

# Uncertainty Representation of Grey Numbers and Grey Sets

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**Abstract**—In the literature there is a presumption that a grey set and an interval-valued fuzzy set are equivalent. This presumption ignores the existence of discrete components in a grey number. In this paper new measurements of uncertainties of grey numbers and grey sets, consisting of both absolute and relative uncertainties, are defined to give a comprehensive representation of uncertainties in a grey number and a grey set. Some simple examples are provided to illustrate that the proposed uncertainty measurement can give an effective representation of both absolute and relative uncertainties in a grey number and a grey set. The relationships between grey sets and interval-valued fuzzy sets are also analysed from the point of view of the proposed uncertainty representation. The analysis demonstrates that grey sets and interval-valued fuzzy sets provide different but overlapping models for uncertainty representation in sets.

**Index Terms**—Grey sets, Fuzzy sets, Relative uncertainty

## I. INTRODUCTION

GREY systems have emerged as an effective approach for modelling systems with partial information [13], [15], [16], [17], [18], [28]. They provide an alternative for representing uncertainty in systems in addition to the more mainstream models like fuzzy sets and rough sets. Grey sets apply the basic concepts of grey numbers in grey systems, and consider the characteristic function values of a set as grey numbers. Grey numbers have been successfully applied into many real world problems, such as manufacturing [19] and hydrology [2]. However, grey numbers and intervals have some similarity and grey sets are considered to be equivalent to interval-valued fuzzy sets [8], [10]. With the increasing applications of grey systems, the combination of grey sets with fuzzy sets and rough sets have been investigated recently [22], [23]. However, all these research works considered grey sets using interval grey numbers only, and such a restriction does indeed make grey sets and interval-valued fuzzy sets equivalent. In fact, grey numbers can have discrete values in its candidate set as well. In an interval there are infinite candidates no matter how small the interval is, but there might be only finite candidates in a grey number. For example, both  $[0.2, 0.4]$  and  $\{0.2, 0.4\}$  are possible for a grey number defined on  $[0, 1]$ , but only the first one is possible with interval representation. In this sense, a different number of candidates will demonstrate different uncertainty in the value selection from the candidate set. By focusing on interval grey numbers only, this important extra uncertainty in grey sets has been completely neglected so far. For grey numbers represented with discrete candidates or mixed candidates (discrete numbers and intervals), the existing measurements may provide misleading results for uncertainty comparisons between grey sets. Obviously, it is necessary to investigate the suitable uncertainty representation of grey numbers and grey sets and their differences from intervals and

interval-valued fuzzy sets.

The paper is structured as follows. In the next Section a brief overview of grey numbers, grey sets and interval-valued fuzzy sets is provided. Then section III defines the concept of absolute and relative uncertainties of grey numbers and grey sets, and discusses the relationship between interval-valued fuzzy sets and grey sets. Based on the proposed absolute and relative uncertainties, Section IV presents the concept of uncertainty pairs of grey numbers and grey sets. Some simple examples are also employed to illustrate the effectiveness of the proposed uncertainty representation. Finally, in Section V we draw out the conclusions.

## II. PRELIMINARIES

We first define some relevant concepts.

Grey systems were proposed by Professor Julong Deng in 1982 [7]. In grey systems, the information is classified into three categories: white with completely certain information, grey with insufficient information, and black with totally unknown information. Grey numbers are the basic concepts in grey systems. The concept of generalised grey numbers [14], [24], [26] demonstrates their difference from intervals.

*Definition 1 (Generalised grey numbers):* Let  $\Omega \subset \mathfrak{R}$  be the universe,  $g^\pm \in \Omega$  be an unknown real number within a union set of closed or open intervals

$$g^\pm \in \bigcup_{i=1}^n [a_i^-, a_i^+] \subseteq \Omega$$

$i = 1, 2, \dots, n$ ,  $n$  is an integer and  $0 < n < \infty$ ,  $a_i^-, a_i^+ \in \Omega$  and  $a_{i-1}^+ < a_i^- \leq a_i^+ < a_{i+1}^-$ . For any interval  $[a_i^-, a_i^+] \subseteq \bigcup_{i=1}^n [a_i^-, a_i^+] \subseteq \Omega$ ,  $p_i$  is the probability for  $g^\pm \in [a_i^-, a_i^+]$ . If the following conditions hold

- $p_i > 0$
- $\sum_{i=1}^n p_i = 1$

then we call  $g^\pm$  a generalised grey number.  $g^- = \inf a_i^-$  and  $g^+ = \sup a_i^+$  are called as the lower and upper limits of  $g^\pm$ .

If  $g^- = g^+$ ,  $g^\pm$  has no uncertainty at all and is called a white number; on the contrary, if  $\bigcup_{i=1}^n [a_i^-, a_i^+] = \Omega$ , there is nothing known about  $g^\pm$  and it is called a black number.

The degree of greyness of a grey number measures the significance of uncertainty in a grey number. For example, three different definitions for the degree of greyness of a generalised grey number have been proposed [14], [16], [26].

*Definition 2 (Degree of greyness of a generalised grey number):* Let  $\Omega \subset \mathfrak{R}$  be the universe and  $g^\pm \in \bigcup_{i=1}^n [a_i^-, a_i^+] \subseteq \Omega$ ,  $d_{min}, d_{max} \in \Omega$  are the minimum and maximum values of  $\Omega$ .  $\mu$  is a measurement

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defined on  $\Omega$ . The degree of greyness of  $g^\pm$  is defined as

$$g_1^\circ(g^\pm) = \frac{\mu(g^\pm)}{\mu(\Omega)} \quad (1)$$

$$g_2^\circ(g^\pm) = \frac{|g^+ - g^-|}{|d_{max} - d_{min}|} \quad (2)$$

$$g_3^\circ(g^\pm) = \frac{1}{\hat{g}} \sum_{i=1}^n \hat{a}_i \frac{\mu(a_i^\pm)}{\mu(\Omega)} \quad (3)$$

where,  $a_i^\pm$  denotes the grey number represented by the interval  $[a_i^-, a_i^+]$ ,  $\hat{g}$  and  $\hat{a}_i$  refer to the mathematical expectation of  $g^\pm$  and  $a_i^\pm$ .

Here,  $g_2^\circ(g^\pm)$  focuses on the hesitation (gap between the upper and lower limits) and  $g_3^\circ(g^\pm)$  highlights the components of a generalised grey number.

