The University of Nottingham

SCHOOL OF COMPUTER SCIENCE and IT

A LEVEL 1 MODULE, AUTUMN 2007–2008

MATHEMATICS FOR COMPUTER SCIENCE

Time allowed 90 minutes

Candidates must NOT start writing their answers until told to do so.

Model Solutions

1 **(Compulsory)**

For each statement, say whether it is true or false. Each correct answer carries one mark. Incorrect answers will be given a mark of -0.5, with a minimum of 0 for each section. Answers of "don't know" carry a mark of 0.

(a) Which of the following statements are correct?

(i) A relation R is reflexive if and only if $[xRx]$.

(ii) Equality is reflexive, symmetric, and associative.

(iii) The greater-than relation $(>)$ is anti-symmetric.

(iv) A partial ordering is a reflexive, transitive, and anti-symmetric relation.

(v) The divides relation is a total ordering.

(i) True

(ii) True

(iii) True

(iv) True

(v) False

(b) In the following questions, S is a set and $\emptyset$ denotes the empty set.

(i) $|2^{(2^{\emptyset})}| = 2$

(ii) $[\emptyset \in S]$

(iii) $[\emptyset \subseteq S]$

(iv) $[\emptyset \in 2^S]$

(v) $\{\emptyset, \{1\}\} = \{\{1,1\}, \emptyset, \emptyset\}$

(c) (i) Conjunction is idempotent.

(ii) Conjunction distributes through equivales $(\equiv)$.

(iii) False is the unit for conjunction.

(iv) True is the zero for disjunction.

(v) Minimum is associative.

(i) True.

(ii) False.

(iii) False.

(iv) True.

(v) True.

(d) In this question, assume the convention is that x and y are reals, and m and n are integers.

(i) $[m+1 \leq n \equiv m < n]$

(ii) $[x+1 > \lceil x \rceil]$

(iii) $[\lfloor x \rfloor + 1 \equiv \lceil x \rceil]$

(iv) $[\lfloor x \rfloor + \lfloor y \rfloor \equiv \lfloor x+y \rfloor]$

(v) $[2 \times \min(x,y) \equiv \min(2 \times x, 2 \times y)]$

(i) True
(ii) True
(iii) False
(iv) False
(v) True

(e) (i) $[q \Rightarrow (p \Rightarrow q)]$
(ii) $[(p \Rightarrow q) \wedge p \equiv p \wedge q]$
(iii) $[\neg(p \vee q) \equiv \neg p \vee \neg q]$
(iv) $[(p \vee q) \vee r \equiv p \vee (q \vee r)]$
(v) $[\neg(p \Rightarrow q) \equiv p \Leftarrow q]$

(i) False
(ii) True
(iii) False
(iv) True
(v) False

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(a) Construct truth tables for the following Boolean expressions:

(i) $p \Rightarrow (q \Rightarrow p)$

p	q	p	$\Rightarrow$	(	q	$\Rightarrow$	p	)
t	t		t			t		
t	f		t			t		
f	t		t			f		
f	f		t			t		

(ii) $(p \Rightarrow q) \wedge (p \Leftarrow q) \equiv (p \equiv q)$

p	q	(	p	$\Rightarrow$	q	)	$\wedge$	(	p	$\Leftarrow$	q	)	$\equiv$	(	p	$\equiv$	q	)
t	t			t			t			t			t			t		
t	f			f			f			t			t			f		
f	t			t			f			f			t			f		
f	f			t			t			t			t			t		

(iii) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg$	(	p	$\vee$	q	)	$\equiv$	$\neg p$	$\wedge$	$\neg q$
t	t	f			t			t	f	f	f
t	f	f			t			t	f	f	t
f	t	f			t			t	t	f	f
f	f	t			f			t	t	t	t

(b) State for each step in the following calculation whether the step is valid or not. If the step is valid, name the rule that justifies the step.

$$
\begin{array}{ll}
& \neg((p \vee q) \vee p) \equiv \neg p \vee \neg q \\
= & \{ \text{ Step 1 } \} \\
& \neg(p \vee (q \vee p)) \equiv \neg p \vee \neg q \\
= & \{ \text{ Step 2 } \} \\
& \neg(p \vee (p \vee q)) \equiv \neg p \vee \neg q \\
= & \{ \text{ Step 3 } \} \\
& \neg((p \vee p) \vee q) \equiv \neg p \vee \neg q \\
= & \{ \text{ Step 4 } \} \\
& \neg(p \vee q) \equiv \neg p \vee \neg q \\
= & \{ \text{ Step 5 } \} \\
& ((p \vee q) \equiv \text{false}) \equiv \neg p \vee \neg q \\
= & \{ \text{ Step 6 } \} \\
& (p \equiv \text{false}) \vee (q \equiv \text{false}) \equiv \neg p \vee \neg q \\
= & \{ \text{ Step 7 } \} \\
& \neg p \vee \neg q \equiv \neg p \vee \neg q \\
= & \{ \text{ Step 8 } \} \\
& \text{true}
\end{array}
$$

