R}.j \sqsubseteq \mathcal{R}.k \equiv (\mathcal{R}.k)_> \subseteq (\mathcal{R}.j)_> \ .$$

□

As suggested by the notation, the relation $\sqsubseteq$ is a provisional ordering on the elements of any indexed bag of relations; it is anti-symmetric whenever $\mathcal{R}$ is an indexed bag of polar rectangles by virtue of lemma 65 and definition 163 of "polar".

Definition 163 defines a *bag* of rectangles rather than a *set* of rectangles. (Recall that a set is an injective bag: see definition 159.) Generally it is easier to construct a bag rather than a set of polar rectangles that cover a given relation. Nevertheless, sets are more desirable than bags. Our theory of indexes of a relation provides the mechanism to construct sets rather than bags. See theorem 166. Note that a definiens $\mathcal{D}$ of an indexed set $\mathcal{R}$ is also a set (because $\mathcal{R}.j = \mathcal{R}.k$ equivales $j = k$).

The adjective "polar" alludes to the property that the left and right domains of a covering are "polar" opposites: the larger the one, the smaller the other. The notion was introduced by Riguet [Rig51] in the context of a theorem on "relations de Ferrers". More precisely, Riguet introduced the notion of a *linear* polar covering. For further details of Riguet's theorem see the section on staircase relations in [Bac21].

In the case of the empty relation, $\perp\!\!\!\perp$, there are two distinct polar coverings according to our definition. One is the empty function (the unique function with source the empty set) and the second is the constanct function with source the unit type that returns $\perp\!\!\!\perp$. The former is the preferred covering because it means that, for all relations R, all elements of a polar covering of R are proper (different from $\perp\!\!\!\perp$). We call such polar coverings *proper* coverings and, from now on, make the assumption that all coverings are proper.

Theorem 166 Suppose R is a relation of type $A{\sim}B$ and suppose J is a (coreflexive) index of $R{\succ}$. Define the function $\mathcal{R}$ by

$$\mathcal{R} \;=\; \langle b : b \subseteq J : R \circ b \circ R\backslash R \rangle \;\; .$$

Then $\mathcal{R}$ is an injective, polar covering of R. (Note: the source of the function $\mathcal{R}$ is the subset of B corresponding to the points given by the range restriction on the dummy b.)

Proof The elements of $\mathcal{R}$ are obviously rectangles because its index set is a set of points. (See lemma 138.) The "polar" property is established as follows. For all b, b' such that $b \subseteq R{>}$ and $b' \subseteq R{>}$,

$$
\begin{array}{cl}
 & (R \circ b' \circ R\backslash R){>} \;\subseteq\; (R \circ b \circ R\backslash R){>} \\
= & \{ \quad \text{assumption: } b \subseteq R{>} \text{ and } b' \subseteq R{>}\text{, domains} \quad \} \\
 & (b' \circ R\backslash R){>} \;\subseteq\; (b \circ R\backslash R){>} \\
= & \{ \quad \text{lemma 156 with } R,a,a' := R\backslash R\,,b\,,b' \quad \} \\
 & b \circ \top\!\!\!\top \circ b' \;\subseteq\; (R\backslash R)/(R\backslash R)
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad (13) \quad \} \\
& b \circ \top\!\top \circ b' \subseteq R\backslash R \\
= & \{ \quad \text{lemma 156} \quad \} \\
& (R \circ b)_< \subseteq (R \circ b')_< \\
= & \{ \quad I \subseteq R\backslash R\,,\ \text{domains} \quad \} \\
& (R \circ b \circ R\backslash R)_< \subseteq (R \circ b' \circ R\backslash R)_< \quad .
\end{array}
$$

The property $R = \cup\mathcal{R}$ is established as follows.

$$
\begin{array}{ll}
& \cup\mathcal{R} \\
= & \{ \quad \text{definition of } \mathcal{R} \text{ and saturation axiom (129)} \quad \} \\
& R \circ J \circ R\backslash R \\
= & \{ \quad R = R \circ R_\succ \ \text{ and } \ J = J \circ R_> \ \text{ (since } J \text{ is a coreflexive index } R\text{)} \quad \} \\
& R \circ R_\succ \circ J \circ R_> \circ R\backslash R \\
= & \{ \quad \text{lemma 37} \quad \} \\
& R \circ R_\succ \circ J \circ R_\succ \circ R\backslash R \\
= & \{ \quad J \text{ is an index } R\,,\ \text{definition 78(d)} \quad \} \\
& R \circ R_\succ \circ R\backslash R \\
= & \{ \quad R = R \circ R_\succ \quad \} \\
& R \circ R\backslash R \\
= & \{ \quad \text{cancellation: (11)} \quad \} \\
& R \quad .
\end{array}
$$

This completes the proof that $R = \cup\mathcal{R}$. The final task is to show that the function $\mathcal{R}$ is injective. To this end, suppose b and b' are points such that $b \subseteq J$ and $b' \subseteq J$. We have to show that

$$
b = b' \quad \Leftarrow \quad R \circ b \circ R\backslash R \ = \ R \circ b' \circ R\backslash R \quad .
$$

We have

$$
\begin{array}{ll}
& R \circ b \circ R\backslash R \ = \ R \circ b' \circ R\backslash R \\
= & \{ \quad \mathcal{R} \text{ is a polar covering (proved above), lemma 164} \quad \} \\
& (R \circ b \circ R\backslash R)_< \ = \ (R \circ b' \circ R\backslash R)_<
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad (R\backslash R)_{<} = I\,,\ \text{domains} \quad \} \\
 & (R\circ b)_{<} \ = \ (R\circ b')_{<} \\
= & \{ \quad b\subseteq J\subseteq R_{\succ}\ \text{ and }\ b'\subseteq J\subseteq R_{\succ}\,,\ \text{lemma 158} \quad \} \\
 & b\circ\top\!\!\top\circ b' \ \subseteq \ J\circ\top\!\!\top\circ J \cap R_{\succ} \\
= & \{ \quad \text{theorem 23(b)},\ J\ \text{is a coreflexive, so}\ J_{<} = J = J_{>} \quad \} \\
 & b\circ\top\!\!\top\circ b' \ \subseteq \ J\circ R_{\succ}\circ J \\
= & \{ \quad J\ \text{is a (coreflexive) index of}\ R_{\succ}\,,\ \text{definition 105(b) with}\ P := R_{\succ} \quad \} \\
 & b\circ\top\!\!\top\circ b' \ \subseteq \ J \\
= & \{ \quad b\ \text{and}\ b'\ \text{are points},\ J\ \text{is a coreflexive, (153) with}\ a,a' := b,b' \quad \} \\
 & b = b' \ .
\end{array}
$$

□

Example 167 The less-than relation on real numbers has a polar covering. Specifically, suppose x is a real number. Let $lt.x$ denote (the coreflexive representing) $\{y : y\in\mathbb{R} : y<x\}$ and $al.x$ denote (the coreflexive representing) $\{y : y\in\mathbb{R} : x\leq y\}$. Theorem 166 predicts that

$$\langle x : x\in\mathbb{R} : lt.x\circ\top\!\!\top\circ al.x\rangle$$

is a polar covering of the less-than relation. (The only non-trivial part is to check that the at-most relation $\leq$ equals $<\backslash<$.)

This covering is, of course, not unique. More significantly, it is *not* non-redundant since

$$\left\langle \forall u,v \ : \ u<x\leq v \ : \ x\neq\tfrac{1}{2}(u+x) \ \wedge \ u<\tfrac{1}{2}(u+x)\leq v \right\rangle \ .$$

For any real number x, it is possible to remove the rectangle defined by x without affecting the supremum.

□

Given the straightforwardness of theorem 166, it is inevitable that our focus is not on the polarity of coverings but on the existence of *non-redundant* coverings. The adjective “non-redundant” is meant to express the property that removal of any element from a covering $\mathcal{R}$ will have the effect of strictly reducing $\cup\mathcal{R}$. Example 167 demonstrates that the less-than relation on real numbers has a polar covering but, as we shall see, the less-than relation on real numbers is an example of a relation for which there is no non-redundant covering.

The notation "$\mathcal{D}$" in definition 163 is chosen primarily to express the property that $\mathcal{D}.k$ uniquely "defines" (or "identifies") $\mathcal{R}.k$. Conveniently, it also expresses the property that the relation covered by a definiens (the relation $\cup\mathcal{D}$) is always difunctional: see lemma 169.

A polar covering is not *obviously redundant* in the sense that, for all elements U and V of $\mathcal{R}$,

$$U\subseteq V \;\equiv\; U=V \;\;.$$

(The easy proof is left to the reader.) That is, it is not possible to identify two elements U and V such that U is a proper subset of V and, thus, U can be removed from $\mathcal{R}$ without affecting $\cup\mathcal{R}$. Example 167 shows that the less-than relation on real numbers has a polar covering that has non-obvious redundancies. Example 168 is an example of a finite relation for which the polar covering constructed by theorem 166 has a non-obvious redundancy.

Example 168 Fig. 4 shows a relation R of type $\{A,B,C\}\sim\{\alpha,\beta,\gamma,\delta\}$. The four relations depicted in fig. 5 are rectangles of type $\{A,B,C\}\sim\{\alpha,\beta,\gamma,\delta\}$ (as indicated by the surrounding rectangular boxes); for greater clarity only edges connecting nodes in their left and right domains have been displayed.

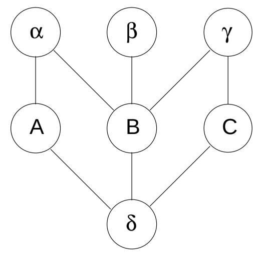


Figure 4: A Relation of Type $\{A,B,C\}\sim\{\alpha,\beta,\gamma,\delta\}$

These four rectangles are the elements of the polar covering constructed by theorem 166. The (reflexive-transitive reduction of the) ordering on the elements of the covering is depicted by arrowed brown lines. Take care to note how the depicted edges correspond to the ordering of the left domains of the rectangles:

$$\{B\}\subseteq\{A,B\} \;\wedge\; \{B\}\subseteq\{B,C\} \;\wedge\; \{A,B\}\subseteq\{A,B,C\} \;\wedge\; \{B,C\}\subseteq\{A,B,C\} \;\;,$$

and to the "polar" ordering of their right domains:

$$\{\alpha,\beta,\gamma,\delta\}\supseteq\{\alpha,\delta\} \;\wedge\; \{\alpha,\beta,\gamma,\delta\}\supseteq\{\beta,\delta\} \;\wedge\; \{\alpha,\delta\}\supseteq\{\delta\} \;\wedge\; \{\beta,\delta\}\supseteq\{\delta\} \;\;.$$

The top rectangle is redundant (but not "obviously" so). By removing this rectangle, one obtains a non-redundant polar covering: this is the polar covering that is the dual of

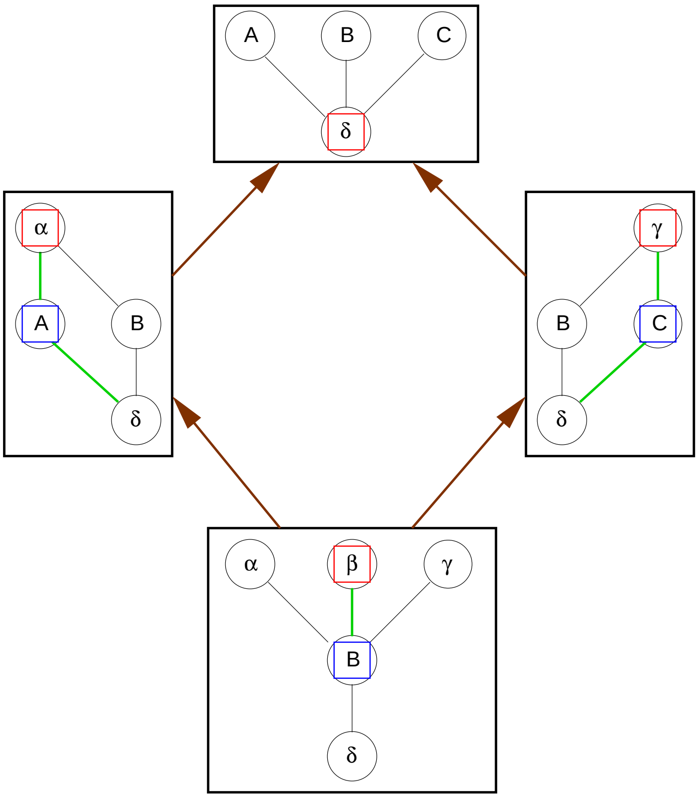


Figure 5: Polar Covering

the covering detailed in theorem 166 (thus indexed by $\{A,B,C\}$ rather than $\{\alpha,\beta,\gamma,\delta\}$). The definiens of this covering is depicted by the bold green edges in fig. 5.

The red and blue squares surrounding instances of the elements of $\{A,B,C\}$ and $\{\alpha,\beta,\gamma,\delta\}$ should be ignored for the moment. We return to this example later;
□

11.3 A Definiens is a Difunction

Crucial to establishing non-redundancy of a covering is the construction of a definiens. Those familiar with the theory of difunctions will immediately recognise that a definiens of a covering is necessarily a difunction (because a relation is a difunction iff it is the union of a set of completely disjoint rectangles). Because we don't need the full theory here, we present just the relevant property and its proof:

Lemma 169 Suppose $\mathcal{D}$ is a function such that

$$\begin{array}{ll} & \langle \forall k \;::\; \mathsf{rectangle}.(\mathcal{D}.k)\rangle \\ \wedge & \langle \forall j,k \;::\; \mathcal{D}.j \neq \mathcal{D}.k \;\equiv\; (\mathcal{D}.j)_< \cap (\mathcal{D}.k)_< = \perp\!\!\perp \;\wedge\; (\mathcal{D}.j)_> \cap (\mathcal{D}.k)_> = \perp\!\!\perp \rangle \;\; . \end{array}$$

Then $\cup\mathcal{D}$ is a difunction.

Proof Recalling lemma 64 (every rectangle is a difunction), we know that

$$(170) \quad \langle \forall k \;::\; \mathcal{D}.k \circ (\mathcal{D}.k)^{\cup} \circ \mathcal{D}.k \;\subseteq\; \mathcal{D}.k \rangle \;\; .$$

Aiming to exploit this property, we calculate:

$$\begin{array}{ll} & (\cup\mathcal{D})^{\cup} \circ \cup\mathcal{D} \\ = & \{ \quad \text{distributivity} \quad \} \\ & \langle \cup j,k :: (\mathcal{D}.j)^{\cup} \circ \mathcal{D}.k \rangle \\ = & \{ \quad \text{range disjunction: } \mathcal{D}.j = \mathcal{D}.k \vee \mathcal{D}.j \neq \mathcal{D}.k \quad \} \\ & \langle \cup j,k : \mathcal{D}.j = \mathcal{D}.k : (\mathcal{D}.j)^{\cup} \circ \mathcal{D}.k \rangle \;\cup\; \langle \cup j,k : \mathcal{D}.j \neq \mathcal{D}.k : (\mathcal{D}.j)^{\cup} \circ \mathcal{D}.k \rangle \\ = & \{ \quad \mathcal{D} \text{ is, by definition, a completely disjoint bag rectangles} \\ & \quad\;\; \text{lemma 162} \quad \} \\ & \langle \cup j,k : \mathcal{D}.j = \mathcal{D}.k : (\mathcal{D}.j)^{\cup} \circ \mathcal{D}.k \rangle \\ = & \{ \quad \text{Leibniz, nesting} \quad \} \\ & \langle \cup k :: \langle \cup j : \mathcal{D}.j = \mathcal{D}.k : (\mathcal{D}.k)^{\cup} \circ \mathcal{D}.k \rangle \rangle \\ \subseteq & \{ \quad \text{by reflexivity of the subset relation,} \end{array}$$

$$\langle\forall j\ :\ \mathcal{D}.j = \mathcal{D}.k\ :\ (\mathcal{D}.k)^{\cup}\circ\mathcal{D}.k\ \subseteq\ (\mathcal{D}.k)^{\cup}\circ\mathcal{D}.k\rangle$$

monotonicity, definition of supremum }

$$\langle\cup k :: (\mathcal{D}.k)^{\cup}\circ\mathcal{D}.k\rangle\ .$$

Thus,

$$\begin{array}{ll} & \cup\mathcal{D}\circ(\cup\mathcal{D})^{\cup}\circ\cup\mathcal{D} \\ \subseteq & \{\quad \text{above, monotonicity, distributivity}\quad\} \\ & \langle\cup j,k :: \mathcal{D}.j\circ(\mathcal{D}.k)^{\cup}\circ\mathcal{D}.k\rangle \\ \subseteq & \{\quad \text{similar calculation to that above}\quad\} \\ & \langle\cup k :: \mathcal{D}.k\circ(\mathcal{D}.k)^{\cup}\circ\mathcal{D}.k\rangle \\ \subseteq & \{\quad (170)\quad\} \\ & \langle\cup k :: \mathcal{D}.k\rangle \\ = & \{\quad \text{definition}\quad\} \\ & \cup\mathcal{D}\ . \end{array}$$

It follows, by definition of a difunction, that $\cup\mathcal{D}$ is a difunction.
□

12 The Diagonal

This section anticipates the study of block-ordered relations in section 13. We introduce the notion of the "diagonal" of a relation in section 12.1 and formulate some basic properties in section 12.2.