Obviously,  $g_1^\circ$ ,  $g_2^\circ$  and  $g_3^\circ$  satisfy the following properties

- $g_i^\circ = 0$  if  $g_i^+ = g_i^-$  for  $i=1,2,3$
- $g_i^\circ = 1$  if  $n = 1$  and  $[a_1^-, a_1^+] = \Omega$  for  $i=1,2,3$
- if  $g^\pm$  is a continuous grey number, then  $g_1^\circ = g_2^\circ = g_3^\circ$

It illustrates that there is no difference between the three representations when a grey number is in the form of a continuous interval. However, they are different when more than one component is involved in a grey number. The value of  $g_1^\circ$  and  $g_3^\circ$  may change but  $g_2^\circ$  keeps constant as long as the upper and lower limits do not change. This is due to their different focuses.  $g_2^\circ$  counts mainly on the uncertainty caused by the hesitation gap between the two limit values, but  $g_1^\circ$  and  $g_3^\circ$  pay more attention to the uncertainty caused by the components. Due to the simplicity of  $g_2^\circ$  in comparison with other representations, without specific explanation, we will use  $g_2^\circ$  as the representation of the degree of greyness hereafter.

With the concept of grey numbers, we can now express the uncertain relationships between an element and a set.

*Definition 3 (Grey sets [25], [26], [27]):* For a set  $A \subseteq U$ , if the characteristic function value of  $x$  with respect to  $A$  can be expressed with a grey number  $g_A^\pm(x) \in \bigcup_{i=1}^n [a_i^-, a_i^+] \in D[0, 1]^\pm$

$$\chi_A : U \rightarrow D[0, 1]^\pm$$

then  $A$  is a grey set.

Here,  $D[0, 1]^\pm$  refers to the set of all grey numbers within the interval  $[0, 1]$ . Similar to the expression of a fuzzy set [12], [29], a grey set  $A$  is represented with its relevant elements and their associated grey numbers for characteristic function:

$$A = g_A^\pm(x_1)/x_1 + g_A^\pm(x_2)/x_2 + \dots + g_A^\pm(x_n)/x_n$$

Similar to the case for a grey number, the uncertainty caused by the distance between upper and lower limits of the characteristic function values of a grey set can be measured using its degree of greyness. Considering the specific feature of grey sets, the degree of greyness for an element and a set are defined here.

*Definition 4 (Degree of greyness for an element [26]):* Let  $U$  be the finite universe of discourse,  $x$  be an element and  $x \in U$ . For a grey set  $A \subseteq U$ , the characteristic function value of  $x$  with respect to  $A$  is  $g_A^\pm(x) \in D[0, 1]^\pm$ . The degree of greyness  $g_A^\circ(x)$  of element  $x$  for set  $A$  is expressed as

$$g_A^\circ(x) = |g^+ - g^-|$$

Based on the degree of greyness for an element, the degree of greyness for a set is defined as follows.

*Definition 5 (Degree of greyness for a set [26]):* Let  $U$  be the finite universe of discourse,  $A$  be a grey set and  $A \subseteq U$ .  $x_i$  is an element relevant to  $A$  and  $x_i \in U$ .  $i = 1, 2, 3, \dots, N$  and  $N$  is the cardinality of  $U$ . The degree of greyness of set  $A$  is defined as

$$g^\circ(A) = \frac{\sum_{i=1}^n g_A^\circ(x_i)}{N}$$

A grey set can also be considered as an extension to a fuzzy set. In this sense, it is closely related with interval-valued fuzzy sets. Here we give the definition of interval-valued fuzzy sets.

*Definition 6 (Interval-valued fuzzy sets [20]):* Let  $D[0, 1]$  be the set of all closed subintervals of the interval  $[0, 1]$ .  $U$  is the universe of discourse,  $x$  is an element and  $x \in U$ . An interval-valued fuzzy set in  $U$  is given by set  $A$

$$A = \{ \langle x, M_A(x) \rangle : x \in U \}$$

with  $M_A : U \rightarrow D[0, 1]$ .

The membership of an individual element is thus reflected by an interval instead of a single value. An intuitionistic fuzzy set [1] is mathematically equivalent to an interval-valued fuzzy set although some semantic differences still exist [3], [4], [5], [6], [8], [9], [21].

If we restrict the grey number in a grey set as a single continuous interval only, then a grey set is equivalent to an interval-valued fuzzy set in the mathematical expression. In this sense, they overlap with each other for grey numbers represented with single continuous intervals although there might still be semantic difference when an interval in interval-valued fuzzy sets is interpreted differently from a grey number. This is the reason why many people consider grey sets identical with interval-valued fuzzy sets.

### III. ABSOLUTE AND RELATIVE UNCERTAINTY

The crucial difference between an interval and a grey number, as shown in Definition 1, is the possible existence of gaps between component intervals in the construction of a grey number. Without these gaps, a grey number is mathematically equivalent to an interval in its representation. However, the involvement of these gaps makes a grey number completely different from an interval. For the existing degree of greyness defined in Equation (1), (2) and (3), their measurements of the uncertainties in a grey number or a grey set are still not ideal. It can be demonstrated with an example.

*Example 1:*  $a \in [0, 1]$ ,  $b \in [0.9, 1]$ ,  $c \in \{[0, 0.001], 1\}$  and  $d \in \{0, 1\}$  are four different grey numbers defined on  $[0, 1]$  representing the opinion of four different people on their intention to support or oppose a specific party in an election. Following Equations (1), (2) and (3), we have

$$g_1^\circ(a) = 1, \quad g_1^\circ(b) = 0.1, \quad g_1^\circ(c) = 0.001, \quad g_1^\circ(d) = 0$$

$$g_2^\circ(a) = 1, \quad g_2^\circ(b) = 0.1, \quad g_2^\circ(c) = 1, \quad g_2^\circ(d) = 1$$

$$g_3^\circ(a) = 1, \quad g_3^\circ(b) = 0.1, \quad g_3^\circ(c) = 0.000001, \quad g_3^\circ(d) = 0$$

In this example,  $a$  is a black number, so it is reasonable to have a greater degree of greyness than anyone else. It shows that  $a$  has no idea if he/she supports this party or not, and it is possible for  $a$  to support nobody (0.5) as well.  $b$  is a clear supporter of the party,  $c$  and

$d$  will not adopt a middle way, which means they will either support this party or oppose it. However,  $c$  and  $d$  can go either way which demonstrates significant uncertainty. Therefore, from this context, we can draw a conclusion that  $a$  has the highest value of uncertainty,  $c$  and  $d$  are more uncertain than  $b$ .

Obviously, the three representations give an identical result for  $a$  and  $b$  due to their single interval representation. However, they are different for  $c$  and  $d$  where more than one component is involved. According to  $g_1^\circ(b)$  and  $g_1^\circ(c)$ ,  $b$  has greater uncertainty than  $c$ . It conflicts with our conclusion from the context. In this case, only  $g_2^\circ$  satisfies this conclusion.