$$
\begin{array}{ll}
& \neg((p \vee q) \vee p) \equiv \neg p \vee \neg q \\
= & \{ \text{ Associativity of disjunction } \} \\
& \neg(p \vee (q \vee p)) \equiv \neg p \vee \neg q \\
= & \{ \text{ Symmetry of disjunction } \} \\
& \neg(p \vee (p \vee q)) \equiv \neg p \vee \neg q \\
= & \{ \text{ Associativity of disjunction } \} \\
& \neg((p \vee p) \vee q) \equiv \neg p \vee \neg q \\
= & \{ \text{ Idempotence of disjunction } \} \\
& \neg(p \vee q) \equiv \neg p \vee \neg q \\
= & \{ \text{ Definition of negation } \} \\
& ((p \vee q) \equiv \text{false}) \equiv \neg p \vee \neg q \\
= & \{ \text{ Invalid! } \} \\
& (p \equiv \text{false}) \vee (q \equiv \text{false}) \equiv \neg p \vee \neg q \\
= & \{ \text{ Definition of negation } \} \\
& \neg p \vee \neg q \equiv \neg p \vee \neg q \\
= & \{ \text{ Reflexivity of equivales } \} \\
& \text{true}
\end{array}
$$

3 The game of Dominoes is played using 28 pieces called bones or dominoes. Each bone is a rectangular tile with a line dividing its face into two square ends. Each end is marked with a number of spots, arranged as they are on a six-sided dice, or is blank. There are thus seven possible end markings. There are no duplicate bones; i.e., each bone has a unique combination of end markings.

Now consider a *generalized* version of Dominoes where each end is marked with up to n spots.

(a) How many bones are there in the generalized version where both ends are marked the same way? Give your answer as an expression in n along with a brief explanation.

There are $n+1$ possible markings. Thus there are $n+1$ bones where both ends are marked the same way.

(b) How many bones are there in *total* in the generalized version; i.e., including both bones where the ends are marked in differently and where they are marked the same? Give your answer as an expression in n. Explain your construction and simplify as far as possible.

There are $(n+1)\ \mathrm{C}\ 2$ ways to mark the two ends differently (the orientatin of the bone does not matter). Additionally, we have already established that there are $n+1$ bones where the two ends have the same marking. Thus, for n spots, the number of bones $b(n)$ is given by

$$\begin{aligned} b(n) &= ((n+1)\ \mathrm{C}\ 2) + (n+1) \\ &= \frac{(n+1)\times n}{2} + (n+1) \\ &= \frac{(n+1)\times(n+2)}{2} \end{aligned}$$

Sanity check: We know that there should be 28 bones for $n=6$

$$b(6) = (6+1)\times(6+2)/2 = 7\times 8/2 = 56/2 = 28$$

(c) Assume that the rule of the generalized game with markings of up to n spots is that a player initially get a hand of $n+1$ bones. (E.g., for the 6-spot version, players get 7 bones initially, for the 9 spot version, they get 10, etc.) Consider a single player. How many different hands are there? Again, give an expression in n for the number of possibilities, along with an explanation.

The number of possible hands is simply given by the number of ways to choose $n+1$ bones among the $b(n)$ possible bones. Thus there are

$$b(n)\ \mathrm{C}\ (n+1) = \frac{b(n)!}{(b(n)-(n+1))!(n+1)!}$$

possibilities.

(d) Generalize your result from (c) to two players. I.e., if there are two players each getting $n+1$ bones, how many possibilities are there? Give an expression in n, simplifying as far as possible, along with an explanation. You may assume that n is sufficiently large that there are sufficiently many bones for both players.

We can think as follows. Let the first player choose $n+1$ bones from the $b(n)$ available ones. We have already established that the number of possibilities is $b(n)\ \mathrm{C}\ (n+1)$. Then let the second player choose $n+1$ bones from the $b(n)-(n+1)$ that remains. There are $(b(n)-(n+1))\ \mathrm{C}\ (n+1)$ ways of doing this. The total number of possibilities is obtained by multiplication. Thus, for two players the number of possibilities is

$$\begin{aligned} &((b(n)\ \mathrm{C}\ (n+1)) \times (b(n)-(n+1))\ \mathrm{C}\ (n+1) \\ &= \frac{b(n)!}{(b(n)-(n+1))!\times(n+1)!} \times \frac{(b(n)-(n+1))!}{(b(n)-(n+1)-(n+1))!\times(n+1)!} \\ &= \frac{b(n)!}{(b(n)-2\times(n+1))!((n+1)!)^2} \end{aligned}$$

(e) Generalize your result from (d) to r players. I.e., if there are r players each getting $n+1$ bones, how many possibilities are there? Give an expression in n, simplifying as far as possible, along with an explanation. You may assume that the relation between n and r is such that there are sufficiently many bones for everyone.