In section 11.2, we introduced the notion of a polar covering of a relation. Theorem 166 shows how to construct a polar covering for any given relation but example 167 demonstrates that the construction does not always produce a non-redundant covering. In section 12.4, we explore conditions under which the diagonal of the relation guarantees the non-redundancy of the covering.

12.1 Definition and Examples

Straightforwardly from the definition of factors, properties of converse and set intersection,

(171) $\quad R \text{ is difunctional} \quad\equiv\quad R \;=\; R\cap(R\backslash R/R)^{\cup}\ .$

More generally, we have:

Lemma 172 For all R, $R\cap(R\backslash R/R)^{\cup}$ is difunctional.

Proof Let S denote $R\cap(R\backslash R/R)^{\cup}$. We have to prove that S is difunctional. That is, by definition,

$$S\circ S^{\cup}\circ S \subseteq S \quad .$$

Since the right side is an intersection, this is equivalent to

$$S\circ S^{\cup}\circ S \subseteq R \quad\wedge\quad S\circ S^{\cup}\circ S \subseteq (R\backslash R/R)^{\cup} \quad .$$

The first is (almost) trivial:

$$\begin{array}{cl}
 & S\circ S^{\cup}\circ S \\
\subseteq & \{ \quad S\subseteq R,\ S\subseteq(R\backslash R/R)^{\cup}, \\
 & \quad\ \ \text{converse, monotonicity} \quad \} \\
 & R\circ R\backslash R/R\circ R \\
\subseteq & \{ \quad \text{cancellation} \quad \} \\
 & R \quad .
\end{array}$$

In the above calculation, the trick was to replace the outer occurrences of S on the left side by R and the middle occurrence by $(R\backslash R/R)^{\cup}$. The replacement is done the opposite way around in the second calculation.

$$\begin{array}{cl}
 & S\circ S^{\cup}\circ S \subseteq (R\backslash R/R)^{\cup} \\
\Leftarrow & \{ \quad S\subseteq(R\backslash R/R)^{\cup},\ S\subseteq R, \text{ monotonicity and transitivity} \quad \} \\
 & (R\backslash R/R)^{\cup}\circ R^{\cup}\circ(R\backslash R/R)^{\cup} \subseteq (R\backslash R/R)^{\cup} \\
= & \{ \quad \text{converse} \quad \} \\
 & R\backslash R/R\circ R\circ R\backslash R/R \subseteq R\backslash R/R \\
= & \{ \quad \text{Galois connection} \quad \} \\
 & R\circ R\backslash R/R\circ R\circ R\backslash R/R\circ R \subseteq R \\
= & \{ \quad \text{cancellation, monotonicity and transitivity} \quad \} \\
 & \mathsf{true} \quad .
\end{array}$$

□

We call the relation $R\cap(R\backslash R/R)^{\cup}$ the *diagonal* of R; Riguet [Rig51] calls it the “différence” of the relation. (Riguet’s definition does not use factors but is equivalent.)

Definition 173 (Diagonal) The *diagonal* of relation R is the relation $R\cap(R\backslash R/R)^{\cup}$. For brevity, $R\cap(R\backslash R/R)^{\cup}$ will be denoted by ΔR.
□

Many readers will be familiar with the decomposition of a preorder into a partial ordering on a set of equivalence classes. The diagonal of a preorder T is the equivalence relation $T\cap T^{\cup}$. More generally:

Example 174 The diagonal of a provisional preorder T is $T\cap T^{\cup}$. This is because, for an arbitrary relation T,

$$T\cap(T\backslash T/T)^{\cup} \quad = \quad T \;\cap\; T{<}\circ(T\backslash T/T)^{\cup}\circ T{>} \quad .$$

But, if T is a provisional preorder,

$$T{<}\circ(T\backslash T/T)^{\cup}\circ T{>} \quad = \quad T^{\cup} \quad .$$

(See lemmas 54 and 57.)
□

Example 175 A particular instance of example 174 is if G is the edge relation of a finite graph. Then $\Delta(G^*)$ is $G^*\cap(G^{\cup})^*$, the relation that holds between nodes a and b if there is a path from a to b and a path from b to a in the graph. Thus $\Delta(G^*)$ is the equivalence relation that holds between nodes that are in the same strongly connected component of G.
□

Example 176 In this example, we consider three versions of the less-than relation: the homogeneous less-than relation on integers, which we denote by $<_{\mathbb{Z}}$, the homogeneous less-than relation on real numbers, which we denote by $<_{\mathbb{R}}$, and the heterogeneous less-than relation on integers and real numbers, which we denote by ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$. Specifically, the relation ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$ relates integer m to real number x when $m<x$ (using the conventional over-loaded notation). We also subscript the at-most symbol $\leq$ in the same way in order to indicate the type of the relation in question.

The diagonal of the less-than relation on integers is the predecessor relation (i.e. it relates integer m to integer n exactly when $n=m+1$). This is because $<_{\mathbb{Z}}\backslash<_{\mathbb{Z}} = \leq_{\mathbb{Z}}$, and $\leq_{\mathbb{Z}}/<_{\mathbb{Z}}$ relates integer m to integer n exactly when $m\leq_{\mathbb{Z}}n+1$ (where the subscript $\mathbb{Z}$ indicates the type of the ordering). The diagonal is thus functional and injective.

The diagonal of the less-than relation on real numbers is the empty relation. This is because $<_{\mathbb{R}}\backslash<_{\mathbb{R}} = \leq_{\mathbb{R}}$, $\leq_{\mathbb{R}}/<_{\mathbb{R}} = \leq_{\mathbb{R}}$ and $<_{\mathbb{R}}\cap\geq_{\mathbb{R}}=\perp\!\!\!\perp_{\mathbb{R}}$. (Again, the subscript indicates the type of the ordering.)

The diagonal of the heterogeneous less-than relation ${}_{\mathbb{Z}}{<}_{\mathbb{R}}$ relates integer m to real number x when $m<x\leq m{+}1$. This is equivalent to $\lceil x \rceil = m{+}1$. The diagonal is thus a difunctional relation characterised by —in the sense of theorem 111— the ceiling function $\langle x{::}\lceil x \rceil\rangle$ and the successor function $\langle m :: m{+}1\rangle$.

We leave the reader to check the details of this example. See also examples 167 and 217.
□

The following example introduces a general mechanism for constructing illustrative examples of the concepts introduced later. The example exploits the observation that ΔR is injective if the preorder $R\backslash R$ is anti-symmetric; that is, ΔR is injective if $R\backslash R$ is a partial ordering. (Equivalently, ΔR is functional if R/R is a partial ordering.) We leave the straightforward proof to the reader.

Example 177 Suppose $\mathcal{X}$ is a finite type. We use dummy x to range over elements of type $\mathcal{X}$. Let $\mathcal{S}$ denote a subset of $2^{\mathcal{X}}$. Let in denote the membership relation of type $\mathcal{X}{\sim}\mathcal{S}$. That is, if S is a subset of $\mathcal{S}$, $x{\circ}\top\!\!\top{\circ}S \subseteq \mathsf{in}$ exactly when x is an element of the set S. The relation $\mathsf{in}\backslash\mathsf{in}$ is the subset relation of type $\mathcal{S}{\sim}\mathcal{S}$.

(Conventionally, in is denoted by the symbol "$\in$" and one writes $x{\in}S$ instead of $x{\circ}\top\!\!\top{\circ}S \subseteq \mathsf{in}$. Also, the relation $\mathsf{in}\backslash\mathsf{in}$ is conventionally denoted by the symbol "$\subseteq$". That is, if S and S' are both elements of $\mathcal{S}$, $S{\circ}\top\!\!\top{\circ}S' \subseteq \mathsf{in}\backslash\mathsf{in}$ exactly when $S\subseteq S'$. Were we to adopt conventional practice, the overloading of the notation that occurs is likely to cause confusion, so we choose to avoid it.)

The relation $\mathsf{in}\backslash\mathsf{in}$ is anti-symmetric. As a consequence, $\Delta\mathsf{in}$ is injective. (Equivalently, $(\Delta\mathsf{in})^{\cup}$ is functional.) Specifically, for all x of type $\mathcal{X}$ and S of type $\mathcal{S}$,

$$x{\circ}\top\!\!\top{\circ}S \subseteq \Delta\mathsf{in} \quad\equiv\quad x{\circ}\top\!\!\top{\circ}S \subseteq \mathsf{in} \;\wedge\; \langle\forall S' : x{\circ}\top\!\!\top{\circ}S' \subseteq \mathsf{in} : S{\circ}\top\!\!\top{\circ}S' \subseteq \mathsf{in}\backslash\mathsf{in}\rangle \quad ,$$

where dummy S' ranges over elements of $\mathcal{S}$. Using conventional notation, the right side of this equation is recognised as the definition of a minimum, and one might write

$$x\;[\![\Delta\mathsf{in}]\!]\;S \quad\equiv\quad S \risingdotseq \langle \mathsf{MIN}\,S' {:} x{\in}S' {:} S'\rangle$$

where the venturi tube "$\risingdotseq$" indicates an equality assuming the well-definedness of the expression on its right side.

Fig. 6 shows a particular instance. The set $\mathcal{X}$ is the set of numbers from 0 to 3. The set $\mathcal{S}$ is a subset of $2^{\{0,1,2,3\}}$; the chosen subset and the relation $\mathsf{in}\backslash\mathsf{in}$ for this choice are depicted by the directed graph forming the central portion of fig. 6. The relation $\Delta\mathsf{in}$ of type $\mathcal{X}{\sim}\mathcal{S}$ is depicted by the undirected graph whereby the atoms of the relation are depicted as rectangles. Note that the numbers 0 and 3 are not related by $\Delta\mathsf{in}$ to any of the elements of $\mathcal{S}$.
□

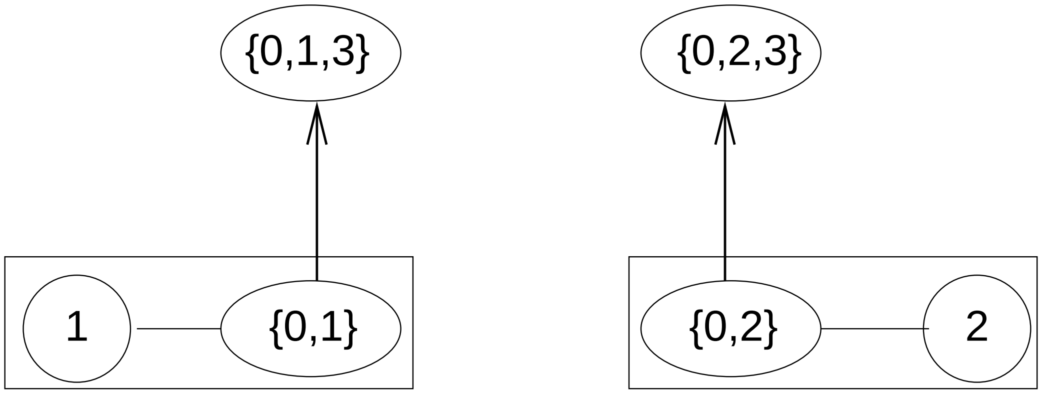


Figure 6: Diagonal of an Instance of the Membership Relation

12.2 Basic Properties

Primarily for notational convenience, we note a simple property of the diagonal:

Lemma 178

$$(\Delta R)^{\cup} \;=\; \Delta(R^{\cup}) \quad .$$

Proof

$$
\begin{array}{ll}
 & (\Delta R)^{\cup} \\
= & \{ \quad \text{definition and distributivity} \quad \} \\
 & R^{\cup} \cap R\backslash R/R \\
= & \{ \quad \text{factors} \quad \} \\
 & R^{\cup} \cap (R^{\cup} \backslash R^{\cup} / R^{\cup})^{\cup} \\
= & \{ \quad \text{definition} \quad \} \\
 & \Delta(R^{\cup}) \quad .
\end{array}
$$

□

A consequence of lemma 178 is that we can write $\Delta R^{\cup}$ without ambiguity. This we do from now on.

Very straightforwardly, the relation $R \circ R^{\cup}$ is a per if R is difunctional. For a difunctional relation R, the relation $R \circ R^{\cup}$ is the left per domain of R. (Symmetrically, $R^{\cup} \circ R$ is the right per domain of R. See theorem 49, parts (iv) and (vi).) Thus $\Delta R \circ (\Delta R)^{\cup}$ is the left per domain of the diagonal of R. The following lemma is the basis of the construction, in certain cases, of an economic representation of the diagonal of R and, hence, of R itself.

Lemma 179 For all relations R,

$$(\Delta R)_{\prec} \;=\; (\Delta R)_{<} \circ R_{\prec} \;=\; R_{\prec} \circ (\Delta R)_{<} \;\; .$$

Dually,

$$(\Delta R)_{\succ} \;=\; R_{\succ} \circ (\Delta R)_{>} \;=\; (\Delta R)_{>} \circ R_{\succ} \;\; .$$

Proof We prove the first equation by mutual inclusion. First,

$$\begin{array}{ll}
 & (\Delta R)_{\prec} \subseteq (\Delta R)_{<} \circ R_{\prec} \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 49; definition: (28)} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \subseteq (\Delta R)_{<} \circ R /\!\!/ R \\
\Leftarrow & \{ \quad \text{domains and monotonicity} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \subseteq R /\!\!/ R \\
= & \{ \quad \text{definition of } R /\!\!/ R \text{, converse and factors} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R \subseteq R \\
= & \{ \quad \Delta R \subseteq R \,;\; \Delta R^{\cup} \subseteq R \backslash R / R \text{ and cancellation} \quad \} \\
 & \mathsf{true} \;\; .
\end{array}$$

Second,

$$\begin{array}{ll}
 & (\Delta R)_{<} \circ R_{\prec} \subseteq (\Delta R)_{\prec} \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 49} \quad \} \\
 & (\Delta R)_{<} \circ R_{\prec} \subseteq \Delta R \circ \Delta R^{\cup} \\
\Leftarrow & \{ \quad \text{domains and definition: (28)} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R /\!\!/ R \subseteq \Delta R \circ \Delta R^{\cup} \\
\Leftarrow & \{ \quad \text{monotonicity and converse} \quad \} \\
 & R /\!\!/ R \circ \Delta R \subseteq \Delta R \\
= & \{ \quad \text{definition of diagonal} \quad \} \\
 & R /\!\!/ R \circ \Delta R \subseteq R \;\;\wedge\;\; R /\!\!/ R \circ \Delta R \subseteq (R \backslash R / R)^{\cup} \\
\Leftarrow & \{ \quad \Delta R \subseteq R \text{ ; converse} \quad \} \\
 & R /\!\!/ R \circ R \subseteq R \;\;\wedge\;\; \Delta R^{\cup} \circ R /\!\!/ R \subseteq R \backslash R / R \\
= & \{ \quad \text{cancellation; factors} \quad \} \\
 & \mathsf{true} \;\;\wedge\;\; R \circ \Delta R^{\cup} \circ R /\!\!/ R \circ R \subseteq R
\end{array}$$

$\Leftarrow \quad \{ \quad \text{cancellation and } \Delta R^{\cup} \subseteq R\backslash R/R \quad \}$

$R \circ R\backslash R/R \circ R \subseteq R$

$= \quad \{ \quad \text{cancellation} \quad \}$

true .

The remaining three equalities are simple consequences of the properties of converse, pers and coreflexives.