When more than one component is involved,  $g_1^\circ$  and  $g_3^\circ$  focus mainly on their relative distribution. There are two reasons for their inappropriate results:

- No consideration to the absolute distance between the upper and lower limits;
- The measurement for continuous intervals is not effective for discrete numbers.

Contrary to  $g_1^\circ$  and  $g_3^\circ$ ,  $g_2^\circ$  highlights the significance of the upper and lower limits but cannot reflect the significance of the distribution of components. Therefore, it is insensitive to the changed distribution of components which is demonstrated by its result with  $a$  and  $c$ .

Obviously, the uncertainty in a grey number consists of two different parts: the absolute uncertainty which is determined by the upper and lower limits and the relative uncertainty which depends on the distribution of components of a grey number. Although a grey number is not determined within its candidate set, its absolute and relative uncertainties are determined as long as its lower limit, upper limit and probability distribution of its candidates are known. Among the existing representations,  $g_2^\circ$  focuses mainly on the absolute uncertainty, but  $g_1^\circ$  and  $g_3^\circ$  pay more attention to the relative uncertainty. They converge to the same value when the specific grey number is represented as a single interval. When more than one component is involved, however, each of them gives a different value and a biased measurement.  $g_1^\circ$  and  $g_3^\circ$  combined absolute and relative uncertainty together in some way, and demonstrate absolute uncertainty for a single interval and relative uncertainty when more components are involved. However, its representation of relative uncertainty fails to integrate continuous components and discrete components together as shown in example (1).

Absolute uncertainty and relative uncertainty are two different aspects of the uncertainty in a grey number, and they have different significance as well. In most real world cases, the absolute uncertainty is more important than the relative uncertainty. For example, we would consider  $b$  in our previous example much more certain than  $c$ . In this sense, the absolute uncertainty should always be taken as the first priority in our comparison, and the relative uncertainty comes in the second position and will be meaningful only when the absolute uncertainty cannot separate one from another.

As shown in Definition 1, a grey number may contain both interval components and discrete components (when the lower and upper bounds of an interval are identical). The uncertainty representation of a grey number has to be able to give consistent results for its representation as a continuous interval, a discrete set of values and a mix of intervals and discrete values in a continuous domain. In this sense, the relative uncertainty representation of a grey number is expected to satisfy the following properties.

*Definition 7 (Properties for the relative uncertainty representation of a grey number):* Let  $\Omega \subset \mathfrak{R}$  be the universe and  $g^\pm \in \bigcup_{i=1}^n [a_i^-, a_i^+] \subseteq \Omega$  be a grey number, and  $i = 1, 2, \dots, n$ ,  $n$  is an integer and  $0 < n < \infty$ ,  $a_i^-, a_i^+ \in \Omega$  and  $a_{i-1}^+ < a_i^- \leq a_i^+ < a_{i+1}^-$ . For any interval  $[a_i^-, a_i^+] \subseteq \bigcup_{i=1}^n [a_i^-, a_i^+] \subseteq \Omega$ ,  $p_i$  is the probability for  $g^\pm \in [a_i^-, a_i^+]$ .  $\delta(g^\pm)$  is the relative uncertainty of  $g^\pm$  and satisfies the following properties

- 1)  $0 \leq \delta(g^\pm) \leq 1$ ;
- 2)  $\delta(g^\pm) = 0$  iff  $g^\pm$  is a white number;
- 3)  $\delta(g^\pm) = 1$  iff  $g^\pm$  is a continuous grey number;
- 4)  $\delta(g^\pm) \neg \max\{p_1, p_2, \dots, p_n\}$ ;

Here,  $\neg$  indicates a negative relationship between its two operands: a reduction of the right-hand side increases the left-hand side.

The first property is a requirement of normalisation. The second property is required to make the uncertainty representation meaningful and consistent. A white number has the same lower and upper limits, and there is no uncertainty in its candidate selection, so we have  $\delta(g^\pm) = 0$ ; similarly,  $\delta(g^\pm) = 0$  means that there is no uncertainty on the selected candidate, then the candidate is a single number which has the same value for its lower and upper limits, thus the grey number is actually a white number. The third property shows that the relative uncertainty reaches its maximum value when the grey number is represented by a continuous interval. If it is represented by a continuous interval, every number in that interval is a candidate so it should have the maximum selection uncertainty. Similarly, if it has the maximum selection uncertainty, every number between its lower and upper limits should be a qualified candidate, then it has to be a continuous interval. The last property indicates that the maximum probability for one candidate interval to cover the number represented by a grey number should have a negative influence on the uncertainty of a grey number. This is reasonable: a higher probability means a better chance for the grey number to be whitenised into a narrow scope, so less uncertainty in comparison with the one with a smaller probability.

According to Definition 2 and Example 1, although  $g_1^\circ$ ,  $g_2^\circ$  and  $g_3^\circ$  do not satisfy the third property in Definition 7, they give an effective representation for absolute uncertainty of a grey number represented as a continuous interval.  $g_2^\circ$  is determined only by the lower and upper limits of a grey number and provides a simple and effective representation for the absolute uncertainty of a grey number. However, it does not consider the relative uncertainty at all. If the candidate set of a grey number contains discrete numbers or more than one interval with gaps in between,  $g_1^\circ$  and  $g_3^\circ$  highlight the relative uncertainty, but do not consider its absolute uncertainty. Their relative uncertainty representation is effective only when both the candidate values and the universe are the same type. In fact, as shown in Definition 1, a grey number could have discrete candidate values defined in a continuous universe, and the candidate values can even be a mix of intervals and discrete numbers.  $g_1^\circ$  and  $g_3^\circ$  defined in Definition 2 fail to represent these situations by violating the last three properties in Definition 7. Following Equation (1), a discrete set of candidate values defined on a continuous domain will lead to  $g_1^\circ(g^\pm) = 0$  when the relevant grey number is not a white number. For a continuous grey number,  $g_1^\circ(g^\pm) < 1$  if the relevant grey number is not a black number. At the same time, for discrete candidate values,  $g_1^\circ$  does not consider the candidate probability at all. Therefore,  $g_1^\circ$  violates the last three properties in Definition 7.  $g_3^\circ$  has similar problems. To overcome the limitation of  $g_1^\circ$  and  $g_3^\circ$  in their relative uncertainty representation, we will define a new measurement for the relative uncertainty in a grey number.