We just continue in the fashion outlined above, letting each player in turn choose from however many bones that remain. Thus, for r players the number of possibilities is

$$\begin{aligned}
&\prod_{i=0}^{r-1}(b(n)-i\times(n+1))\ \mathrm{C}\ (n+1)\\
&= \prod_{i=0}^{r-1}\frac{(b(n)-i\times(n+1))!}{(b(n)-i\times(n+1)-(n+1))!(n+1)!}\\
&= \prod_{i=0}^{r-1}\frac{(b(n)-i\times(n+1))!}{(b(n)-(i+1)\times(n+1))!(n+1)!}\\
&= \frac{b(n)!}{(b(n)-r\times(n+1))!((n+1)!)^r}
\end{aligned}$$

Question 4:

i)

$\frac{1}{2}(n-\frac{1}{2}m) < \frac{1}{2}(m-\frac{1}{2}n)$

$=$ { cancellation of multiplication by 4 }

$2\times n-m < 2\times m-n$

$=$ { cancellation of addition of $m+n$ }

$3\times n < 3\times m$

$=$ { cancellation of multiplication by 3 }

$n < m$

ii)

$m\leq n \vee m+2<n+2$

$=$ { at-most is less than or equal to,
cancellation of addition of 2 }

$m<n \vee m=n \vee m<n$

$=$ { symmetry and idempotence of disjunction }

$m<n \vee m=n$

$=$ { at-most is less than or equal to }

$m\leq n$

b)i) Lots of possibilities. Eg $x=0$ and $y=0.5$.

ii)Lots of possibilities. Eg $n=3$ and $x=0.4$. iii) $\lceil x\rceil-1 < x \leq \lceil x\rceil$.

iv) x must be an integer. v) x must be an integer.

c)

$n\times\lceil x\rceil \ \leq\ k$

$=$ { cancellation, n is strictly positive. CORRECT }

$\lceil x\rceil \ \leq\ k/n$

$=$ { definition of the ceiling function.

INcorrect. The right side must be an integer. }

$x \ \leq\ k/n$

$=$ { cancellation, n is strictly positive. CORRECT }

$n\times x \ \leq\ k$

$=$ { definition of the ceiling function. CORRECT }

$\lceil n\times x\rceil \ \leq\ k$.

If $x=0.5$, $k=1$ and $n=2$, $\lceil x\rceil \leq k/n$ is false but $x\leq k/n$ is true.

Question 6 i)

$\langle \Pi k : 0\leq k<6 \wedge \text{odd}.k : k+1\rangle$

$=$ { range splitting }

$(1+1)\times(3+1)\times(5+1)$

$=$ { arithmetic }

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ii)

$\langle \forall k : 0\leq k<5 : 0\leq k^2-2\times k\rangle$

$=$ { range disjunction on k equal to 1 }

$0\leq 1^2-2\times 1 \wedge \langle \forall k : 0\leq k<5 : 0\leq k^2-2\times k\rangle$

$=$ { arithmetic }

$0\leq -1 \wedge \langle \forall k : 0\leq k<5 : 0\leq k^2-2\times k\rangle$

$=$ { $0\leq -1 \equiv$ false , false is the zero of conjunction }

false

iii)

$\langle \exists j : 2\leq j\leq 3 : 3\times j=2\times(j+1)\rangle$

$=$ { range splitting }

$3\times 2=2\times(2+1) \vee 3\times 3=2\times(3+1)$

$=$ { arithmetic }

true $\vee$ $3\times 3=2\times(3+1)$

$=$ { true is the zero of disjunction }

true

iv)

$\langle MIN\, k : 0 \leq k < 5 : 3\times k - k^2 \rangle$

= { range splitting }

$(3\times 0 - 0^2)\downarrow(3\times 1 - 1^2)\downarrow(3\times 2 - 2^2)\downarrow(3\times 3 - 3^2)\downarrow(3\times 4 - 4^2)$

= { arithmetic }

$0 \downarrow 2 \downarrow 2 \downarrow 0 \downarrow -4$

= { definition of $\downarrow$ }

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v)

$\langle \Sigma\, k : 100 \leq k < 110 : 1 \rangle$

= { counting elements of a range }

$110-100$

= { arithmetic }

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b) i)

$even.\langle \Pi\, k : 1 \leq k \leq 500 \ \wedge\ odd.k : k^2 \rangle$

= { distributivity }

$\langle \exists\, k : 1 \leq k \leq 500 \ \wedge\ odd.k : even.k \vee even.k \rangle$

= { idempotency of disjunction, trading }

$\langle \exists\, k : 1 \leq k \leq 500 : odd.k \wedge even.k \rangle$

= { $odd.k \wedge even.k \equiv$ false }

$\langle \exists\, k : 1 \leq k \leq 500 : \text{false} \rangle$

= { trading, empty range }

false

ii)

$\langle \exists j : 2\times j = 3\times(j+1) : 3\times j = 2\times(j+1) \rangle$

= { arithmetic }

$\langle \exists j : j = -3 : j = 2 \rangle$

= { trading, empty range }

false

iii)

$\langle \forall j : 0 \leq j \leq 100 \leq j-1 : 3\times j = 2\times(j+1) \rangle$

= { by transitivity of $\leq$,
$j \leq 100 \leq j-1$ is false }

$\langle \forall j : \text{false} : 3\times j = 2\times(j+1) \rangle$

$=$ { empty range }

true