□

The following corollary of lemma 179 proves to be crucial later:

Lemma 180 For all relations R,

$$(\Delta R)_{\prec} = R_{\prec} \equiv (\Delta R)_{<} = R_{<} \quad .$$

Dually,

$$(\Delta R)_{\succ} = R_{\succ} \equiv (\Delta R)_{>} = R_{>} \quad .$$

Proof The proof is by mutual implication:

$(\Delta R)_{<} = R_{<}$

$\Rightarrow \quad \{ \quad \text{lemma 179 and Leibniz} \quad \}$

$(\Delta R)_{\prec} = R_{<} \circ R_{\prec}$

$= \quad \{ \quad \text{dual of (36)} \quad \}$

$(\Delta R)_{\prec} = R_{\prec}$

$\Rightarrow \quad \{ \quad \text{Leibniz} \quad \}$

$((\Delta R)_{\prec})_{<} = (R_{\prec})_{<}$

$= \quad \{ \quad \text{dual of (36) with } R{:=}\Delta R \text{ and } R{:=}R \quad \}$

$(\Delta R)_{<} = R_{<}$.

□

12.3 Reduction to the Core

In this section our goal is to prove that if J is an index of relation R then ΔJ is an index of ΔR. Instantiating definition 100 with $J,R{:=}\Delta J,\Delta R$ the properties we have to prove are as follows.

(a) $\Delta J \subseteq \Delta R$,

(b) $\Delta R \circ \Delta J^{\cup} \circ \Delta R \;=\; \Delta R$.

(c) $(\Delta J)_< \circ \Delta R \circ \Delta R^{\cup} \circ (\Delta J)_< \;=\; (\Delta J)_<$,

(d) $(\Delta J)_> \circ \Delta R^{\cup} \circ \Delta R \circ (\Delta J)_> \;=\; (\Delta J)_>$.

Of these, the hardest to prove is (b). For properties (a), (c) and (d), all we need is that J is an arbitrary index of R. For property (b), we use the fact that an index of an arbitrary relation R is defined to be $J{\circ}R{\circ}K$ where J is an index of $R{\prec}$ and K is an index of $R{\succ}$.

We begin with the easier properties.

Lemma 181 Suppose J is an index of R. Then

$$\Delta J \subseteq \Delta R \quad .$$

Proof

$$
\begin{array}{ll}
 & \Delta J \subseteq \Delta R \\
= & \{ \quad \text{definition 173} \quad \} \\
 & J \cap (J\backslash J/J)^{\cup} \;\subseteq\; R \cap (R\backslash R/R)^{\cup} \\
= & \{ \quad \text{domains} \quad \} \\
 & J \cap J_< \circ (J\backslash J/J)^{\cup} \circ J_> \;\subseteq\; R \cap (R\backslash R/R)^{\cup} \\
\Leftarrow & \{ \quad J \text{ is an index of } R \text{, so } J \subseteq R \text{; monotonicity} \quad \} \\
 & J_< \circ (J\backslash J/J)^{\cup} \circ J_> \;\subseteq\; (R\backslash R/R)^{\cup} \\
= & \{ \quad \text{converse} \quad \} \\
 & J_> \circ J\backslash J/J \circ J_< \;\subseteq\; R\backslash R/R \\
= & \{ \quad \text{factors} \quad \} \\
 & R \circ J_> \circ J\backslash J/J \circ J_< \circ R \;\subseteq\; R \\
= & \{ \quad J \text{ is an index of } R \text{, definition 78(b); per domains} \quad \} \\
 & R{\prec} \circ J \circ R{\succ} \circ J_> \circ J\backslash J/J \circ J_< \circ R{\prec} \circ J \circ R{\succ} \;\subseteq\; R{\prec} \circ R \circ R{\succ} \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & J \circ R{\succ} \circ J_> \circ J\backslash J/J \circ J_< \circ R{\prec} \circ J \;\subseteq\; R \quad .
\end{array}
$$

Continuing with the left side of the inclusion:

$$
\begin{array}{ll}
 & J \circ R{\succ} \circ J{>} \circ J\backslash J/J \circ J{<} \circ R{\prec} \circ J \\
= & \{ \quad \text{domains} \quad \} \\
 & J \circ J{>} \circ R{\succ} \circ J{>} \circ J\backslash J/J \circ J{<} \circ R{\prec} \circ J{<} \circ J \\
= & \{ \quad J \text{ is an index of } R\text{; definition 78(c) and (d)} \quad \} \\
 & J \circ J{>} \circ J\backslash J/J \circ J{<} \circ J \\
\subseteq & \{ \quad \text{domains and cancellation} \quad \} \\
 & J \\
\subseteq & \{ \quad J \text{ is an index of } R\text{; definition 78(a)} \quad \} \\
 & R \quad .
\end{array}
$$

□

Lemma 182 Suppose J is an index of R. Then

$$(\Delta J){<} \circ \Delta R \circ \Delta R^{\cup} \circ (\Delta J){<} \quad = \quad (\Delta J){<} \quad .$$

Dually,

$$(\Delta J){>} \circ \Delta R^{\cup} \circ \Delta R \circ (\Delta J){>} \quad = \quad (\Delta J){>} \quad .$$

Proof

$$
\begin{array}{ll}
 & (\Delta J){<} \circ \Delta R \circ \Delta R^{\cup} \circ (\Delta J){<} \\
= & \{ \quad \Delta R \text{ is a difunction, theorem 49} \quad \} \\
 & (\Delta J){<} \circ (\Delta R){\prec} \circ (\Delta J){<} \\
= & \{ \quad \text{lemma 179 (and symmetry)} \quad \} \\
 & (\Delta J){<} \circ (\Delta R){<} \circ R{\prec} \circ (\Delta R){<} \circ (\Delta J){<} \\
= & \{ \quad \text{by lemma 181 and monotonicity, } (\Delta J){<} \subseteq (\Delta R){<} \quad \} \\
 & (\Delta J){<} \circ R{\prec} \circ (\Delta J){<} \\
= & \{ \quad (\Delta J){<} \subseteq J{<} \text{ (since } \Delta J \subseteq J\text{)} \quad \} \\
 & (\Delta J){<} \circ J{<} \circ R{\prec} \circ J{<} \circ (\Delta J){<} \\
= & \{ \quad J \text{ is an index of } R\text{, definition 78(c)} \quad \} \\
 & (\Delta J){<} \circ J{<} \circ (\Delta J){<} \\
= & \{ \quad (\Delta J){<} \subseteq J{<} \text{ (since } \Delta J \subseteq J\text{)} \quad \} \\
 & (\Delta J){<} \quad .
\end{array}
$$

□

In order to prove (b), we prove a more general theorem on cores. First, a lemma:

Lemma 183 Suppose R, C, λ and ρ are as in definition 90. Then

$$R_{>}\circ R\backslash R/R\circ R_{<} \;=\; \rho^{\cup}\circ C\backslash C/C\circ\lambda \quad .$$

Proof

$$
\begin{array}{ll}
 & R_{>}\circ R\backslash R/R\circ R_{<} \\
= & \{ \quad (36) \quad \} \\
 & (R_{\succ})_{>}\circ R\backslash R/R\circ (R_{\prec})_{<} \\
= & \{ \quad R_{\prec} \;=\; \lambda^{\cup}\circ\lambda\,,\; R_{\succ} \;=\; \rho^{\cup}\circ\rho\,, \text{and domains} \quad \} \\
 & \rho_{>}\circ R\backslash R/R\circ\lambda_{>} \\
= & \{ \quad \text{lemma 92} \quad \} \\
 & \rho_{>}\circ(\lambda^{\cup}\circ C\circ\rho)\backslash(\lambda^{\cup}\circ C\circ\rho)/(\lambda^{\cup}\circ C\circ\rho)\circ\lambda_{>} \\
= & \{ \quad \text{lemma 44 with } f,g,U,V,W := \rho,\lambda,C,C,C \quad \} \\
 & \rho^{\cup}\circ(\lambda_{<}\circ C)\backslash C/(C\circ\rho_{<})\circ\lambda \\
= & \{ \quad C = \lambda\circ R\circ\rho^{\cup}\,;\ \text{so } \lambda_{<}\circ C = C = C\circ\rho_{<} \quad \} \\
 & \rho^{\cup}\circ C\backslash C/C\circ\lambda \quad .
\end{array}
$$

□

Theorem 184 Suppose R, C, λ and ρ are as in definition 90. Then

$$\Delta R \;=\; \lambda^{\cup}\circ\Delta C\circ\rho \quad\wedge\quad \Delta C \;=\; \lambda\circ\Delta R\circ\rho^{\cup} \quad .$$

In words, if λ and ρ witness that C is a core of R, then λ and ρ witness that ΔC is a core of ΔR.

Proof

$$
\begin{array}{ll}
 & \Delta R \\
= & \{ \quad \text{definition} \quad \} \\
 & R\cap(R\backslash R/R)^{\cup} \\
= & \{ \quad \text{domains and converse} \quad \} \\
 & R \;\cap\; (R_{>}\circ R\backslash R/R\circ R_{<})^{\cup}
\end{array}
$$

$$
\begin{array}{cl}
= & \{ \quad \text{lemma 183} \quad \} \\
 & R \cap (\rho^{\cup} \circ C \backslash C / C \circ \lambda)^{\cup} \\
= & \{ \quad \text{lemma 92} \quad \} \\
 & \lambda^{\cup} \circ C \circ \rho \ \cap \ (\rho^{\cup} \circ C \backslash C / C \circ \lambda)^{\cup} \\
= & \{ \quad \text{distributivity of converse and functional relations} \quad \} \\
 & \lambda^{\cup} \circ (C \cap (C \backslash C / C)^{\cup}) \circ \rho \\
= & \{ \quad \text{definition 173} \quad \} \\
 & \lambda^{\cup} \circ \Delta C \circ \rho \ .
\end{array}
$$

Hence

$$
\begin{array}{cl}
 & \lambda \circ \Delta R \circ \rho^{\cup} \\
= & \{ \quad \text{above} \quad \} \\
 & \lambda \circ \lambda^{\cup} \circ \Delta C \circ \rho \circ \rho^{\cup} \\
= & \{ \quad \lambda \text{ and } \rho \text{ are functional} \quad \} \\
 & \lambda_{<} \circ \Delta C \circ \rho_{<} \\
= & \{ \quad \Delta C \subseteq C \text{ ; so } (\Delta C)_{<} \subseteq C_{<} \text{ and } (\Delta C)_{>} \subseteq C_{>} \\
 & \quad\quad \text{lemma 94 and domains} \quad \} \\
 & \Delta C \ .
\end{array}
$$

□

We are now in a position to prove the final property (b) above.

Lemma 185 Suppose J is an index of R. Then

$$\Delta R \circ \Delta J^{\cup} \circ \Delta R \ = \ \Delta R \ .$$

Proof We begin by noting that theorem 184 applies with C instantiated to J and λ and ρ defined by $\lambda = J_{<} \circ R_{\prec}$ and $\rho = J_{>} \circ R_{\succ}$. This is because J is a core of R : see theorem 91. So

$$
\begin{array}{cl}
 & \Delta R \circ \Delta J^{\cup} \circ \Delta R \\
= & \{ \quad \text{theorem 184 with } C,\lambda,\rho := J \text{ , } J_{<} \circ R_{\prec} \text{ , } J_{>} \circ R_{\succ} \quad \} \\
 & \Delta R \circ (\lambda \circ \Delta R \circ \rho^{\cup})^{\cup} \circ \Delta R \\
= & \{ \quad \text{converse} \quad \} \\
 & \Delta R \circ \rho \circ \Delta R^{\cup} \circ \lambda^{\cup} \circ \Delta R
\end{array}
$$

$= \quad \{ \quad$ definition of ρ and λ, $(J_< \circ R_\prec)^\cup = R_\prec \circ J_< \quad \}$

$\Delta R \circ J_> \circ R_\succ \circ \Delta R^\cup \circ R_\prec \circ J_< \circ \Delta R$

$= \quad \{ \quad$ per domains $\quad \}$

$\Delta R \circ (\Delta R)_\succ \circ J_> \circ R_\succ \circ \Delta R^\cup \circ R_\prec \circ J_< \circ (\Delta R)_\prec \circ \Delta R$

$= \quad \{ \quad$ lemma 179 $\quad \}$

$\Delta R \circ (\Delta R)_> \circ R_\succ \circ J_> \circ R_\succ \circ \Delta R^\cup \circ R_\prec \circ J_< \circ R_\prec \circ (\Delta R)_< \circ \Delta R$

$= \quad \{ \quad$ lemma 87 $\quad \}$

$\Delta R \circ (\Delta R)_> \circ R_\succ \circ \Delta R^\cup \circ R_\prec \circ (\Delta R)_< \circ \Delta R$

$= \quad \{ \quad$ lemma 179 $\quad \}$

$\Delta R \circ (\Delta R)_\succ \circ \Delta R^\cup \circ (\Delta R)_\prec \circ \Delta R$

$= \quad \{ \quad$ per domains $\quad \}$

$\Delta R \circ \Delta R^\cup \circ \Delta R$

$= \quad \{ \quad \Delta R$ is difunctional, theorem 49 $\quad \}$

ΔR .

□

Putting all the lemmas together, we have:

Theorem 186 Suppose J is an index of R. Then ΔJ is an index of ΔR.

Proof Lemmas 181, 182 and 185 combined with definition 100 (instantiated with $J,R := \Delta J, \Delta R$).

□

We conclude with a beautiful theorem.

Theorem 187 Suppose J is an index of R. Then

$$\Delta J = J_< \circ \Delta R \circ J_> \quad \wedge \quad \Delta R = R_\prec \circ \Delta J \circ R_\succ \ .$$

Proof We first prove, by mutual implication, that the two equations are equivalent. Assume that

$$\Delta R = R_\prec \circ \Delta J \circ R_\succ \ .$$

Then,

$J_< \circ \Delta R \circ J_>$

$= \quad \{ \quad$ assumption $\quad \}$

$$J_< \circ R_\prec \circ \Delta J \circ R_\succ \circ J_>$$
$$= \quad \{ \quad \Delta J \subseteq J \text{, so } (\Delta J)_< \subseteq J_< \text{ and } (\Delta J)_> \subseteq J_>\text{; domains} \quad \}$$
$$J_< \circ R_\prec \circ J_< \circ \Delta J \circ J_> \circ R_\succ \circ J_>$$
$$= \quad \{ \quad J \text{ is an index of } R\text{, definition 78(c) and (d)} \quad \}$$
$$J_< \circ \Delta J \circ J_>$$
$$= \quad \{ \quad \text{reverse of middle step} \quad \}$$
$$\Delta J \quad .$$

Conversely, assume

$$\Delta J = J_< \circ \Delta R \circ J_> \quad .$$

Then,

$$R_\prec \circ \Delta J \circ R_\succ$$
$$= \quad \{ \quad \text{assumption} \quad \}$$
$$R_\prec \circ J_< \circ \Delta R \circ J_> \circ R_\succ$$
$$= \quad \{ \quad \text{lemma 179} \quad \}$$
$$R_\prec \circ J_< \circ (\Delta R)_< \circ R_\prec \circ \Delta R \circ R_\succ \circ (\Delta R)_> \circ J_> \circ R_\succ$$
$$= \quad \{ \quad \text{lemma 181 and domains} \quad \}$$
$$R_\prec \circ J_< \circ R_\prec \circ \Delta R \circ R_\succ \circ J_> \circ R_\succ$$
$$= \quad \{ \quad \text{definition 78(c) and 78(d)} \quad \}$$
$$R_\prec \circ \Delta R \circ R_\succ$$
$$= \quad \{ \quad \text{lemma 179 and domains} \quad \}$$
$$\Delta R \quad .$$

Combining the two calculations, the two equations are equivalent and, therefore, it suffices to prove just one of them[4]. We prove the second by mutual inclusion:

$$\Delta R$$
$$= \quad \{ \quad \Delta R \text{ is difunctional} \quad \}$$
$$\Delta R \circ \Delta R^\cup \circ \Delta R$$

[4]It is not necessary to prove the equivalence of the two statements in order to prove the theorem; we could have omitted the second calculation. But some redundancy in proofs enhances their reliability.