*Definition 8 (Relative uncertainty of a grey number):* Let  $g^\pm \in \bigcup_{i=1}^n [a_i^-, a_i^+]$  be a grey number, and  $i = 1, 2, \dots, n$ ,  $n$  is an integer and  $0 < n < \infty$ ,  $a_i^-, a_i^+ \in \mathfrak{R}$  and  $a_{i-1}^+ < a_i^- \leq a_i^+ < a_{i+1}^-$ . For any interval  $[a_i^-, a_i^+] \subseteq \bigcup_{i=1}^n [a_i^-, a_i^+]$ ,  $p_i$  is the probability for  $g^\pm \in [a_i^-, a_i^+]$  and  $|d_i| = |a_i^+ - a_i^-|$ . The relative uncertainty  $\delta(g^\pm)$  of  $g^\pm$  is defined as:

$$\delta(g^\pm) = 1 - \lim_{\epsilon \rightarrow 0} \frac{|D| + \epsilon - \sum_{i=1}^n |d_i|}{|D| + \epsilon} \max\{p_1, p_2, \dots, p_n\}$$

Here,  $|D| = |g^+ - g^-|$ ,  $g^- = \inf a_i^-$ ,  $g^+ = \sup a_i^+$  and  $\epsilon > 0$ .

Different from the absolute uncertainty, the relative uncertainty  $\delta(g^\pm)$  for a grey number  $g^\pm$  measures its uncertainty in selecting a candidate from its defined set of candidates only. Obviously, the less candidates we have, the lower the relative uncertainty should be. It is also relevant to the probability distribution of the candidate intervals. A higher probability in one candidate will reduce the other candidate's chance to be selected, and hence decreases the relative uncertainty.

As suggested by its name, a relative uncertainty is meaningful only in the sense of relative measurement. For a grey number with fixed domain, its absolute uncertainty is meaningful for the whole domain, but its relative uncertainty is only meaningful for the specific diameter (distance between its upper and lower limits) of a grey number. The relative uncertainty of a grey number indicates the uncertainty in selecting a value from the specific candidate set of the grey number. Therefore, its value reflects the nature of the candidate set of a grey number, as demonstrated by Theorem 1.

*Theorem 1:* The relative uncertainty  $\delta$  of a grey number  $g^\pm$  has the following properties.

- $0 \leq \delta \leq 1$ ;
- $\delta = 0$  iff  $g^\pm$  is a white number;
- $\delta = 1$  iff  $g^\pm$  is a continuous grey number.
- $\delta \neg \max\{p_1, p_2, \dots, p_n\}$ ;
- $\delta = 1 - \max\{p_1, p_2, \dots, p_n\}$  iff  $g^\pm$  is a discrete grey number;
- $\delta = 1 - \frac{1}{n}$ , if  $g^\pm$  is a discrete grey number with  $n$  candidates and all have the same significance;

*Proof:* The first item can be easily derived from definition 8. From Definition 8, we have

$$\delta = 1 - \lim_{\epsilon \rightarrow 0} \frac{|D| + \epsilon - \sum_{i=1}^n |d_i|}{|D| + \epsilon} \max\{p_1, p_2, \dots, p_n\}$$

if  $g^\pm$  is a white number, we have

$$|D| = \sum_{i=1}^n |d_i| = 0$$

and

$$\max\{p_1, p_2, \dots, p_n\} = 1$$

Then, we have

$$\delta = 1 - \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon} = 0$$

If  $\delta = 0$ , we have

$$\lim_{\epsilon \rightarrow 0} \frac{|D| + \epsilon - \sum_{i=1}^n |d_i|}{|D| + \epsilon} \max\{p_1, p_2, \dots, p_n\} = 1$$

Thus

$$\sum_{i=1}^n |d_i| = 0 \text{ and } \max\{p_1, p_2, \dots, p_n\} = 1$$

$\sum_{i=1}^n |d_i| = 0$  means only discrete candidates are available, and  $\max\{p_1, p_2, \dots, p_n\} = 1$  indicates the maximum probability is 1. Therefore, only 1 discrete candidate exists, so it is a white number.

If  $g^\pm$  is a continuous grey number, we have

$$\sum_{i=1}^n |d_i| = |D|$$

Thus, we have

$$\delta = 1 - \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{|D| + \epsilon} \max\{p_1, p_2, \dots, p_n\} = 1$$

If  $\delta = 1$ , we have

$$\lim_{\epsilon \rightarrow 0} \frac{|D| + \epsilon - \sum_{i=1}^n |d_i|}{|D| + \epsilon} \max\{p_1, p_2, \dots, p_n\} = 0$$

For a grey number, we know  $\max\{p_1, p_2, \dots, p_n\} > 0$ , so we have

$$\sum_{i=1}^n |d_i| = |D|$$

From Definition 1, it is clear that  $g^\pm$  is a continuous grey number.

Let  $\max\{p_1, p_2, \dots, p_n\}$  increase to  $\max\{p_1, p_2, \dots, p_n\} + \Delta$ , where  $0 < \Delta \leq 1 - \max\{p_1, p_2, \dots, p_n\}$

According to Definition 8, we have

$$\delta_1 = 1 - \lim_{\epsilon \rightarrow 0} \frac{|D| + \epsilon - \sum_{i=1}^n |d_i|}{|D| + \epsilon} (\max\{p_1, p_2, \dots, p_n\} + \Delta)$$

$$\delta_1 = \delta - \lim_{\epsilon \rightarrow 0} \frac{|D| + \epsilon - \sum_{i=1}^n |d_i|}{|D| + \epsilon} \Delta$$

Obviously, we have

$$\delta_1 < \delta$$

For a discrete grey number  $g^\pm$ , we have

$$|D| = \sum_{i=1}^n |d_i| = 0$$

Thus

$$\begin{aligned} \delta &= 1 - \lim_{\epsilon \rightarrow 0} \frac{|D| + \epsilon}{|D| + \epsilon} \max\{p_1, p_2, \dots, p_n\} \\ &= 1 - \max\{p_1, p_2, \dots, p_n\} \end{aligned}$$

From this conclusion, it is obvious that  $\delta = 1 - \frac{1}{n}$  if  $g^\pm$  is a discrete grey number with  $n$  candidates and all have the same significance. ■

The first four properties in Theorem 1 are the exact properties required in Definition 7. This theorem proves that the proposed relative uncertainty  $\delta$  satisfies the required properties in Definition 7.

*Example 2:*  $a \in [0.2, 0.6]$ ,  $b \in \{0.2, 0.6\}$  and  $c \in \{0.2, 0.4, 0.6\}$  are three grey numbers defined on  $[0, 1]$ . Their corresponding absolute uncertainties are

$$g_2^\circ(a) = 0.4, g_2^\circ(b) = 0.4, g_2^\circ(c) = 0.4$$

With Equation (1) and (3), we have

$$g_1^\circ(a) = 0.4, g_1^\circ(b) = 0, g_1^\circ(c) = 0$$

$$g_3^\circ(a) = 0.4, g_3^\circ(b) = 0, g_3^\circ(c) = 0$$

Obviously, there is no difference in their absolute uncertainty. If we consider  $g_1^\circ$  and  $g_3^\circ$  as their relative uncertainties, they do not satisfy the last 3 properties required in Definition 7.  $b$  and  $c$  are obviously not white numbers, but we have  $g_1^\circ(b) = 0$ ,  $g_1^\circ(c) = 0$ ,  $g_3^\circ(b) = 0$  and  $g_3^\circ(c) = 0$ .  $a$  is a continuous grey number, but we got  $g_1^\circ(a) = 0.4 < 1$  and  $g_3^\circ(a) = 0.4 < 1$ . The number of candidates are completely different in  $a$ ,  $b$  and  $c$ : there are infinite candidates in  $a$ , but only two candidates in  $b$  and three candidates in  $c$ . The probability for each candidate to be the represented number has a significant difference between  $b$  and  $c$ , but both  $g_1^\circ$  and  $g_3^\circ$  failed to reveal it.

For  $a$ , there is only one interval which covers all possible candidates for the represented number, we have

$$\max\{p_1, p_2, \dots, p_n\} = 1 \quad \text{for } a$$

For  $b$ , there are two discrete numbers as candidates, and we do not have more information on their probability distribution. According to the Jayness Maximum Entropy Principle [11], we can take uniform distribution in this case. Thus, we have

$$\begin{aligned} \max\{p_1, p_2, \dots, p_n\} &= 0.5 \quad \text{for } b \\ \max\{p_1, p_2, \dots, p_n\} &= 0.33 \quad \text{for } c \end{aligned}$$

Therefore, we have

$$\delta(a) = 1, \delta(b) = 0.5, \delta(c) = 0.67$$

For the proposed measure, there is no violation to the required properties in Definition 7. Although they share the same absolute uncertainty, their relative uncertainty is completely different.  $c$  and  $b$  have much better certainty than  $a$  in terms of their relative uncertainty, and  $c$  is more uncertain than  $b$ . This example shows that the measure can capture this type of uncertainty.

As we can see that the relative uncertainty  $\delta$  of  $g^\pm$  is different from its absolute uncertainty  $g_2^\circ$ , and they are complementary to each other. It demonstrates the difference between a grey number and an interval. Firstly, an interval does not necessarily represent a single value, but a grey number does; secondly, when an interval is interpreted as a representation of a single unknown number, it is still only a special case of a grey number where its relative uncertainty is 1.

Similar to the degree of greyness (absolute uncertainty), we can also define the relative uncertainty of an element of a grey set and the relative uncertainty of a grey set.

*Definition 9 (Relative uncertainty of an element in a grey set):* Let  $U$  be the finite universe of discourse,  $x$  be an element and  $x \in U$ . For a grey set  $A \subseteq U$ , the characteristic function value of  $x$  with respect to  $A$  is represented as a grey number  $g_A^\pm(x) \in D[0, 1]^\pm$ . The relative uncertainty  $\delta_A(x)$  of element  $x$  for set  $A$  is expressed as

$$\delta_A(x) = \delta(g_A^\pm(x))$$

With the absolute uncertainty representation in Equation (2), the degree of greyness of an element measures the difference between its upper and lower limits of its characteristic function value, but it does not tell how uncertain it is to get a value from such a presentation. Although the representations in Equation (1) and (3) do reveal some information on selection, they are not comprehensive enough to consider both continuous and discrete components at the same time. The relative uncertainty of an element fills this gap and provides a measure to the uncertainty to get a value between the

upper and lower limits applicable both to continuous and discrete components. In a similar way, we can measure the relative uncertainty of a grey set.

*Definition 10 (Relative uncertainty of a grey set):* Let  $U$  be the finite universe of discourse,  $A$  be a grey set and  $A \subseteq U$ .  $x_i$  is an element relevant to  $A$  and  $x_i \in U$ .  $i = 1, 2, 3, \dots, N$  and  $N$  is the cardinality of  $U$ . The relative uncertainty of set  $A$  is defined as

$$\delta(A) = \frac{\sum_{i=1}^N \delta_A(x_i)}{N}$$

As a measure of the uncertainty of a grey set, the relative uncertainty of a grey set reflects the degree of uncertainty when we whitenise a grey set based on the grey values of their characteristic functions. The lower the relative uncertainty is, the better chance we have in choosing a right candidate. Therefore, the uncertainty of a grey set does not relate only with its absolute uncertainty, but also to its relative uncertainty as well. For the same absolute uncertainty, the grey set with a lower relative uncertainty has less uncertainty. The relative uncertainty reveals some uncertainty which is not captured by the absolute uncertainty. Although two grey sets share the same absolute uncertainty when their upper and lower limits are identical, their relative uncertainties may not be the same.

*Example 3:* For a given universe  $U = \{x, y, z\}$ , there are two grey sets:

$$\begin{aligned} A &= \frac{[0.1, 0.4]}{x} + \frac{[0.4, 0.7]}{y} + \frac{[0.7, 1]}{z} \\ B &= \frac{\{0.1, 0.4\}}{x} + \frac{\{0.4, 0.7\}}{y} + \frac{\{0.7, 1\}}{z} \end{aligned}$$

Obviously,  $A$  is a grey set with continuous grey numbers (which can also be considered as an interval-valued fuzzy set), and  $B$  is a grey set with discrete grey numbers. We can calculate their absolute uncertainty

$$g_2^\circ(A) = 0.3, g_2^\circ(B) = 0.3$$

Clearly, the two grey sets share the same absolute uncertainty in this specific case. However, although they share similar boundaries of their grey numbers in their characteristic functions, the number of candidates in their grey numbers are completely different:  $A$  has infinite candidates but  $B$  has only 2 candidates. This difference can be revealed by their relative uncertainties. Similar to Example 2, we take average probability distribution when we do not have more information. According to Definition 8 and 10, we have

$$\begin{aligned} \delta_A(x) = 1, \delta_A(y) = 1, \delta_A(z) &= 1 \\ \delta_B(x) = 0.5, \delta_B(y) = 0.5, \delta_B(z) &= 0.5 \end{aligned}$$

Following Definition 10, we have

$$\delta(A) = 1, \delta(B) = 0.5$$

We can also calculate  $g_1^\circ$  and  $g_3^\circ$

$$g_1^\circ(A) = 0.3, g_1^\circ(B) = 0, g_3^\circ(A) = 0.3, g_3^\circ(B) = 0$$

$A$  is a set represented by continuous grey numbers and  $B$  is a set using discrete grey numbers. According to Definition 7, we expect  $g_1^\circ(A) = 1$  and  $g_1^\circ(B) > 0$ . Which is clearly not the case. Therefore, the proposed relative uncertainty provides a better solution than  $g_1^\circ$  and  $g_3^\circ$ .