$$
\begin{array}{ll}
= & \{ \quad \text{lemma 185, converse} \quad \} \\
& \Delta R \circ \Delta R^{\cup} \circ \Delta J \circ \Delta R^{\cup} \circ \Delta R \\
= & \{ \quad \Delta R \ \text{ is difunctional, theorem 49(iv) and (vi)} \quad \} \\
& (\Delta R)_{\prec} \circ \Delta J \circ (\Delta R)_{\succ} \\
= & \{ \quad \text{lemma 179} \quad \} \\
& (\Delta R)_{<} \circ R_{\prec} \circ \Delta J \circ R_{\succ} \circ (\Delta R)_{>} \\
\subseteq & \{ \quad \text{domains are coreflexive} \quad \} \\
& R_{\prec} \circ \Delta J \circ R_{\succ} \\
\subseteq & \{ \quad \text{lemma 181 and monotonicity} \quad \} \\
& R_{\prec} \circ \Delta R \circ R_{\succ} \\
= & \{ \quad \text{lemma 179, domains} \quad \} \\
& \Delta R \quad .
\end{array}
$$

□

12.4 Non-Redundant Polar Coverings

We have shown in theorem 166 how to construct an injective polar covering of a given relation R. Now we consider circumstances in which the covering is non-redundant. In the case that R is difunctional, it is straightforward to show that the covering constructed in theorem 166 is non-redundant and is its own definiens. (We omit the proof because it is a special case of theorem 188.) This suggests that, in general, a covering of the diagonal of a relation R can be used as the definiens of a covering of R. This, however, is not the case: see example 195. It is true so long as the diagonal is sufficiently large. Specifically:

Theorem 188 Suppose R is a relation and suppose $(\Delta R)_{>} = R_{>}$. Suppose J is an index of $R_{\succ}$. Then the function $\mathcal{D}$ defined by

$$\mathcal{D} \;=\; \langle b : b \subseteq J : \Delta R \circ b \circ \Delta R \backslash \Delta R \rangle$$

is an injective, polar covering of ΔR. Moreover, if $(\Delta R)_{>} = R_{>}$, for all points b and b' such that $b \subseteq J$ and $b' \subseteq J$,

$$b \neq b' \quad \equiv \quad (\Delta R \circ b \circ \Delta R \backslash \Delta R)_{<} \circ (\Delta R \circ b' \circ \Delta R \backslash \Delta R)_{<} \;=\; \bot\!\!\bot$$

and

$$b \neq b' \quad \equiv \quad (\Delta R \circ b \circ \Delta R \backslash \Delta R)_{>} \circ (\Delta R \circ b' \circ \Delta R \backslash \Delta R)_{>} \;=\; \bot\!\!\bot \quad .$$

It follows that, if $(\Delta R)_{>} = R_{>}$, $\mathcal{D}$ is a completely disjoint, injective, polar covering of ΔR.

Proof That $\mathcal{D}$ is an injective covering of ΔR is an application of theorem 166 with $R := \Delta R$: it suffices to note that the assumption $(\Delta R)_{>} = R_{>}$ is equivalent to the assumption $(\Delta R)_{\succ} = R_{\succ}$, by lemma 180, and so J is an index of $(\Delta R)_{\succ}$.

We use lemma 162 to show that $\mathcal{D}$ is completely disjoint. First, the elements are non-empty because $\mathcal{D}$ is a polar covering. That is,

$$(189) \quad \langle \forall b \;:\; b \subseteq J \;:\; \Delta R \circ b \circ \Delta R \backslash \Delta R \neq \perp\!\!\perp \rangle \quad .$$

For the second proof obligation (see lemma 162), assume that $b \neq b'$. We begin by noting that we can exploit (52) to rewrite the definition of $\mathcal{D}$. Specifically,

$$\begin{array}{ll}
& \Delta R \circ b \circ \Delta R \backslash \Delta R \\
= & \{ \quad b \subseteq J \subseteq (\Delta R)_{>} \quad \} \\
& \Delta R \circ b \circ (\Delta R)_{>} \circ \Delta R \backslash \Delta R \\
= & \{ \quad \Delta R \text{ is difunctional, (52)} \quad \} \\
& \Delta R \circ b \circ (\Delta R)_{\succ} \quad .
\end{array}$$

That is,

$$(190) \quad \mathcal{D} \;=\; \langle b : b \subseteq J : \Delta R \circ b \circ (\Delta R)_{\succ} \rangle \quad .$$

We use this defintion of $\mathcal{D}$ to prove that its elements are completely disjoint. First, the left domains. We have, for all points b and b' such that $b \subseteq J$ and $b' \subseteq J$,

$$\begin{array}{ll}
& \Delta R \circ b \circ (\Delta R)_{\succ} \circ (\Delta R \circ b' \circ (\Delta R)_{\succ})^{\cup} \\
= & \{ \quad \text{converse, } (\Delta R)_{\succ} \text{ is a per, } b' \text{ is coreflexive} \quad \} \\
& \Delta R \circ b \circ (\Delta R)_{\succ} \circ b' \circ \Delta R^{\cup} \\
= & \{ \quad b \subseteq J \text{ and } b' \subseteq J,\ b,\ b' \text{ and } J \text{ are coreflexive} \quad \} \\
& \Delta R \circ b \circ J \circ (\Delta R)_{\succ} \circ J \circ b' \circ \Delta R^{\cup} \\
= & \{ \quad J \text{ is an index of } (\Delta R)_{\succ}\text{, lemma 104 with } R := (\Delta R)_{\succ} \quad \} \\
& \Delta R \circ b \circ J \circ b' \circ \Delta R^{\cup} \\
= & \{ \quad b \subseteq J \text{ and } b' \subseteq J,\ b,\ b' \text{ and } J \text{ are coreflexive} \quad \} \\
& \Delta R \circ b \circ b' \circ \Delta R^{\cup} \\
= & \{ \quad \text{assumption: } b \neq b'\text{, (127)} \quad \} \\
& \perp\!\!\perp \quad .
\end{array}$$

That is,

$$(191) \quad \langle \forall b,b' : b \subseteq J \wedge b \neq b' : (\Delta R \circ b \circ (\Delta R){\succ}) \circ (\Delta R \circ b' \circ (\Delta R){\succ})^{\cup} = \perp\!\!\perp \rangle \quad .$$

The calculation for the right domains is similar. We have:

$$
\begin{array}{ll}
 & (\Delta R \circ b \circ (\Delta R){\succ})^{\cup} \circ (\Delta R \circ b' \circ (\Delta R){\succ}) \\
= & \{ \quad \text{converse} \quad \} \\
 & (\Delta R){\succ} \circ b \circ \Delta R^{\cup} \circ \Delta R \circ b' \circ (\Delta R){\succ} \\
= & \{ \quad \text{theorem 49} \quad \} \\
 & (\Delta R){\succ} \circ b \circ (\Delta R){\succ} \circ b' \circ (\Delta R){\succ} \\
= & \{ \quad b \circ (\Delta R){\succ} \circ b' = b \circ b' \ \text{(see last calculation)} \quad \} \\
 & (\Delta R){\succ} \circ b \circ b' \circ (\Delta R){\succ} \\
= & \{ \quad \text{assumption: } b \neq b',\ (127) \quad \} \\
 & \perp\!\!\perp \quad .
\end{array}
$$

That is, applying lemma 161,

$$(192) \quad \langle \forall b,b' : b \subseteq J \wedge b \neq b' : (\Delta R \circ b \circ (\Delta R){\succ})^{\cup} \circ (\Delta R \circ b' \circ (\Delta R){\succ}) = \perp\!\!\perp \rangle \quad .$$

The combination of (189), (191) and (192) together with lemma 162 establishes that the elements of $\mathcal{D}$ are completely disjoint.
□

It is now easy to see that $\mathcal{D}$ is a definiens of the injective polar covering of R defined in theorem 166:

Theorem 193 Suppose R is a relation such that $(\Delta R){>} = R{>}$. Suppose also that J is a coreflexive index of $R{\succ}$. Then the indexed bag $\mathcal{R}$ of rectangles defined by

$$\mathcal{R} \ = \ \langle b : b \subseteq J : R \circ b \circ R \backslash R \rangle$$

is a non-redundant, injective, polar covering of R. (In particular, $\mathcal{R}$ is an indexed set.) A definiens of the covering is the indexed set $\mathcal{D}$ defined by

$$\mathcal{D} \ = \ \langle b : b \subseteq J : \Delta R \circ b \circ \Delta R \backslash \Delta R \rangle \quad .$$

Moreover, by theorem 188, $\mathcal{D}$ is a covering of ΔR.

Proof Theorem 166 shows that $\mathcal{R}$ is an injective, polar covering of R. It remains to show that it is non-redundant as witnessed by the function $\mathcal{D}$.

We must first prove that, for all points b such that $b \subseteq J$, $\mathcal{D}.b \subseteq \mathcal{R}.b$. To this end, we use (190) as definition of $\mathcal{D}$. Assume b is a point such that $b \subseteq J$. Then

$$
\begin{array}{ll}
 & \mathcal{D}.b \subseteq \mathcal{R}.b \\
= & \{ \quad (190) \text{ and definition of } \mathcal{R} \quad \} \\
 & \Delta R \circ b \circ (\Delta R)\succ \ \subseteq R \circ b \circ R\backslash R \\
\Leftarrow & \{ \quad \Delta R \subseteq R \text{, monotonicity} \quad \} \\
 & (\Delta R)\succ \subseteq R\backslash R \\
= & \{ \quad \text{factors} \quad \} \\
 & R \circ (\Delta R)\succ \ \subseteq R \\
= & \{ \quad \text{assumption: } (\Delta R)> = R> \text{; so, by lemma 180, } (\Delta R)\succ = R\succ \quad \} \\
 & R \circ R\succ \ \subseteq R \\
= & \{ \quad \text{per domains} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

That the elements of $\mathcal{D}$ form a completely disjoint set of rectangles was shown in theorem 188. It remains to show that $\mathcal{D}$ “defines” $\mathcal{R}$. We have, for all points b and b' such that $b \subseteq J$ and $b' \subseteq J$,

$$
\begin{array}{ll}
 & \mathcal{R}.b = \mathcal{R}.b' \\
= & \{ \quad \mathcal{R} \text{ is injective (theorem 166)} \quad \} \\
 & b = b' \\
= & \{ \quad \mathcal{D} \text{ is injective (theorem 188)} \quad \} \\
 & \mathcal{D}.b = \mathcal{D}.b' \quad .
\end{array}
$$

□

Example 194

Fig. 7 pictures a small example of the theorems in this section. Fig. 7(a) depicts a (core) relation R of type $\{\alpha,\beta,\gamma\}\sim\{A,B\}$; other parts of the figure depict the result of applying different functions to the relation R. (Heterogeneous relations are depicted as bipartite graphs whereas homogeneous relations are depicted as directed graphs.) Specifically, these are as follows.

$$
\begin{array}{llll}
(a) & R \quad , & (b) & \Delta R \quad , \\
(c) & R\backslash R \quad , & (d) & R/R \quad , \\
(e) & R \circ A \circ R\backslash R \quad , & (f) & R \circ B \circ R\backslash R \quad ,
\end{array}
$$

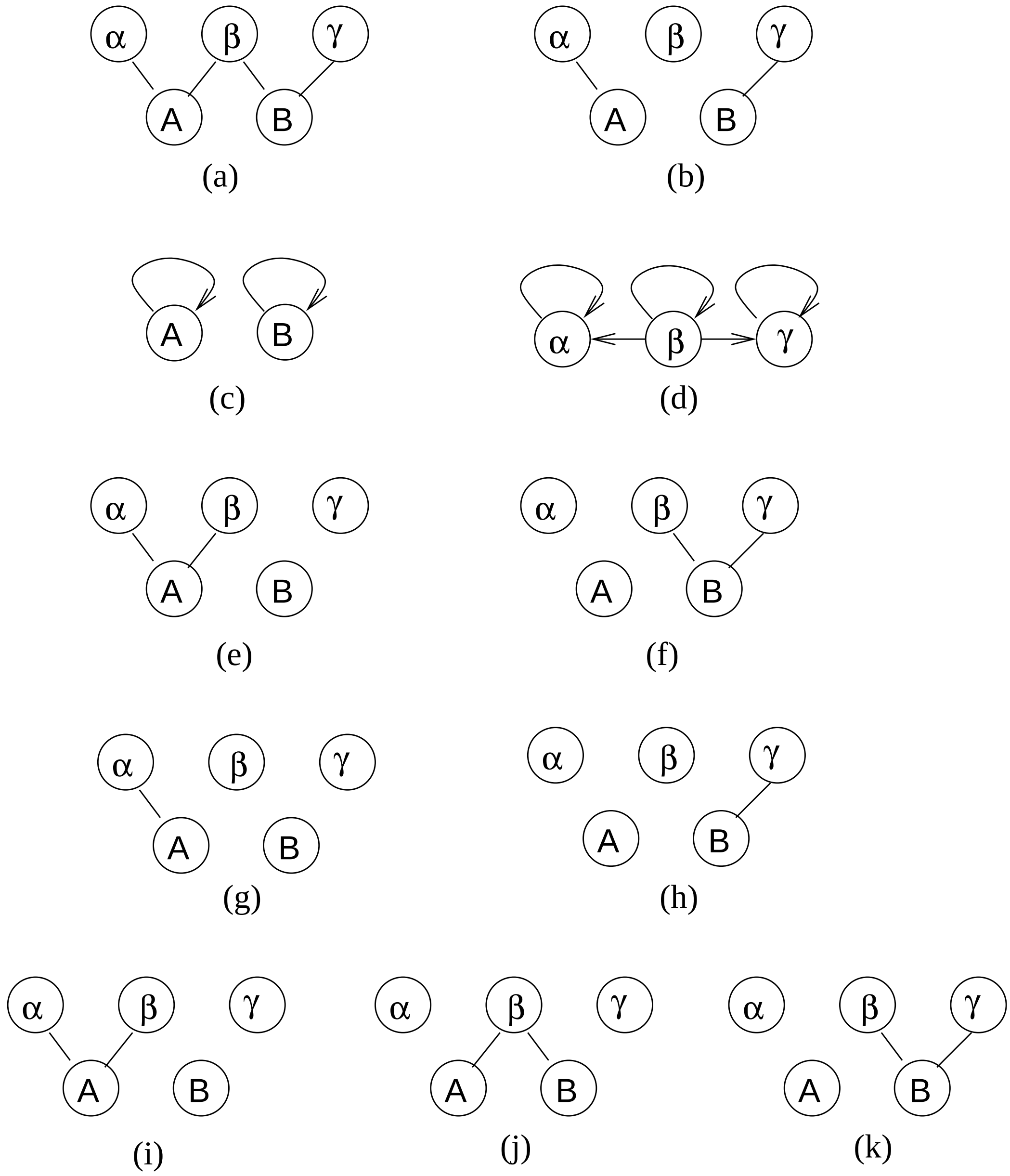


Figure 7: A Small Example

$$(g)\ \Delta R \circ A \circ R_{\succ}\quad ,\qquad (h)\ \Delta R \circ B \circ R_{\succ}\quad ,$$

$$(i)\ R/R \circ \alpha \circ R\ \ ,\ \ (j)\ R/R \circ \beta \circ R\ \ ,\ \ (k)\ R/R \circ \gamma \circ R\ \ .$$

We have chosen to depict the relation as a graph (rather than a boolean matrix) because —for very small examples such as this— it is much easier for a human being to perform the necessary calculations by manipulating the graphs. For example, computing the composition of two relations is executed by chasing edges.

The example has been chosen deliberately to illustrate a number of aspects simultaneously. Note particularly that, for the relation depicted, $(\Delta R)_{>} = R_{>}$ but $(\Delta R)_{<} \neq R_{<}$. This means that theorem 193 is applicable but its dual is not.

Considering the application of theorem 166, note that the combination of figs. 7(e) and 7(f) covers the relation R; also the relation depicted by 7(g) uniquely identifies the rectangle $R \circ A \circ R\backslash R$ shown in fig. 7(e) whilst 7(h) uniquely identifies the rectangle $R \circ A \circ R\backslash R$ shown in fig. 7(f). In contrast, figs. 7(i), (j) and (k) depict the relations $R/R \circ \alpha \circ R$, $R/R \circ \beta \circ R$ and $R/R \circ \gamma \circ R$ but none of these is identified by any subrectangle: the rectangles depicted by figs. 7(i) and (k) are disjoint but both have a non-empty intersection with the rectangle depicted by fig. 7(j).

□

Example 194 is an example of a relation R such that $(\Delta R)_{>} = R_{>}$ but $(\Delta R)_{<} \neq R_{<}$. It is thus the case that, for this example,

$$R \;=\; \langle \cup b : b \subseteq (\Delta R)_{>} : R \circ b \circ R\backslash R \rangle \quad .$$

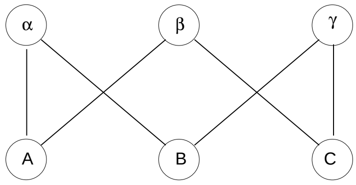
(Note the range restriction on the dummy b.) Curiously, in spite of the fact that $(\Delta R)_{<} \neq R_{<}$, it is also the case that

$$R \;=\; \langle \cup a : a \subseteq (\Delta R)_{<} : R/R \circ a \circ R \rangle \quad .$$

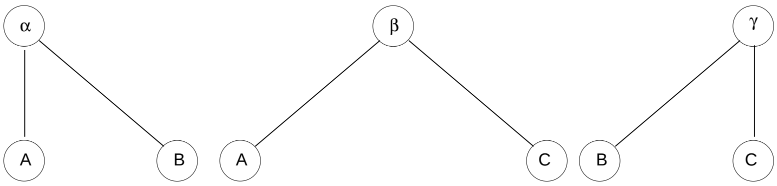
(Again, note the range restriction on the dummy a. To check the validity of the equation, it suffices to observe that the relation R is the union of the relations depicted by figs. 7(i) and (k).) This is also a non-redundant polar covering of R. One might thus conjecture that, in all cases, the diagonal ΔR is the key to finding a non-redundant polar covering of a given relation R. However, this is not always the case, as evidenced by the following example.

Example 195

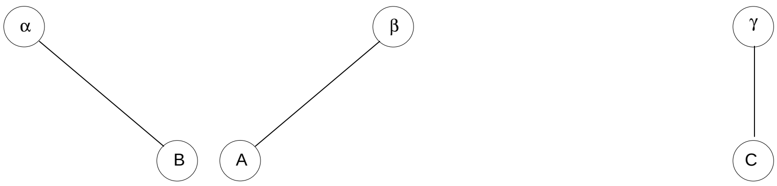
The top diagram of fig. 8 pictures a relation R of type $\{A,B,C\}\sim\{\alpha,\beta,\gamma\}$ such that ΔR is the empty relation. The example is a simplification of the example on p.161 of [KGJ00].



(a) Relation



(b) Non−redundant covering



(c) A Definiens

Figure 8: Empty Diagonal and Non-Redundant Covering

The three components of the polar covering predicted by theorem 166 are depicted in the second row. (The index set of the covering is $\{\alpha,\beta,\gamma\}$.) Note that the covering is non-redundant: the third row pictures a function that satisfies the definition of a definiens of the covering. This contradicts [KGJ00, theorem 1,p.159]: each of the edges in this third row is what [KGJ00] calls an “isolated point” in a “maximal rectangle” but none is a “point” in the diagonal.
□

13 Block-Ordered Relations

In general, dividing a subset of a set A into blocks is formulated by specifying a functional relation f, say, with source[5] the set A; elements $a0$ and $a1$ are in the same block equivales $f.a0$ and $f.a1$ are both defined and $f.a0 = f.a1$. In mathematical terminology, a functional relation f defines the *partial equivalence relation* $f^{\cup} \circ f$ and the “blocks” are the equivalence classes of $f^{\cup} \circ f$. (Partiality means that some elements may not be in an equivalence class.)

[5] In the terminology we use, a relation of type $A{\sim}B$ has *target* A and *source* B.

Given functional relations f and g with sources A and B, respectively, and equal left domains, relation R of type $A{\sim}B$ is said to be *block-structured* by f and g if there is a relation S such that $R = f^{\cup} \circ S \circ g$. Informally, whether or not a and b are related by R depends entirely on the "block" $(f.a\,,\,g.b)$ to which they belong. Note that it is not required that f and g be total functions: it suffices that $f{>} = R{<}$ and $g{>} = R{>}$. The type of S is $C{\sim}C$ where C includes $\{a\colon a \circ f{>} = a\colon f.a\}$ (equally $\{b\colon b \circ f{>} = b\colon g.b\}$).

Definition 196 (Block-Ordered Relation) Suppose T is a relation of type $C{\sim}C$, f is a relation of type $C{\sim}A$ and g is a relation of type $C{\sim}B$. Suppose further that T is a provisional ordering, i.e. that

(197) $\quad T \cap T^{\cup} \subseteq I \quad \wedge \quad T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \wedge \quad T \circ T \subseteq T \ .$

Suppose also that f and g are functional and onto the domain of T. That is, suppose

(198) $\quad f \circ f^{\cup} \ = \ f{<} \ = \ T \cap T^{\cup} \ = \ g{<} \ = \ g \circ g^{\cup} \ .$

Then we say that the relation $f^{\cup} \circ T \circ g$ is a *block-ordered relation*. A relation R of type $A{\sim}B$ is said to be *block-ordered* by f, g and T if $R = f^{\cup} \circ T \circ g$ and $f^{\cup} \circ T \circ g$ is a block-ordered relation.
□

The archetypical example of a block-ordered relation is a preorder. Informally, if R is a preorder, its symmetric closure $R \cap R^{\cup}$ is an equivalence relation, and the relation R defines a partial ordering on the equivalence classes. Equivalently, if a representative element is chosen for each equivalence class, the relation R is a partial ordering on the representatives. Theorem 201 makes this precise.

Assume that T is a provisional preorder. That is, by definition 53 and lemma 57,

(199) $\quad T{<} = T{>} \quad \wedge \quad T{<} \subseteq T \quad \wedge \quad T{>} \subseteq T \quad \wedge \quad T \circ T \subseteq T \ .$

Also, by lemma 59,

(200) $\quad T \cap T^{\cup} \ = \ T{\prec} \ = \ T{\succ} \ .$

Theorem 201 Suppose T is a provisional preorder and suppose J is a (coreflexive) index of $T{\prec}$. Then $J \circ T \circ J$ is an index of T and is a provisional ordering. Hence, T is a block-ordered relation.

Proof That $J \circ T \circ J$ is an index of T is the combination of (200) and theorem 107. So, it remains to show that $J \circ T \circ J$ is a provisional ordering. That is, we must show that $J \circ T \circ J \cap (J \circ T \circ J)^{\cup} \subseteq I$.

$$J\circ T\circ J \cap (J\circ T\circ J)^{\cup}$$

$=$ { J is coreflexive, distributivity }

$$J\circ(T\cap T^{\cup})\circ J$$

$\subseteq$ { (200) }

$$J\circ T_{\prec}\circ J$$

$=$ { J is an index of $T_{\prec}$, definition 105(b) with $P := T_{\prec}$ }

$$J$$

$\subseteq$ { J is coreflexive }

$$I \ .$$

□

Identifying a block-ordering of a relation —if it exists— is important for efficiency. Although a relation is defined to be a set of pairs, relations —even relations on finite sets— are rarely stored as such; instead some base set of pairs is stored and an algorithm used to generate, on demand, additional information about the relation. This is particularly so of ordering relations. For example, a test $m<n$ on integers m and n in a computer program is never implemented as a table lookup; instead an algorithm is used to infer from the basic relations $0<1$ together with the internal representation of m and n what the value of the test is. In the case of block-structured relations, functional relations f and g define partial equivalence relations $f^{\cup}\circ f$ and $g^{\cup}\circ g$ on their respective sources. (The relations $f^{\cup}\circ f$ and $g^{\cup}\circ g$ are partial because f and g are not required to be total.) Combining the functional relations with an ordering relation on their (common) target is an effective way of implementing a relation (assuming the ordering relation is also implemented effectively).

Example 202 Suppose G is the edge relation of a finite graph. The relation G^* is, of course, a preorder and so is block-ordered. The block-ordering of G^* given by theorem 201 is, however, not very useful. For practical purposes a block-ordering constructed from G (rather than G^*) is preferable. Here we outline how this is done.

Recall from example 175, that the diagonal $\Delta(G^*)$ is the relation $G^*\cap(G^{\cup})^*$ and that this is an equivalence relation on the nodes of G, whereby the equivalence classes are the *strongly connected components* of G. Let N denote the nodes of G and C denote the set of strongly connected components of G. By theorem 109, there is a function sc of type $C\leftarrow N$ such that

$$(203) \quad G^*\cap(G^{\cup})^* \ = \ sc^{\cup}\circ sc \ .$$

The relation $\mathcal{A}$ defined by

$$sc\circ G\circ sc^{\cup} \ \cap \ \neg I_C$$

is a homogeneous relation on the strongly connected components of G, i.e. a relation of type $C{\sim}C$. Informally, it is a graph obtained from the graph G by coalescing the nodes in a strongly connected component of G into a single node whilst retaining the edges of G that connect nodes in distinct strongly connected components[6]. A fundamental theorem is that

$$(204)\quad G^* \;=\; \mathsf{sc}^{\cup}\circ\mathcal{A}^*\circ\mathsf{sc} \;\;.$$

Moreover, $\mathcal{A}$ is acyclic. That is,

$$(205)\quad I_C\cap\mathcal{A}^+ \;=\; \perp\!\!\perp \;\;.$$

(See [BDGv22, Bac22] for the details of the proof of (204) and (205). In fact the theorem is valid for all relations G; finiteness is not required.)

The relation $\mathcal{A}^*$ is, of course, transitive. It is also reflexive; combined with its acyclicity, it follows that

$$(206)\quad \mathcal{A}^*\cap(\mathcal{A}^*)^{\cup} \;=\; I_C \;\;.$$

That is, $\mathcal{A}^*$ is a (total) provisional ordering on C. The conclusion is that G^* is block-ordered by sc, sc and $\mathcal{A}^*$.

Informally, a finite graph can always be decomposed into its strongly connected components together with an acyclic graph connecting the components.

Although the informal interpretation of this theorem is well-known, the formal proof is non-trivial. Although not formulated in the same way, it is essentially the "transitive reduction" of an arbitrary (not necessarily acyclic) graph formulated by Aho, Garey and Ullman [AGU72, Theorem 2].

The decomposition (204) is (implicitly) exploited when computing the inverse $\mathbf{A}^{-1}$ of a real matrix $\mathbf{A}$ in order to minimise storage requirements: using an elimination technique, a so-called "product form" is computed for each strongly connected component, whilst the process of "forward substitution" is applied to the acyclic-graph structure.
□

It is important to note the very strict requirement (198) on the functionals f and g. Were this requirement to be omitted (retaining only that f and g are functional relations *into* —not onto— the domain of T), there would be no guarantee of non-redundancy. As we shall see, our definition of block-ordering does guarantee the existence of a non-redundant polar covering (theorem 228) but not vice-versa (corollary 231). This suggests that the requirement may be too strong.

Theorem 207 makes precise the statement that block orderings —where they exist— are unique "up to isomorphism".

[6] Although we don't go into details, for any function f of appropriate type, the graph $f\circ G\circ f^{\cup}$ is "pathwise homomorphic" [McN67] to G.

Theorem 207 Suppose T is a provisional ordering. That is, suppose

$$T \cap T^{\cup} \subseteq I \quad \wedge \quad T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \wedge \quad T \circ T \subseteq T \ .$$

Suppose also that f and g are functional and onto the domain of T. That is, suppose

$$f \circ f^{\cup} \ = \ f< \ = \ T \cap T^{\cup} \ = \ g< \ = \ g \circ g^{\cup} \ .$$

Suppose further[7] that S, h and k satisfy the same properties as T, f and g (respectively) and that

(208) $\quad f^{\cup} \circ T \circ g \ = \ h^{\cup} \circ S \circ k \ .$

Then

(209) $\quad f> = h> \ \wedge \ g> = k> \ ,$

(210) $\quad f^{\cup} \circ g = h^{\cup} \circ k \ ,$

(211) $\quad f^{\cup} \circ T^{\cup} \circ g \ = \ h^{\cup} \circ S^{\cup} \circ k \ $, and

(212) $\quad f \circ h^{\cup} \ = \ g \circ k^{\cup} \ .$

Also, letting ϕ denote $f \circ h^{\cup}$ (equally, by (212), $g \circ k^{\cup}$),

(213) $\quad \phi \circ \phi^{\cup} = T \cap T^{\cup} \ \wedge \ \phi^{\cup} \circ \phi = S \cap S^{\cup} \ \wedge \ \phi \circ T = S \circ \phi \ .$

In words, ϕ is an order isomorphism of the domains of T and S.

Proof In combination with the assumption (208), properties (209), (211) and (210) are immediate from (222), (223) and (224), respectively.

Proof of (212) is a step on the way to proving (213). From symmetry considerations, it is an obvious first step.

$$\begin{array}{ll}
 & f \circ h^{\cup} \\
= & \{ \quad \text{assumption:} \quad k \circ k^{\cup} = h< \quad \} \\
 & f \circ h^{\cup} \circ k \circ k^{\cup} \\
= & \{ \quad (210) \quad \} \\
 & f \circ f^{\cup} \circ g \circ k^{\cup} \\
= & \{ \quad \text{assumption:} \quad f \circ f^{\cup} = g< \quad \} \\
 & g \circ k^{\cup} \ .
\end{array}$$

[7] The types of T and S may be different. The types of f and h, and of g and k will then also be different. As in lemma 221, the requirement is that the types are compatible with the type restrictions on the operators in all assumed properties. The symbol "I" in (213) is overloaded: if the type of T is $A{\sim}A$ and the type of S is $B{\sim}B$, $\phi \circ \phi^{\cup}$ has type $A{\sim}A$ and $\phi^{\cup} \circ \phi$ has type $B{\sim}B$.

Now,

$$\phi \circ \phi^{\cup}$$
$$= \quad \{ \quad \text{definition of } \phi \text{, converse} \quad \}$$
$$f \circ h^{\cup} \circ h \circ f^{\cup}$$
$$= \quad \{ \quad (212) \quad \}$$
$$g \circ k^{\cup} \circ h \circ f^{\cup}$$
$$= \quad \{ \quad (210) \text{ and converse} \quad \}$$
$$g \circ g^{\cup} \circ f \circ f^{\cup}$$
$$= \quad \{ \quad \text{assumption:} \quad f \circ f^{\cup} = T \cap T^{\cup} = g \circ g^{\cup} \quad \}$$
$$T \cap T^{\cup} \quad .$$

Symmetrically, $\phi^{\cup} \circ \phi = T \cap T^{\cup}$. Finally,

$$T \circ \phi$$
$$= \quad \{ \quad \text{definition of } \phi \quad \}$$
$$T \circ f \circ h^{\cup}$$
$$= \quad \{ \quad \text{assumptions:} \quad f \circ f^{\cup} = T \cap T^{\cup} = g \circ g^{\cup}$$
$$T \;=\; (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \}$$
$$f \circ f^{\cup} \circ T \circ g \circ g^{\cup} \circ f \circ h^{\cup}$$
$$= \quad \{ \quad \text{assumption:} \quad f^{\cup} \circ T \circ g \;=\; h^{\cup} \circ S \circ k \quad \text{, (210) and converse} \quad \}$$
$$f \circ h^{\cup} \circ S \circ k \circ k^{\cup} \circ h \circ h^{\cup}$$
$$= \quad \{ \quad \text{assumption:} \quad h \circ h^{\cup} = S \cap S^{\cup} = k \circ k^{\cup} \quad \}$$
$$f \circ h^{\cup} \circ S$$
$$= \quad \{ \quad \text{definition of } \phi \quad \}$$
$$\phi \circ S \quad .$$

□

13.1 Pair Algebras and Galois Connections

In order to find lots of examples of block-ordered relations one need look no further than the theory of Galois connections (which are, of course, ubiquitous). In this section, we briefly review the notion of a "pair algebra" —due to Hartmanis and Stearns [HS64, HS66]— and its relation to Galois connections.

Hartmanis and Stearns studied a particular practical problem: the so-called “state assignment problem”. This is the problem of how to encode the states and inputs of a sequential machine in such a way that state transitions can be implemented economically using logic circuits. However, as they made clear in the preface of their book [HS66], their contribution was to “information science” in general:

> It should be stressed, however, that although many structure theory results describe possible physical realizations of machines, the theory itself is independent of the particular physical components of technology used in the realization.
>
> ...
>
> The mathematical foundations of this structure theory rest on an algebraization of the concept of “information” in a machine and supply the algebraic formalism necessary to study problems about the flow of this information.