Obviously, although the absolute uncertainty cannot differentiate  $A$  from  $B$  in this case, their relative uncertainties disclose their difference clearly.  $A$  is more uncertain than  $B$  but  $B$  is not completely white in terms of relative uncertainty. In fact,  $A$  can be

considered as an interval-valued fuzzy set as well, so this example does demonstrate the difference between interval-valued fuzzy sets and grey sets in general. The following theorem demonstrates this crucial difference between a grey set and an interval-valued fuzzy set.

*Theorem 2:* The relative uncertainty of a grey set has the following properties.

- $0 \leq \delta_A \leq 1$ ;
- $\delta_A = 0$  iff  $A$  is a white set;
- $\delta_A = 1$  iff  $A$  is an interval-valued fuzzy set;

The proof to this theorem is obvious from the Definition 9, 10 and Theorem 1.

Theorem 2 illustrates the relationship between grey sets, white sets and interval-valued fuzzy sets from the point of view of relative uncertainty. The relative uncertainty of a white set is always 0, and the relative uncertainty of an interval-valued fuzzy set is always 1. For a grey set, its relative uncertainty can be any value between 0 and 1. Obviously, a grey set covers more situations than an interval valued fuzzy set. At the same time, there are also situations where interval-valued fuzzy sets represent completely different situations than a grey set. As a set representation, an interval can also be interpreted as a collection of multiple values where each value is valid and inclusive rather than exclusive. In this sense, there are situations where a grey set does not cover an interval-valued fuzzy set. The comparison between interval-valued fuzzy sets and grey sets are presented in Table I.

#### IV. UNCERTAINTY COMPARISON BETWEEN GREY SETS

As illustrated in Definition 7, absolute uncertainty itself is not sufficient to express the uncertainty of a grey number, and the relative uncertainty is a useful supplement. However, the addition of an extra measurement of uncertainty may cause confusion in their correct usage. It is necessary to know when and where to use each of them. Different from the absolute uncertainty of a grey number, the relative uncertainty is closely associated with the specific diameter of a grey number. Therefore, an uncertainty measurement of grey numbers taking both of them into account is hugely beneficial to their applications. To highlight the necessity to consider both absolute and relative uncertainties together in the uncertainty comparison between grey sets, we define the two together as a pair in uncertainty measurement.

*Definition 11 (Uncertainty pair of a grey number):* Let  $g^\pm$  be a grey number,  $g_2^\circ(g^\pm)$  be its absolute uncertainty and  $\delta(g^\pm)$  be its relative uncertainty. The uncertainty pair of  $g^\pm$  can be expressed as

$$u(g^\pm) = (g_2^\circ(g^\pm), \delta(g^\pm))$$

With the uncertainty pair  $u(g^\pm)$ , the absolute uncertainty and relative uncertainty are both included in the representation. The absolute uncertainty  $g_2^\circ(g^\pm)$  gives a measure on the scale of difference between candidate values, which is an absolute value when the universe is fixed. A greater  $g_2^\circ(g^\pm)$  indicates a significant difference between candidate values, which implies a high uncertainty in the result. The relative uncertainty  $\delta(g^\pm)$ , reveals the relative uncertainty for a given grey number. A greater  $\delta(g^\pm)$  means more candidates involved which demonstrates less certainty for one candidate to be selected. The pair  $(g_2^\circ(g^\pm), \delta(g^\pm))$  catches both uncertainties in scale and selection and gives a better representation of uncertainties in a grey number.

The uncertainty pair employs both the absolute uncertainty and relative uncertainty of a grey number to represent uncertainties. This raises the question of how to compare uncertainties between two grey numbers. It is therefore necessary to construct a suitable comparison between two uncertainty pairs. As aforementioned, the absolute uncertainty of a grey number is an absolute value for a given universe, but its relative uncertainty is always associated with the specific grey number and it is a relative measurement. In this sense, the absolute uncertainty is directly comparable between different grey numbers defined on the same universe, but their relative uncertainty can only be compared when they share the same absolute uncertainty. We now define the comparison between two uncertainty pairs as Definition 12.

*Definition 12 (Comparison of uncertainty pairs):* Let  $a$  and  $b$  be two grey numbers defined on the same universe  $\Omega$ ,  $u(a) = (g_2^\circ(a), \delta(a))$  and  $u(b) = (g_2^\circ(b), \delta(b))$  be their corresponding uncertainty pairs. The comparison of  $u(a)$  and  $u(b)$  satisfies the following items

- if  $g_2^\circ(a) < g_2^\circ(b)$ , then  $u(a) < u(b)$ ;
- if  $g_2^\circ(a) = g_2^\circ(b)$  and  $\delta(a) < \delta(b)$ , then  $u(a) < u(b)$ ;
- if  $g_2^\circ(a) = g_2^\circ(b)$  and  $\delta(a) = \delta(b)$ , then  $u(a) = u(b)$ ;

In this way, we can compare the uncertainty pairs of different grey numbers. We give some simple examples here.

*Example 4:*  $a \in [0.2, 0.6]$ ,  $b \in \{0.2, 0.4, 0.6\}$ ,  $c \in \{0.2, 0.6\}$ ,  $d \in \{0.2, 0.8\}$ ,  $e \in \{0.1, 0.7\}$  are grey numbers defined on  $[0, 1]$ .

We can calculate their absolute uncertainty, relative uncertainty and uncertainty pairs as follows

$$g_2^\circ(a) = 0.4, g_2^\circ(b) = 0.4, g_2^\circ(c) = 0.4, g_2^\circ(d) = 0.6, g_2^\circ(e) = 0.6$$

$$\delta(a) = 1, \delta(b) = 0.67, \delta(c) = 0.5, \delta(d) = 0.5, \delta(e) = 0.5$$

Thus, we have

$$u(a) = (0.4, 1), u(b) = (0.4, 0.67), u(c) = (0.4, 0.5)$$

$$u(d) = (0.6, 0.5), u(e) = (0.6, 0.5)$$

According to Definition 12, we have

$$u(c) < u(b) < u(a) < u(d) = u(e)$$

When discrete candidates are involved,  $g_1^\circ$ ,  $g_2^\circ$  and  $g_3^\circ$  defined in Definition 2 can only give partial information on the uncertainty of a grey number as aforementioned. They cannot satisfy the required properties for relative uncertainty in Definition 7. However, the uncertainty pair has captured both absolute uncertainty and relative uncertainty and provides a reliable representation. Obviously, two different grey numbers can have the same uncertainty, and they share the same uncertainty only when they have the same absolute uncertainty and the same relative uncertainty. It illustrates that a set is not recoverable from its uncertainty. In uncertainty comparison, the absolute uncertainty is the dominant factor, and the relative uncertainty plays a role only when they share the same absolute uncertainty. In example 4,  $u(a)$  and  $u(b)$  have greater relative uncertainty values than  $u(d)$ , but their absolute uncertainty are less than  $u(d)$ , so they are still lower than  $u(d)$  in terms of their uncertainty pairs.