Hartmanis and Stearns limited their analysis to finite, complete posets, and their analysis was less general than is possible. This work was extended in [Bac98] to non-finite posets and the current section is a short extract.

A Galois connection involves two posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$ and two functions, $F\in\mathcal{A}\leftarrow\mathcal{B}$ and $G\in\mathcal{B}\leftarrow\mathcal{A}$. These four components together form a *Galois connection* iff for all $b\in\mathcal{B}$ and $a\in\mathcal{A}$

(214) $F.b\sqsubseteq a \equiv b\preceq G.a$.

We refer to F as the *lower adjoint* and to G as the *upper adjoint*.

A Galois connection is thus a connection between two functions between posets. Typical accounts of the properties of Galois connections (for e.g. [GHK$^+$80]) focus on the properties of these *functions*. For example, given a function F, one may ask whether F is a lower adjoint in a Galois connection. The question posed by Hartmanis and Stearns was, however, rather different.

To motivate their question, note that the statement $F.b\sqsubseteq a$ defines a *relation* between $\mathcal{B}$ and $\mathcal{A}$. So too does $b\preceq G.a$. The existence of a Galois connection states that these two relations are equal. A natural question is therefore: under which conditions does an arbitrary (binary) relation between two posets define a Galois connection between the sets?

Exploring the question in more detail leads to two separate questions. The first is: suppose R is a relation between posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$. What is a necessary and sufficient condition that there exist a function F such that

$$(a,b)\in R \;\equiv\; F.b\sqsubseteq a \quad ?$$

The second is the dual of the first: given relation R, what is a necessary and sufficient condition that there exist a function G such that

$$(a,b)\in R \;\equiv\; b \preceq G.a \quad ?$$

The conjunction of these two conditions is a necessary and sufficient condition for a relation R to define a Galois connection. Such a relation is called a *pair algebra*.

Example 215 It is easy to demonstrate that the two questions are separate. To this end, fig. 9 depicts two posets and a relation between them. The posets are $\{\alpha,\beta\}$ and $\{A,B\}$; both are ordered lexicographically: the reflexive-transitive reduction of the lexicographic ordering is depicted by the directed edges. The relation of type $\{\alpha,\beta\}\sim\{A,B\}$ is depicted by the undirected edges.

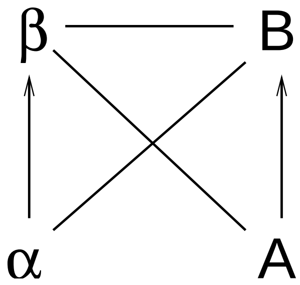


Figure 9: A Relation on Two Posets

Let the relation be denoted by R. Define the function F of type $\{\alpha,\beta\}\leftarrow\{A,B\}$ by $F.B=\alpha$ and $F.A=\beta$. Then it is easy to check that. for $a\in\{\alpha,\beta\}$ and $b\in\{A,B\}$,

$$(a,b)\in R \;\equiv\; F.b \sqsubseteq a \quad .$$

(There are just four cases to be considered.) On the other hand, there is no function G of type $\{A,B\}\leftarrow\{\alpha,\beta\}$ such that

$$(a,b)\in R \;\equiv\; b \preceq G.a \quad .$$

To check that this is indeed the case, it suffices to check that the assignment $G.A=\alpha$ is invalid (because $\alpha\sqsubseteq\alpha$ but $(\alpha,A)\notin R$) and the assignment $G.A=\beta$ is also invalid (because $\alpha\sqsubseteq\beta$ but $(\alpha,A)\notin R$).
□

Example 216 A less artificial, general way to demonstrate that the two questions are separate is to consider the membership relation. Specifically, suppose $\mathcal{S}$ is a set. Then the membership relation, denoted as usual by the —overloaded— symbol "$\in$", is a heterogeneous relation of type $\mathcal{S}\sim 2^{\mathcal{S}}$ (where $2^{\mathcal{S}}$ denotes the type of subsets of $\mathcal{S}$). Now, for all x of type $\mathcal{S}$ and X of type $2^{\mathcal{S}}$,

$$x\in X \equiv \{x\}\subseteq X \quad .$$

The right side of this equation has the form $F.b\sqsubseteq a$ where F is the function that maps an element into a singleton set and the ordering is the subset ordering. Also, its left side has the form $(a,b)\in R$, where the relation R is the membership relation and a and b are x and X, respectively. (This is where the overloading of notation can become confusing, for which our apologies!) It is, however, not possible to express $x\in X$ in the form $x\preceq G.X$ (except in the trivial cases where $\mathcal{S}$ has cardinality at most one). We leave the proof to the reader.
□

Example 217 An example of a Galois connection is the definition of the ceiling function on real numbers: for all real numbers x, $\lceil x\rceil$ is an integer such that, for all integers m,

$$x\leq m \equiv \lceil x\rceil\leq m \quad .$$

To properly fit the definition of a Galois connection, it is necessary to make explicit the implicit coercion from integers to real numbers in the left side of this equation. Specifically, we have, for all real numbers x and integers m,

$$x \leq_{\mathbb{R}} \mathsf{real}.m \quad\equiv\quad \lceil x\rceil \leq_{\mathbb{Z}} m$$

where real denotes the function that "coerces" an integer to a real, and $\leq_{\mathbb{R}}$ and $\leq_{\mathbb{Z}}$ denote the (homogeneous) at-most relations on, respectively, real numbers and integers. If, however, we consider the symbol "$\leq$" on the left side of the equation to denote the heterogeneous at-most relation of type $\mathbb{R}\sim\mathbb{Z}$, the fact that

$$x\leq m \quad\equiv\quad \lceil x\rceil \leq_{\mathbb{Z}} m$$

gives a representation of the (heterogeneous) "$\leq$" relation of type $\mathbb{R}\sim\mathbb{Z}$ as a block-ordered relation: referring to definition 196, the provisional ordering is $\leq_{\mathbb{Z}}$, f is the ceiling function and g is the identity function.

More interesting is if we take the contrapositive. We have, for all real numbers x and integers m,

$$m<x \equiv m\leq\lceil x\rceil-1 \quad .$$

On the right of this equation is the (homogeneous) at-most relation on integers. On the left is the (heterogeneous) less-than relation of type $\mathbb{Z}\sim\mathbb{R}$. The equation demonstrates

that this relation is block-ordered; the "blocks" of real numbers being all the numbers that have the same ceiling. (The functional f is the identity function, the functional g maps real number x to $\lceil x\rceil-1$ and the provisional ordering is the ordering $\leq_{\mathbb{Z}}$.) The example is interesting because the (homogeneous) less-than relation on real numbers is *not* block-ordered. This is because its diagonal is empty. See [Bac21] for details.
□

Returning to the discussion immediately preceding example 215, the two separate questions are each of interest in their own right: a positive answer to either question may predict that a given relation has a block-ordering of a specific form: in the case of the first question, where the functional g in definition 196 is the identity function, and, in the case of the second question, where the functional f in definition 196 is the identity function. In both cases, a further step is to check the requirement on f and g: in the first case, one has to check that the function F is surjective and in the second case that the function G is surjective. (A Galois connection is said to be "perfect" if both F and G are surjective.) For example, the fact that

$$x \leq m \quad \equiv \quad x \leq_{\mathbb{R}} \mathsf{real}.m$$

does *not* define a block-ordering because the function real is not surjective.

The relevant theory predicting exactly when the first of the two questions has a positive answer is as follows. Suppose $(\mathcal{B},\sqsubseteq)$ is a complete poset. Let $\sqcap$ denote the infimum operator for $\mathcal{B}$ and suppose p is a predicate on $\mathcal{B}$. Then we define *inf-preserving* by

(218) $\quad p$ is inf-preserving $\quad\equiv\quad \langle\forall g :: p.(\sqcap g) \equiv \langle\forall x :: p.(g.x)\rangle\rangle$.

So, for a given a, the predicate $\langle b:: (a,b)\in R\rangle$ is inf-preserving equivales

$$\langle\forall g \;::\; (a\,,\sqcap g)\in R \;\equiv\; \langle\forall x :: (a\,,\,g.x)\in R\rangle\rangle \;\;.$$

Then we have:

Theorem 219 Suppose $\mathcal{A}$ is a set and $(\mathcal{B},\sqsubseteq)$ is a complete poset. Suppose $R\subseteq\mathcal{A}\times\mathcal{B}$ is a relation between the two sets. Define F by

(220) $\quad F.a \;=\; \langle\sqcap b \,:\, (a,b)\in R \,:\, b\rangle$.

Then the following two statements are equivalent.

- $\langle\forall\, a,b \;:\; a\in\mathcal{A}\wedge b\in\mathcal{B} \;:\; (a,b)\in R \;\equiv\; F.a\sqsubseteq b\rangle$.
- For all a, the predicate $\langle b:: (a,b)\in R\rangle$ is inf-preserving.

□

The answer to the second question is, of course, obtained by formulating the dual of theorem 219.

In general, for most relations occurring in practical information systems the answer to the pair-algebra questions will be negative: the required inf- and sup-preserving properties just do not hold. However, a common way to define a pair algebra is to extend a given relation to a relation between sets in such a way that the infimum and supremum preserving properties are automatically satisfied. Hartmanis and Stearns' [HS64, HS66] solution to the state assignment problem was to consider the lattice of partitions of a given set; in so-called "concept analysis", the technique is to extend a given relation to a relation between rectangles.

An important property of Galois connections is the (well-known) theorem we call the "unity of opposites": if F and G are the adjoint functions in a Galois connection of the posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$, then there is an isomorphism between the posets $(F.\mathcal{B}, \sqsubseteq)$ and $(G.\mathcal{A}, \preceq)$. ($F.\mathcal{B}$ denotes the "image" of the function F, and similarly for $G.\mathcal{A}$.) Knowledge of the unity-of-opposites theorem suggests theorem 207, which expresses an isomorphism between different representations of block-ordered relations.

13.2 Analogie Frappante

In this section, we relate block-orderings to diagonals. The main results are theorems 228 and 235. We have named theorem 235 the "analogie frappante" because it generalises Riguet's "analogie frappante" connecting "relation de Ferrers" to diagonals.

Lemma 221 Suppose T is a provisional ordering of type $C{\sim}C$. That is, suppose

$$T\cap T^{\cup} \subseteq I_C \quad\wedge\quad T = (T\cap T^{\cup})\circ T\circ(T\cap T^{\cup}) \quad\wedge\quad T\circ T\subseteq T \ .$$

Suppose also that f and g are functional and onto the domain of T. That is, suppose[8] that

$$f\circ f^{\cup} \;=\; f< \;=\; T\cap T^{\cup} \;=\; g< \;=\; g\circ g^{\cup} \ .$$

Let R denote $f^{\cup}\circ T\circ g$. Then

(222) $R< \,=\, f> \;\wedge\; R> \,=\, g>$,

(223) $f^{\cup}\circ T^{\cup}\circ g \;=\; R<\circ(R\backslash R/R)^{\cup}\circ R>$, and

[8]The ordering T must be homogeneous but f and g may be heterogeneous and of different type, so long as both have target C.

(224) $\quad f^{\cup}\circ g \;=\; \Delta R \quad ,$

(225) $\quad R_{<} = (\Delta R)_{<} \;\;\wedge\;\; R_{>} = (\Delta R)_{>} \quad ,$

(226) $\quad R_{\prec} \;=\; \Delta R\circ\Delta R^{\cup} \;=\; f^{\cup}\circ f \;\;\wedge\;\; R_{\succ} \;=\; \Delta R^{\cup}\circ\Delta R \;=\; g^{\cup}\circ g \quad .$

Proof Property (222) is a straightforward application of domain calculus:

$$
\begin{array}{ll}
 & R_{>} \\
= & \{ \quad \text{definition: } R = f^{\cup}\circ T\circ g \quad \} \\
 & (f^{\cup}\circ T\circ g)_{>} \\
= & \{ \quad \text{domains (specifically, } [\,(U\circ V)_{>} = (U_{>}\circ V)_{>}\,] \text{ and } [\,(U^{\cup})_{>} = U_{<}\,]\text{)} \quad \} \\
 & (f_{<}\circ T\circ g)_{>} \\
= & \{ \quad \text{assumption: } T = f_{<}\circ T\circ g_{<} \text{ (so } T = f_{<}\circ T\text{)} \quad \} \\
 & (T\circ g)_{>} \\
= & \{ \quad \text{domains (specifically, } [\,(U\circ V)_{>} = (U_{>}\circ V)_{>}\,]\text{)} \quad \} \\
 & (T_{>}\circ g)_{>} \\
= & \{ \quad \text{lemma 61 and assumption: } T\cap T^{\cup} = g_{<} \quad \} \\
 & g_{>} \quad .
\end{array}
$$

By a symmetric argument, $(f^{\cup}\circ T\circ g)_{<} = f_{>}$.

Now we consider (223). The raison d'être of (223) is that it expresses the left side as a function of $f^{\cup}\circ T\circ g$. In a pointwise calculation a natural step is to use indirect ordering. In a point-free calculation, this corresponds to using factors. That is, we exploit lemma 58:

$$
\begin{array}{ll}
 & f^{\cup}\circ T^{\cup}\circ g \\
= & \{ \quad \text{assumption: } T \text{ is a provisional ordering} \\
 & \quad\;\; \text{lemmas 55, 59 and 58} \quad \} \\
 & f^{\cup}\circ(T\cap T^{\cup})\circ T^{\cup}\backslash T^{\cup}/T^{\cup}\circ(T\cap T^{\cup})\circ g \\
= & \{ \quad \text{assumption: } f_{<} = T\cap T^{\cup} = g_{<} \quad \} \\
 & f^{\cup}\circ T^{\cup}\backslash T^{\cup}/T^{\cup}\circ g \\
= & \{ \quad \text{lemma 44 and assumption: } T = f_{<}\circ T\circ g_{<} \quad \} \\
 & f_{>}\circ(g^{\cup}\circ T^{\cup}\circ f)\backslash(g^{\cup}\circ T^{\cup}\circ f)/(g^{\cup}\circ T^{\cup}\circ f)\circ g_{>} \\
= & \{ \quad \text{(222) and definition of } R \quad \}
\end{array}
$$

$$R{<} \circ R^{\cup} \setminus R^{\cup} / R^{\cup} \circ R{>}$$

$$= \quad \{ \quad \text{factors} \quad \}$$

$$R{<} \circ (R\backslash R/R)^{\cup} \circ R{>} \quad .$$

Note the use of lemma 44. The discovery of this lemma is driven by the goal of the calculation.

The pointwise interpretation of $f^{\cup} \circ g$ is a relation expressing equality between values of f and g. This suggests that, in order to prove (224), we begin by exploiting the anti-symmetry of T:

$$f^{\cup} \circ g$$

$$= \quad \{ \quad f{<} \ = \ T \cap T^{\cup} \ = \ g{<} \ \text{and domains} \quad \}$$

$$f^{\cup} \circ (T \cap T^{\cup}) \circ g$$

$$= \quad \{ \quad \text{distributivity (valid because } f \text{ and } g \text{ are functional)} \quad \}$$

$$f^{\cup} \circ T \circ g \ \ \cap \ \ f^{\cup} \circ T^{\cup} \circ g$$

$$= \quad \{ \quad \text{definition of } R \text{ and (223)} \quad \}$$

$$f^{\cup} \circ T \circ g \ \ \cap \ \ f{>} \circ ((f^{\cup} \circ T \circ g) \setminus (f^{\cup} \circ T \circ g) / (f^{\cup} \circ T \circ g))^{\cup} \circ g{>}$$

$$= \quad \{ \quad \text{(227) (see below)} \quad \}$$

$$f{>} \circ f^{\cup} \circ T \circ g \circ g{>} \ \ \cap \ \ ((f^{\cup} \circ T \circ g) \setminus (f^{\cup} \circ T \circ g) / (f^{\cup} \circ T \circ g))^{\cup}$$

$$= \quad \{ \quad \text{domains (specifically, } f{>} \circ f^{\cup} = f^{\cup} \text{ and } g \circ g{>} = g \text{)} \quad \}$$

$$f^{\cup} \circ T \circ g \ \ \cap \ \ ((f^{\cup} \circ T \circ g) \setminus (f^{\cup} \circ T \circ g) / (f^{\cup} \circ T \circ g))^{\cup}$$

$$= \quad \{ \quad \text{definition of } R \text{ and } \Delta R \quad \}$$

$$\Delta R \quad .$$

A crucial step in the above calculation is the use of the property

$$(227) \quad U \cap p{\circ}V{\circ}q \quad = \quad p{\circ}(U{\cap}V){\circ}q \quad = \quad p{\circ}U{\circ}q \cap V$$

for all relations U and V and coreflexive relations p and q. This is a frequently used property of domain restriction.