Similarly, we can define the uncertainty pairs for a grey set and its element.

*Definition 13 (Uncertainty pair of an element in a grey set):* Let  $U$  be the finite universe of discourse,  $x$  be an element and  $x \in U$ . For a grey set  $A \subseteq U$ , the characteristic function value of  $x$  with

TABLE I  
COMPARISON BETWEEN INTERVAL-VALUED FUZZY SETS AND GREY SETS

Item	Interval-valued fuzzy sets	Grey sets
Characteristic function values	Intervals within [0,1]	Any sets of intervals within [0,1]
Discrete set of numbers	No	Yes
Interpretation	Multiple or single	Single
Distance between two limits	Hesitation	Absolute uncertainty
Relative uncertainty	1	A value between 0 and 1

respect to  $A$  is represented as a grey number  $g_A^\pm(x) \in D[0, 1]^\pm$ . Its absolute uncertainty is  $g_A^\circ(x)$ , and its relative uncertainty is  $\delta_A(x)$ . The uncertainty pair of the element  $x$  for set  $A$  is expressed as

$$u(x) = (g_A^\circ(x), \delta_A(x))$$

We can define the uncertainty pair of a grey set as well.

*Definition 14 (Uncertainty pair of a grey set):* Let  $A$  be a grey set defined in the finite universe of discourse  $U$ ,  $g^\circ(A)$  be its absolute uncertainty, and  $\delta(A)$  be its relative uncertainty. The uncertainty pair of set  $A$  is defined as

$$u_A = (g^\circ(A), \delta(A))$$

With the defined uncertainty pairs, we can represent both the absolute uncertainty in scale and the relative uncertainty in selection. This ability is necessary especially when the absolute uncertainty itself cannot distinguish one set from another.

*Example 5:* Two travellers are planning their trips from city  $C_1$  to city  $C_2$ . They have to arrive before midday to  $C_2$ . There are five trains for  $C_2$  from  $C_1$  in the morning, their departure time slots are: 7:00am, 8:00am, 9:00am, 10:00am and 11:00am. Other trains will arrive late. In addition to travelling by train, they may also drive to  $C_2$  by car, in which case the departure time can be any time between 8:15am and 8:45am. The travellers have three options in their plan: trains, cars or any vehicle. The two travellers are travelling independently. They may contact their host in  $C_2$  to inform them of their decision. However, if they do not contact their host in advance, we have to assume that they can take any option. Due to the limited time slot for car departure (less than one hour), we assume the probability for them to take a car is identical to the probability for them to take one of the trains. We need to consider the uncertainty of these options to make a choice.

For the three available options, we can represent it with three sets:

$$\begin{aligned} &\{7 : 00am, 8 : 00am, 9 : 00am, 10 : 00am, 11 : 00am\} \\ &\quad \{8 : 15am, 8 : 45am\} \\ &\{7 : 00am, 8 : 00am, [8 : 15am, 8 : 45am], 9 : 00am, \\ &\quad 10 : 00am, 11 : 00am\} \end{aligned}$$

Here, the domain is certainly the whole time slots:

$$[7 : 00am, 11 : 00am]$$

Mapping these time slots into  $[0, 1]$ , we have the following grey numbers:

$$\begin{aligned} a &\in \{0, 0.25, 0.5, 0.75, 1\} \\ b &\in [0.3125, 0.4375] \end{aligned}$$

$$c \in \{0, 0.25, [0.3125, 0.4375], 0.5, 0.75, 1\}$$

and their value domain  $D_v$ :

$$D_v = [0, 1]$$

We can calculate their absolute uncertainty and relative uncertainty:

$$g_2^\circ(a) = 1, g_2^\circ(b) = 0.125, g_2^\circ(c) = 1$$

$$\delta(a) = 0.8, \delta(b) = 1, \delta(c) = 0.85$$

Let  $x$  and  $y$  be the two travellers, and the universe  $U = \{x, y\}$ . We can establish the grey sets for all possible travel patterns:

$$\begin{aligned} A &= \{\text{train for } x, \text{ train for } y\} = \frac{a}{x} + \frac{a}{y} \\ &= \frac{\{0, 0.25, 0.5, 0.75, 1\}}{x} + \frac{\{0, 0.25, 0.5, 0.75, 1\}}{y} \end{aligned}$$

$$\begin{aligned} B &= \{\text{train for } x, \text{ car for } y\} = \frac{a}{x} + \frac{b}{y} \\ &= \frac{\{0, 0.25, 0.5, 0.75, 1\}}{x} + \frac{[0.3125, 0.4375]}{y} \end{aligned}$$

$$\begin{aligned} C &= \{\text{train for } x, \text{ any for } y\} = \frac{a}{x} + \frac{c}{y} \\ &= \frac{\{0, 0.25, 0.5, 0.75, 1\}}{x} \\ &\quad + \frac{\{0, 0.25, [0.3125, 0.4375], 0.5, 0.75, 1\}}{y} \end{aligned}$$

$$\begin{aligned} D &= \{\text{car for } x, \text{ train for } y\} = \frac{b}{x} + \frac{a}{y} \\ &= \frac{[0.3125, 0.4375]}{x} + \frac{\{0, 0.25, 0.5, 0.75, 1\}}{y} \end{aligned}$$

$$\begin{aligned} E &= \{\text{car for } x, \text{ car for } y\} = \frac{b}{x} + \frac{b}{y} \\ &= \frac{[0.3125, 0.4375]}{x} + \frac{[0.3125, 0.4375]}{y} \end{aligned}$$

$$\begin{aligned} F &= \{\text{car for } x, \text{ any for } y\} = \frac{b}{x} + \frac{c}{y} \\ &= \frac{[0.3125, 0.4375]}{x} \\ &\quad + \frac{\{0, 0.25, [0.3125, 0.4375], 0.5, 0.75, 1\}}{y} \end{aligned}$$

$$\begin{aligned} G &= \{\text{any for } x, \text{ train for } y\} = \frac{c}{x} + \frac{a}{y} \\ &= \frac{\{0, 0.25, [0.3125, 0.4375], 0.5, 0.75, 1\}}{x} \\ &\quad + \frac{\{0, 0.25, 0.5, 0.75, 1\}}{y} \end{aligned}$$

$$\begin{aligned} H &= \{\text{any for } x, \text{ car for } y\} = \frac{c}{x} + \frac{b}{y} \\ &= \frac{\{0, 0.25, [0.3125, 0.4375], 0.5, 0.75, 1\}}{x} \\ &\quad + \frac{[0.3125, 0.4375]}{y} \end{aligned}$$

$$\begin{aligned}
& + \frac{[0.3125, 0.4375]}{y} \\
I & = \{\text{any for } x, \text{ any for } y\} = \frac{c}{x} + \frac{c}{y} \\
& = \frac{\{0, 0.25, [0.3125, 0.4375], 0.5, 0.75, 1\}}{x} \\
& + \frac{\{0, 0.25, [0.3125, 0.4375], 0.5, 0.75, 1\}}{y}
\end{aligned}$$