The remaining equations (225) and (226) are straightforward. First

$$(\Delta R){<}$$

$$= \quad \{ \quad (224) \quad \}$$

$$(f^{\cup} \circ g){<}$$

$$
\begin{array}{ll}
= & \{ \quad \text{domains and assumption: } f_< = g_< \quad \} \\
& f_> \\
= & \{ \quad \text{assumption: } f_< \ = \ T \cap T^\cup \quad \} \\
& ((T \cap T^\cup) \circ f)_> \\
= & \{ \quad \text{domains and converse} \quad \} \\
& (f^\cup \circ (T \cap T^\cup))_< \\
= & \{ \quad \text{lemma 61 and domains} \quad \} \\
& (f^\cup \circ T)_< \\
= & \{ \quad \text{domains and assumption: } g_< = T \cap T^\cup \\
& \quad\ \ \text{and lemma 61} \quad \} \\
& (f^\cup \circ T \circ g)_< \ .
\end{array}
$$

That is $(\Delta R)_< \ = \ R_<$. The dual equation $(\Delta R)_> \ = \ R_>$ is immediate from the fact that $(\Delta R)^\cup = \Delta(R^\cup)$ and properties of the domain operators. For the per domains, we have:

$$
\begin{array}{ll}
& R_\prec \\
= & \{ \quad R_< = (\Delta R)_< \text{ and } R_> = (\Delta R)_> \text{ (above); lemma 180} \quad \} \\
& (\Delta R)_\prec \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 49 (with } R := \Delta R \text{)} \quad \} \\
& \Delta R \circ \Delta R^\cup \\
= & \{ \quad \text{lemma 221 and definition of } \Delta R \quad \} \\
& f^\cup \circ g \circ (f^\cup \circ g)^\cup \\
= & \{ \quad \text{converse and } f_< \ = \ g_< \ = \ g \circ g^\cup \quad \} \\
& f^\cup \circ f \ .
\end{array}
$$

Again, the dual equation is immediate.

□

Theorem 228 Suppose $R = f^\cup \circ T \circ g$ where f, g and T have the properties stated in definition 196. Then the function $\mathcal{R}$ defined by

(229) $\mathcal{R} \ = \ \langle c \ : \ c \subseteq T \cap T^\cup \ : \ f^\cup \circ T \circ c \circ T \circ g \rangle$

is a non-redundant, injective, polar covering of R, and the function $\mathcal{D}$ defined by

(230) $\mathcal{D} \ = \ \langle c \ : \ c \subseteq T \cap T^\cup \ : \ f^\cup \circ c \circ g \rangle$

is a definiens of $\mathcal{R}$ such that $\cup\mathcal{D} = \Delta R$. That is, a block-ordered relation has a non-redundant, injective, polar covering such that the definiens of the covering is a covering of the diagonal of R.

Proof The theorem is a consequence of lemma 221, theorem 193 and theorem 188. Specifically, lemma 221 (in particular (226) and (225)) states that the conditions required to apply theorem 193 are met with ρ instantiated to g. Thus,

$$\mathcal{R} \;=\; \langle c : c \subseteq g_< : R \circ g^\cup \circ c \circ g \circ R \backslash R \rangle$$

is a non-redundant, injective polar covering of R. The definition of $\mathcal{R}$ is simplified as follows. First,

$$\begin{array}{ll}
 & g \circ R\backslash R \\
= & \{ \quad R = f^\cup \circ T \circ g \quad \} \\
 & g \circ (f^\cup \circ T \circ g)\backslash(f^\cup \circ T \circ g) \\
= & \{ \quad \text{lemma 45 with } R,S,f,g := T\,, T{\circ}g\,, f\,, g \quad \} \\
 & g \circ g^\cup \circ T\backslash(T{\circ}g) \\
= & \{ \quad g \circ g^\cup = g_< \quad \} \\
 & g_< \circ T\backslash(T{\circ}g) \quad .
\end{array}$$

So, for all c such that $c \subseteq g_<$,

$$\begin{array}{ll}
 & R \circ g^\cup \circ c \circ g \circ R\backslash R \\
= & \{ \quad \mathcal{R} \text{ covers } R\text{, so } (R \circ g^\cup \circ c \circ g \circ R\backslash R)_> \subseteq R_> \;;\; R_> = g_> \\
 & \quad\quad \text{(in preparation for lemma 43)} \quad \} \\
 & R \circ g^\cup \circ c \circ g \circ R\backslash R \circ g_> \\
= & \{ \quad R = f^\cup \circ T \circ g \text{ and } g \circ R\backslash R = g_< \circ T\backslash(T{\circ}g) \text{ (see above)} \quad \} \\
 & f^\cup \circ T \circ g \circ g^\cup \circ c \circ g_< \circ T\backslash(T{\circ}g) \circ g_> \\
= & \{ \quad g \circ g^\cup = g_<\text{, assumption: } c \subseteq g_<\text{, lemma 43 with } R,f := T,g \quad \} \\
 & f^\cup \circ T \circ c \circ T\backslash T \circ g \\
= & \{ \quad T \text{ is a provisional ordering, } T \cap T^\cup = g_<\text{,} \\
 & \quad\quad \text{lemma 57} \quad \} \\
 & f^\cup \circ T \circ c \circ T \circ g \quad .
\end{array}$$

Since $g< = T\cap T^{\cup}$ by assumption, we have established (229).

Theorem 193 defines the definiens of the covering as the indexed set $\mathcal{D}$ where

$$\mathcal{D} \;=\; \langle c : c\subseteq g< : \Delta R\circ g^{\cup}\circ c\circ g\circ R\succ\rangle \;\; .$$

But, for all c such that $c\subseteq g<$,

$$\begin{array}{ll} & \Delta R\circ g^{\cup}\circ c\circ g\circ R\succ \\ = & \{ \quad (226) \text{ and } (224) \quad \} \\ & f^{\cup}\circ g\circ g^{\cup}\circ c\circ g\circ g^{\cup}\circ g \\ = & \{ \quad g\circ g^{\cup} = g<, \text{ assumption: } c\subseteq g< \quad \} \\ & f^{\cup}\circ c\circ g \;\; . \end{array}$$

Using the assumption that $g< = T\cap T^{\cup}$ once again, we have established (230). That $\cup\mathcal{D} = f^{\cup}\circ g = \Delta R$ follows from $f^{\cup}\circ g = \Delta R$ and the saturation axiom.
□

Lemma 221 has as immediate corollary that the converse of theorem 228 is invalid.

Corollary 231 There are relations that have a non-redundant polar covering but are not block-ordered.

Proof Examples 194 and 195 are both examples of finite relations that have non-redundant polar coverings. Example 194 has the property that $(\Delta R)< \neq R<$; however, $(\Delta R)> = R>$. Example 195 has an empty diagonal; that is, $(\Delta R)< \neq R<$ (and $(\Delta R)> \neq R>$). So by (the converse of) lemma 221 (specifically, (225)), neither relation is block-ordered.
□

We now prove the converse of lemma 221.

Lemma 232 A relation R is block-ordered if $R< = (\Delta R)<$ and $R> = (\Delta R)>$.

Proof Suppose $R< = (\Delta R)<$ and $R> = (\Delta R)>$. Our task is to construct relations f, g and T such that

$$R = f^{\cup}\circ T\circ g \;\; ,$$

$$T\cap T^{\cup} \subseteq I \;\;\wedge\;\; T \;=\; (T\cap T^{\cup})\circ T\circ(T\cap T^{\cup}) \;\;\wedge\;\; T\circ T\subseteq T \quad \text{and}$$

$$f\circ f^{\cup} \;=\; f< \;=\; T\cap T^{\cup} \;=\; g< \;=\; g\circ g^{\cup} \;\; .$$

Since ΔR is difunctional, theorem 111 is the obvious place to start. Applying the theorem, we can construct f and g such that $\Delta R = f^{\cup}\circ g$ and

$$\Delta R = f^{\cup}\circ g \;\;\wedge\;\; f\circ f^{\cup} \;=\; f< \;=\; g\circ g^{\cup} \;=\; g< \;\; .$$

Using standard properties of the domain operators together with the assumption that $R_{<} = (\Delta R)_{<}$ and $R_{>} = (\Delta R)_{>}$, it follows that

$$R_{<} = f_{>} \;\wedge\; R_{>} = g_{>} \quad .$$

It remains to construct the provisional ordering T. The appropriate construction is suggested by lemma 221, in particular (223). Specifically, we define T by the equation

(233) $\quad T \;=\; g \circ R\backslash R/R \circ f^{\cup} \quad .$

The proof that $R = f^{\cup} \circ T \circ g$ is by mutual inclusion. First note that

(234) $\quad f^{\cup} \circ T \circ g \;=\; \Delta R \circ R\backslash R/R \circ \Delta R$

since

$$\begin{array}{ll}
 & f^{\cup} \circ T \circ g \\
= & \quad \{ \quad (233) \quad \} \\
 & f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \circ g \\
= & \quad \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & \Delta R \circ R\backslash R/R \circ \Delta R \quad .
\end{array}$$

So

$$\begin{array}{ll}
 & f^{\cup} \circ T \circ g \\
\subseteq & \quad \{ \quad (234) \text{ and } \Delta R \subseteq R \quad \} \\
 & R \circ R\backslash R/R \circ R \\
\subseteq & \quad \{ \quad \text{cancellation} \quad \} \\
 & R \quad .
\end{array}$$

Also,

$$\begin{array}{ll}
 & R \subseteq f^{\cup} \circ T \circ g \\
= & \quad \{ \quad (234) \quad \} \\
 & R \subseteq \Delta R \circ R\backslash R/R \circ \Delta R \\
= & \quad \{ \quad \text{per domains: } (33) \quad \} \\
 & R_{\prec} \circ R \circ R_{\succ} \subseteq \Delta R \circ R\backslash R/R \circ \Delta R \\
= & \quad \{ \quad \text{assumption: } R_{<} = (\Delta R)_{<} \text{ and } R_{>} = (\Delta R)_{>} \text{, lemma 180} \quad \}
\end{array}$$

$$
\begin{array}{ll}
& (\Delta R){\prec} \circ R \circ (\Delta R){\succ} \;\subseteq\; \Delta R \circ R\backslash R/R \circ \Delta R \\
= & \{ \quad \Delta R \text{ is difunctional, theorem 49 (with } R := \Delta R\text{)} \quad \} \\
& \Delta R \circ \Delta R^{\cup} \circ R \circ \Delta R^{\cup} \circ \Delta R \;\subseteq\; \Delta R \circ R\backslash R/R \circ \Delta R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
& \Delta R^{\cup} \circ R \circ \Delta R^{\cup} \;\subseteq\; R\backslash R/R \\
\Leftarrow & \{ \quad \Delta R^{\cup} \subseteq R\backslash R/R\text{, monotonicity} \quad \} \\
& R\backslash R/R \circ R \circ R\backslash R/R \;\subseteq\; R\backslash R/R \\
= & \{ \quad \text{factors} \quad \} \\
& R \circ R\backslash R/R \circ R \circ R\backslash R/R \circ R \;\subseteq\; R \\
= & \{ \quad \text{cancellation} \quad \} \\
& \text{true} \quad .
\end{array}
$$

Combining the two inclusions we conclude that indeed $R = f^{\cup} \circ T \circ g$.

We now establish the requirements on T. First,

$$
\begin{array}{ll}
& T \cap T^{\cup} \\
= & \{ \quad \text{definition and converse} \quad \} \\
& g \circ R\backslash R/R \circ f^{\cup} \;\cap\; f \circ (R\backslash R/R)^{\cup} \circ g^{\cup} \\
\subseteq & \{ \quad \text{modular law} \quad \} \\
& f \circ (f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \circ g \;\cap\; (R\backslash R/R)^{\cup}) \circ g^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
& f \circ (\Delta R \circ R\backslash R/R \circ \Delta R \;\cap\; (R\backslash R/R)^{\cup}) \circ g^{\cup} \\
\subseteq & \{ \quad \Delta R \subseteq R\text{, monotonicity and cancellation} \quad \} \\
& f \circ (R \cap (R\backslash R/R)^{\cup}) \circ g^{\cup} \\
= & \{ \quad \Delta R = R \cap (R\backslash R/R)^{\cup} \quad \} \\
& f \circ \Delta R \circ g^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
& f \circ f^{\cup} \circ g \circ g^{\cup} \\
= & \{ \quad f \circ f^{\cup} \;=\; f{<} \;=\; g \circ g^{\cup} \;=\; g{<} \quad \} \\
& f{<} \quad .
\end{array}
$$

Thus $T \cap T^{\cup} \subseteq f{<}$. So $T \cap T^{\cup} \subseteq I$. Now

$$f< \subseteq T \cap T^{\cup}$$
$= \quad \{ \quad$ infima and $f<$ is coreflexive $\quad \}$
$$f< \subseteq T$$
$\Leftarrow \quad \{ \quad$ domains $\quad \}$
$$f \circ f^{\cup} \subseteq T$$
$\Leftarrow \quad \{ \quad$ definition of T and monotonicity $\quad \}$
$$f \subseteq g \circ R\backslash R/R$$
$\Leftarrow \quad \{ \quad f< = g \circ g^{\cup}$, domains and monotonicity $\quad \}$
$$g^{\cup} \circ f \subseteq R\backslash R/R$$
$= \quad \{ \quad f^{\cup} \circ g = \Delta R \quad \}$
$$\Delta R^{\cup} \subseteq R\backslash R/R$$
$= \quad \{ \quad \Delta R = R \cap (R\backslash R/R)^{\cup}$, converse $\quad \}$
$$\mathsf{true} \quad .$$

So, by anti-symmetry we have established that $T \cap T^{\cup} = f<$. Since also $f< = g<$, we conclude from the definition of T and properties of domains that

$$T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad .$$

The final task is to show that T is transitive:

$$T \circ T$$
$= \quad \{ \quad$ definition $\quad \}$
$$g \circ R\backslash R/R \circ f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup}$$
$= \quad \{ \quad \Delta R = f^{\cup} \circ g \quad \}$
$$g \circ R\backslash R/R \circ \Delta R \circ R\backslash R/R \circ f^{\cup}$$
$\subseteq \quad \{ \quad \Delta R \subseteq R \quad \}$
$$g \circ R\backslash R/R \circ R \circ R\backslash R/R \circ f^{\cup}$$
$\subseteq \quad \{ \quad$ factors $\quad \}$
$$g \circ R\backslash R/R \circ f^{\cup}$$
$= \quad \{ \quad$ definition $\quad \}$
$$T \quad .$$

□

It is interesting to reflect on the proof of lemma 232. The hardest part is to find appropriate definitions of f, g and T such that $R = f^{\cup} \circ T \circ g$. The key to constructing f and g is Riguet's "analogie frappante" [Rig51] whereby he introduced the "différence" —the diagonal ΔR— of the relation R. Expressing the diagonal in terms of factors as we have done makes many parts of the calculations very straightforward. One much less straightforward step is the use of lemma 180 in the proof that $R \subseteq f^{\cup} \circ T \circ g$. Note how calculational needs drive the search for the lemma: in order to simplify the inclusion it is necessary to expose the term $R\backslash R/R$ on the right side, and that is precisely what the lemma enables.

We conclude with the theorem that we call the "analogie frappante". It is not the theorem that Riguet presented but we have chosen to give it this name in order to recognise Riguet's contribution.

Theorem 235 (Analogie Frappante) A relation R is block-ordered if and only if $R_{<} = (\Delta R)_{<}$ and $R_{>} = (\Delta R)_{>}$.