According to Definition 4, 5, 9 and 10, we have

$$\begin{aligned}
g^\circ(A) & = 1, \delta(A) = 0.8; g^\circ(B) = 0.5625, \delta(B) = 0.9 \\
g^\circ(C) & = 1, \delta(C) = 0.825; g^\circ(D) = 0.5625, \delta(D) = 0.9 \\
g^\circ(E) & = 0.125, \delta(E) = 1; g^\circ(F) = 0.5625, \delta(F) = 0.925 \\
g^\circ(G) & = 1, \delta(G) = 0.825; g^\circ(H) = 0.5625, \delta(H) = 0.925 \\
g^\circ(I) & = 1, \delta(I) = 0.85
\end{aligned}$$

From Definition 13 and 14, we have

$$\begin{aligned}
u(A) & = (1, 0.8); u(B) = (0.5625, 0.9) \\
u(C) & = (1, 0.825); u(D) = (0.5625, 0.9) \\
u(E) & = (0.125, 1); u(F) = (0.5625, 0.925) \\
u(G) & = (1, 0.825); u(H) = (0.5625, 0.925) \\
u(I) & = (1, 0.85)
\end{aligned}$$

According to Definition 12, we have

$$\begin{aligned}
u(E) & < u(B) = u(D) < u(H) = u(F) \\
u(F) & < u(A) < u(C) = u(G) < u(I)
\end{aligned}$$

The existing degree of greyness  $g_1^\circ$ ,  $g_2^\circ$  and  $g_3^\circ$  can also partially reveal uncertainty.

Replace the absolute uncertainty in grey sets  $A, B, C, D, E, F, G, H$  and  $I$  with  $g_1^\circ$ , we have

$$\begin{aligned}
g_1^\circ(A) & < g_1^\circ(E) = g_1^\circ(F) = g_1^\circ(H) = g_1^\circ(I) \\
g_1^\circ(I) & < g_1^\circ(B) = g_1^\circ(C) = g_1^\circ(D) = g_1^\circ(G)
\end{aligned}$$

According to the values of  $g_2^\circ$ , we have

$$\begin{aligned}
g_2^\circ(E) & < g_2^\circ(F) = g_2^\circ(B) = g_2^\circ(D) = g_2^\circ(H) \\
g_2^\circ(H) & < g_2^\circ(I) = g_2^\circ(G) = g_2^\circ(C) = g_2^\circ(A)
\end{aligned}$$

Replace the absolute uncertainty in grey sets  $A, B, C, D, E, F, G, H$  and  $I$  with  $g_3^\circ$ , we have

$$\begin{aligned}
g_3^\circ(A) & < g_3^\circ(B) = g_3^\circ(D) < g_3^\circ(E) < g_3^\circ(C) \\
g_3^\circ(C) & = g_3^\circ(G) < g_3^\circ(F) = g_3^\circ(H) < g_3^\circ(I)
\end{aligned}$$

Due to the mix of grey numbers with discrete and continuous candidates,  $g_1^\circ$  and  $g_3^\circ$  cannot satisfy the properties in Definition 7.  $g_2^\circ$  cannot differentiate those options with the same absolute uncertainty. Both  $g_1^\circ$  and  $g_3^\circ$  conclude that train travel option  $A$  is the most certain option, and they give very different order for other sets.  $g_1^\circ$  cannot separate most sets.  $g_3^\circ$  suggests that the combination of car and any vehicle in  $H$  and  $F$  is worse than the combination of train and any vehicle in  $C$  and  $G$ . From the context, car travel will certainly arrive in a narrow time slot and the conclusions from  $g_3^\circ$  is obviously not ideal for this order. For the option with the least uncertainty from the point of view of arrival time, car travel will certainly give less uncertainty in arrival time, so the conclusion on the best option  $A$  from both  $g_1^\circ$  and  $g_3^\circ$  is also not acceptable in this

case. Based on the result from the proposed uncertainty pairs, it is clear that the car option  $E$  has the lowest uncertainty in comparison with other possible combinations. The worst case comes as grey set  $I$  when both travellers can take any choice. From the absolute uncertainty, the grey set  $A, C, G$  and  $I$  have the same value, and the grey set  $B, D, F$  and  $H$  share the same value, but the uncertainty pairs can easily identify their difference. It reveals that  $H$  and  $F$  have less uncertainty than  $C$  and  $G$ . This example demonstrates the effectiveness of the uncertainty pairs.

## V. CONCLUSIONS

Different from interval-valued fuzzy sets, a grey set may employ discrete grey numbers as their characteristic function values. Under the same absolute uncertainty, the discrete grey numbers may have different number of candidate values. The present degree of greyness can only reveal absolute uncertainty and partial relative uncertainty. In this article we defined a new relative uncertainty of a grey number and a grey set to give a complete picture of its relative uncertainty. This proposed relative uncertainty has the ability to reveal the relative uncertainty in the presence of both continuous intervals and discrete numbers in the candidate set of a grey number. To reduce confusion with the two different uncertainty measurements (absolute uncertainty and relative uncertainty), the concept of an uncertainty pair of a grey number and a grey set has also been defined. An uncertainty pair employs both the absolute uncertainty and relative uncertainty of a grey number and a grey set to represent their uncertainties. To compare uncertainties between different grey numbers or grey sets, we constructed the rules of comparison according to their different roles and significance in the comparison. A number of simple examples demonstrated the effectiveness of the proposed models. Our work illustrated that neither the absolute uncertainty nor the relative uncertainty alone can effectively capture the uncertainty in a grey number and a grey set, and the combination of both of them is necessary. Furthermore, the relative uncertainty does reveal an important difference between a grey set and an interval-valued fuzzy set as well. Our analysis proved that grey sets and interval-valued fuzzy sets provide different but overlapping models for uncertainty representation in sets.

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