Proof Lemma 221 establishes "only-if" and lemma 232 establishes "if".
□

Example 236 Recall that example 194 is of a relation R such that $R_{<} = (\Delta R)_{<}$ but $R_{>} \neq (\Delta R)_{>}$. Because of the simplicity of the example, it is possible to check, by exhausting all possible assignments to f and g, that the relation is not block-ordered. For suppose, on the contrary, that $R = f^{\cup} \circ T \circ g$, where f, T and g satisfy the conditions for a block-ordering. Then it must be the case that $g.A \neq g.B$ (since $(R{\circ}A)_{<} \neq (R{\circ}B)_{<}$). But also it must be the case that $f.\alpha$, $f.\beta$ and $f.\gamma$ are distinct (because, eg., $(\alpha{\circ}R)_{>} \neq (\beta{\circ}R)_{>}$). This has the consequence that $f_{<} \neq g_{<}$. But, by defining $f.\alpha = x$, $f.\beta = y$, $f.\gamma = z$, $g.A = x$, $g.B = z$ and $y \sqsubseteq x$ and $y \sqsubseteq z$, it is the case that $R = f^{\cup} \circ \sqsubseteq \circ g$. We say that the relation has an "imperfect" block-ordering.
□

Example 237 A generic way to construct examples of relations that are not block-ordered is to exploit example 177. In order to avoid unnecessary repetition, we refer the reader to that example for the definition of the relation in given a finite set $\mathcal{X}$ and a set $\mathcal{S}$ of subsets of $\mathcal{X}$.

(Example 236 is a slightly disguised instance of the generic construction: the nodes A and B can be identified with, respectively, $\{\alpha,\beta\}$ and $\{\beta,\gamma\}$.)

Recall that the diagonal $\Delta\mathsf{in}$ of type $\mathcal{X}{\sim}\mathcal{S}$ is injective. It follows that the size of $(\Delta\mathsf{in})_{<}$ is at most the size of $\mathcal{S}$. If, however, the set $\mathcal{S}$ has $\mathcal{X}$ as one of its elements, the size of $\mathrm{in}_<$ equals the size of $\mathcal{X}$. Theorem 235 thus predicts that, if $\mathcal{X}$ is an element of $\mathcal{S}$, a necessary condition for in to be block-ordered is that the sizes of $\mathcal{X}$ and $\mathcal{S}$ must be equal; conversely, if $\mathcal{X}$ is an element of $\mathcal{S}$, in is not block-ordered if the sizes of $\mathcal{X}$ and $\mathcal{S}$ are different.

Fig. 6 (example 177) shows that, even if the sizes of $\mathcal{X}$ and $\mathcal{S}$ are equal, the relation in may not be block-ordered: as remarked then, for the choice of $\mathcal{S}$ shown in fig. 6, $\mathrm{in}_<$ and $(\Delta\mathrm{in})_<$ are different since 0 and 3 are elements of the former but not the latter.

It is straightforward to construct instances of $\mathcal{X}$ and $\mathcal{S}$ such that the relation in is block-ordered. It suffices to ensure that three conditions are satisfied: $\mathcal{X}$ is an element of $\mathcal{S}$, the sizes of $\mathcal{X}$ and $\mathcal{S}$ are equal, and, for each x in $\mathcal{X}$, the set of sets represented by $(x{\circ}\mathrm{in})_>$ is totally ordered. Fig. 10 is one such. Referring to definition 196, the functional f is $\Delta\mathrm{in}^{\cup}$ (depicted by rectangles) and the functional g is $\mathrm{I}_{\mathcal{S}}$; the ordering relation is the subset relation $\mathrm{in}\backslash\mathrm{in}$ (depicted by the directed graph).

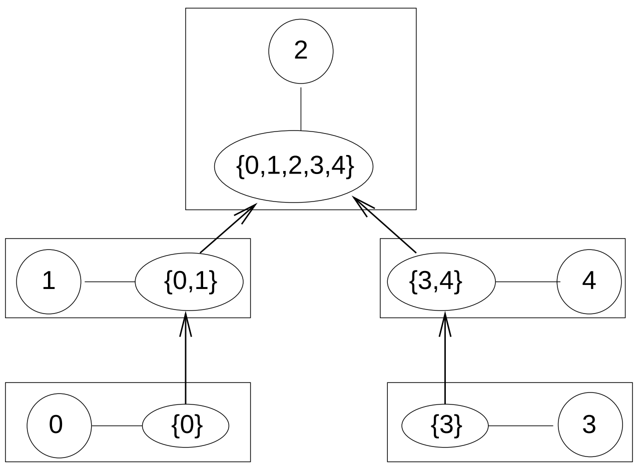


Figure 10: A Block-Ordered Membership Relation

□

The following theorem is a corollary of theorem 184. In combination with theorem 235 it states that a relation is block-ordered iff its core is block-ordered. Testing whether or not a given relation is block-ordered can thus be decomposed into computing the core of the relation and then testing whether or not that is block-ordered.

Theorem 238 Suppose R is an arbitrary relation and suppose C is a core of R as witnessed by λ and ρ . Then

$$R_< = (\Delta R)_< \quad \equiv \quad C_< = (\Delta C)_< \quad .$$

Dually,

$$R_> = (\Delta R)_> \quad \equiv \quad C_> = (\Delta C)_> \quad .$$

Proof Suppose R, C, λ and ρ are as in definition 90. Then

$$
\begin{array}{ll}
 & C{<} \;=\; (\Delta C){<} \\
= & \{ \quad \text{definition 90 and theorem 184} \quad \} \\
 & (\lambda \circ R \circ \rho^{\cup}){<} \;=\; (\lambda \circ \Delta R \circ \rho^{\cup}){<} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & (\lambda^{\cup} \circ (\lambda \circ R \circ \rho^{\cup}){<}){<} \;=\; (\lambda^{\cup} \circ (\lambda \circ \Delta R \circ \rho^{\cup}){<}){<} \\
= & \{ \quad \text{domains} \quad \} \\
 & (\lambda^{\cup} \circ \lambda \circ R \circ \rho^{\cup}){<} \;=\; (\lambda^{\cup} \circ \lambda \circ \Delta R \circ \rho^{\cup}){<} \\
= & \{ \quad \lambda^{\cup} \circ \lambda \circ R \;=\; R{\prec} \circ R \;=\; R\,, \\
 & \quad\;\; (\rho^{\cup}){<} \;=\; (\rho^{\cup} \circ \rho){<} \;=\; (R{\succ}){<} \;=\; R{>}\,, \text{ and domains} \quad \} \\
 & R{<} \;=\; (\lambda^{\cup} \circ \lambda \circ \Delta R \circ \rho^{\cup}){<} \\
= & \{ \quad (\rho^{\cup}){<} \;=\; (\rho^{\cup} \circ \rho){<} \text{ and domains} \quad \} \\
 & R{<} \;=\; (\lambda^{\cup} \circ \lambda \circ \Delta R \circ \rho^{\cup} \circ \rho){<} \\
= & \{ \quad \text{theorem 184} \quad \} \\
 & R{<} \;=\; (\lambda^{\cup} \circ \Delta C \circ \rho){<} \\
= & \{ \quad \text{theorem 184} \quad \} \\
 & R{<} \;=\; (\Delta R){<} \quad .
\end{array}
$$

Similarly,

$$
\begin{array}{ll}
 & R{<} \;=\; (\Delta R){<} \\
= & \{ \quad \text{definition 90, theorem 184 and Leibniz} \quad \} \\
 & (\lambda^{\cup} \circ C \circ \rho){<} \;=\; (\lambda^{\cup} \circ \Delta C \circ \rho){<} \\
\Rightarrow & \{ \quad \text{Leibniz and domains} \quad \} \\
 & (\lambda \circ \lambda^{\cup} \circ C \circ \rho){<} \;=\; (\lambda \circ \lambda^{\cup} \circ \Delta C \circ \rho){<} \\
= & \{ \quad \rho{<} \;=\; (\rho \circ \rho^{\cup}){<} \text{ and domains} \quad \} \\
 & (\lambda \circ \lambda^{\cup} \circ C \circ \rho \circ \rho^{\cup}){<} \;=\; (\lambda \circ \lambda^{\cup} \circ \Delta C \circ \rho \circ \rho^{\cup}){<} \\
= & \{ \quad \text{theorem 184 (applied twice)} \quad \} \\
 & C{<} \;=\; (\Delta C){<} \quad .
\end{array}
$$

The property

$$R{<} \;=\; (\Delta R){<} \quad \equiv \quad C{<} \;=\; (\Delta C){<}$$

follows by mutual implication. The dual follows by instantiating R to $R^\cup$ and applying the properties of converse.

□

By combining the definition of block-ordering with theorem 184, it is immediately clear that R is block-ordered if its core C is a provisional ordering. In general, a core of a block-ordered relation will not be a provisional ordering. This is because the types of the targets of the components λ and ρ in the definition of a core are not required to be the same; on the other hand, orderings are required to be homogeneous relations. However by carefully restricting the choice of core, it is possible to construct a core that is indeed a provisional ordering.

Theorem 239 Suppose R is an arbitrary relation. Then if R is block-ordered it has a core that is a provisional ordering.

Proof Suppose R is block-ordered. That is, suppose that f, g and T are relations such that T is a provisional ordering,

$$R = f^\cup \circ T \circ g$$

and

$$f \circ f^\cup \;=\; f_< \;=\; T \cap T^\cup \;=\; g_< \;=\; g \circ g^\cup \quad .$$

Then, by lemma 221, $R_\prec = f^\cup \circ f$ and , $R_\succ = g^\cup \circ g$. Thus f and g satisfy the conditions for witnessing a core C of R. (Cf. definition 90 with $\lambda,\rho := f,g$.) Consequently,

$$
\begin{array}{cl}
 & C \\
= & \{ \quad \text{definition 90} \quad \} \\
 & f \circ R \circ g^\cup \\
= & \{ \quad R = f^\cup \circ T \circ g \quad \} \\
 & f \circ f^\cup \circ T \circ g \circ g^\cup \\
= & \{ \quad f \circ f^\cup \;=\; f_< \;=\; T \cap T^\cup \;=\; g_< \;=\; g \circ g^\cup \quad \} \\
 & (T \cap T^\cup) \circ T \circ (T \cap T^\cup) \\
= & \{ \quad T \text{ is a provisional ordering, lemma 61 and domains} \quad \} \\
 & T \quad .
\end{array}
$$

We conclude that C is the provisional ordering T.

□

Combining theorem 239 with theorem 93, we conclude that any core of a block-ordered relation is isomorphic to a provisional ordering. Loosely speaking, block-ordered relations are provisional orderings up to isomorphism and reduction to the core.

Example 240 From the Galois connection, for all reals x and integers m,

$$\lceil x \rceil \leq m \;\equiv\; x \leq m$$

defining the ceiling function, we deduce that the heterogeneous relation ${}_{\mathbb{R}}{\leq}_{\mathbb{Z}}$ has core the provisional ordering $\leq_{\mathbb{Z}}$. This is because the ceiling function is surjective. Its core in *not* the ordering $\leq_{\mathbb{R}}$ because the coercion real from integers to reals is not surjective. (See also example 217.)

On the other hand, if a Galois connection

$$F.b \sqsubseteq a \equiv b \preceq G.a$$

of posets $(\mathcal{A}, \sqsubseteq)$ and $(\mathcal{B}, \preceq)$ is "perfect" (i.e. both F and G are surjective), both the orderings $\sqsubseteq$ and $\preceq$ are cores of the defined heterogeneous relation. That the orderings are isomorphic is an instance of the unity-of-opposites theorem [Bac02].
$\square$

14 Conclusion

A major advantage of point-free relation algebra is the combination of concision with precision. But there are numerous circumstances where pointwise reasoning is unavoidable. Unbridled use of pointwise reasoning is, in our view, unwelcome because of the potential lack of concision, with the danger of an accompanying loss of precision. In this paper, we have shown how the pointwise reasoning that is necessary to formulate the properties of polar coverings of a relation and of block-ordered relations can be conducted in a way that avoids such dangers. Doing so has led to the introduction of the concepts of an index and a core of a relation which we believe may have important practical applications when dealing with very large volumes of data. Our definitions of indexes and cores of a (heterogeneous) relation are point-free.

The primary contribution of the paper has been to show how the addition of a simple axiom to relation algebra —essentially, it is possible to choose a representative element of every equivalence class of a partial equivalence relation— has far-reaching consequences in enabling pointwise reasoning, whilst not sacrificing the combination of concision and precision. Some may criticise the axiom for being non-constructive, but the criticism has little practical relevance. For finite pers, it is straightforward to construct an index and, indeed, in practice this is done as a matter of course. For example, the two-phase algorithm attributed to R.Kosaraju and M.Sharir by Aho, Hopcroft and Ullman [AHU82] for constructing the strongly connected components of a graph computes a representative element (called a "delegate" in [Bac19]) of each component in the second phase.

One focus of this investigation has been on showing that the so-called "all-or-nothing" rule introduced by Glück [Glü17] is a consequence of our axiom. There are other ways of facilitating pointwise reasoning in relation algebra. Bird and De Moor [BdM97] argue that the introduction of "tabulations" and a "unit" (as formulated by Freyd and Ščedrov [Fv90]) "makes it possible to mimic pointwise proofs in a categorical setting". But Bird and De Moor do not give any practical application of tabulations[9]. Separately from tabulations, Bird and De Moor [BdM97, section 4.6, p.103] introduce so-called "power allegories". This involves the introduction of "power-objects", the "power transpose" of a relation, and a "membership relation". Subsequently, they do make significant practical use of these notions in their derivation of algorithms. However, as we have shown elsewhere [BDGv22, Bac22], these notions can be derived from the all-or-nothing rule.

In fact, the only practical application of pointwise reasoning in this paper is in section

[9]Indeed, their only use of tabulations is in an erroneous proof. [BdM97, theorem 5.1] asserts that (in a tabular allegory) a functor is a relator iff it preserves converse. However, the penultimate step in the "proof" asserts that application of a functor to a "simple" relation preserves the "simple" property. A traditional pointwise argument makes clear that the step has no justification.

11 on coverings of a relation. In this case, points are unavoidable because they define the individual rectangles in the covering. In contrast, our investigation of the diagonal of a relation and block-ordered relations, and the formulation of the analogie frappante connecting the two, is entirely point-free. At the same time, we make extensive use of our (point-free) formulation of indexes and cores in order to significantly improve previous calculations of the same results [Bac21].

As shown elsewhere, the direct use of pointwise reasoning (as formulated here) does combine concision with precision in an elegant way in the construction of the characterisations of pers and difunctions. (See section 8.) Specifically, [BO22] compares the explicit use of points with the use of the power transpose of a relation. This paper offers an alternative third way. We leave the reader to make the judgement on which method is to be preferred.

Finally, a few words on notation. The very rich algebraic properties of the converse of a relation mean that many notions and properties come in pairs, each element of the pair being the dual mirror-image of the other. For example, we have defined both the left domain and right domain of a relation. Some authors emphasise such mirroring by their choice of notation. Freyd and Ščedrov [Fv90], for example, denote the source and target of a relation R by $\Box R$ and $R\Box$, respectively.

A consequence of this is that it is possible to get away with defining just one of a pair of operators, leaving its mirror image to have an "obvious" definition in terms of relational converse. Doing this systematically would mean introducing the notation $R{<}$ for the left domain of relation R and then using the notation $(R^{\cup}){<}$ to denote the right domain of R. Similarly, one might introduce just the left factor R/S and then write $(S^{\cup}/R^{\cup})^{\cup}$ for the right factor $R\backslash S$. This is, of course, very undesirable because then the associativity of the operators (the rule that $R\backslash(S/T)$ and $(R\backslash S)/T$ are equal, which we exploit by using the notation $R\backslash S/T$) becomes the very cumbersome

$$((S/T)^{\cup}/R^{\cup})^{\cup} \;=\; (S^{\cup}/R^{\cup})^{\cup}/T \quad .$$

Even worse is when a symmetric notation is used for an operator that has both left and right variants — as is done by both Freyd and Ščedrov [Fv90] and Schmidt and Ströhlein [SS93, p.80] in the case of the so-called "symmetric division/quotient" of a relation. By writing $\frac{R}{S}$ (or $R{\div}S$), the reader may be misled into supposing that either the operator has no mirror image or that the mirror image is $\frac{S}{R}$ (or $S{\div}R$). The main drawback, however, is that the notation gives —literally and figuratively— a one-sided view of relation algebra that inhibits progress. The notions of "index" and "core" of a relation are, so far as we know, novel; the insight leading to their introduction is the simple formula

$$R \;=\; R{\prec} \circ R \circ R{\succ}$$

combined with well-known properties of (partial) equivalence relations. It is, in our view, a striking example of the sort of insight that is obscured using Freyd and Ščedrov's or Schmidt and Ströhlein's notation.

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