An Example

- Map coloring
  - A region in a map has several sub-regions
  - Given a set of colors
  - Assign each sub-region a color so that no adjacent sub-regions have the same color
An Example

- Map coloring
  - Can we use 3 colors to color the map below (Australia)?
  - How about 100 sub-regions?
An Example

- Map coloring
  - The above map can be modelled as a graph

- Node: sub-region
- Edge: constraint (adjacency)
- Problem modelled as graph coloring: adjacent nodes have different colors
A constraint satisfaction problem (CSP) consists of

- a set of *variables* \( \{x_1, x_2, \ldots, x_i\} \);
- a finite set of *domain* \( D \) (possible values) associated with each variable;
- a set of *constraints* \( C \) restricting the values that the variables can simultaneously take

CSP is to assign values to variables so that all constraints are satisfied.
The domain of a variable is a set of possible values that can be assigned to the variable.

- Finite domain vs. infinite domain
  - {1 \ldots 500}
  - {red, green, blue}
  - {yes, no}
  - Temperatures

- In this module we consider the CSPs with finite domains.
CSP - Domain

The domain of a variable is a set of possible values that can be assigned to the variable

- Discrete vs. continuous
  - \{1 \ldots 500\}
  - ...

- Temperatures
  - ...

CSP - Domain

- The domain of a variable is a set of possible values that can be assigned to the variable.

- Different types
  - Boolean: \{yes, no\}
  - Symbols: \{red, green, blue\}
  - Reals: Time of days, Temperatures
  - Integers: 1, 2, ... 500
CSP - Constraints

- Properties of objects and relations between them
  - Formulated as predicates in logic
  - A constraint on a set of variables
    - Restriction on the values the variables can simultaneously take
    - What values are (dis-)allowed among the variables
CSP - Constraints

Examples of constraints

- The demand will be more than five thousands units in August
- Optimise the benefit of products
- John prefers to work at only weekends
- Schedule these employees to cover all shifts
CSP - Constraints

- Constraint is a set of relations upon domains of variables
  - Functions
  - Matrices
  - Inequalities
    - $x_1 \neq x_2$ or $x_1 < x_2$
    - $3x + 6y = 0$, $y + x < 7$
    - alldifferent(x,y,z)
CSP - Constraints

Formal definition

A constraint $C_{i,j,k...}$ is a subset of possible relations between values of variables $x_i, x_j, x_k, ...$; $C_{i,j,k...}$ is a subset of $D_i, D_j, D_k, ...$
CSP - Constraints

Types of constraints

- Unary: affects only one variable
  - $x \geq 1, \text{WA} = \text{green}$

- Binary: affects two variables
  - $x \geq y, \text{WA} \neq \text{NT}$

- High-order: affects more than two variables
  - \text{alldifferent} (WA, NT, Q)
  - $x = y = z$
CSP - Constraints

- How the values of one variable restrict those of the others
  - $3x + 6y = 0$
  - $x \in \{0, 1, 2, 3\}, y \in \{-2, -1, 0\}$
  - $x > y$

$x = 2, y = -1$
$x = 0, y = 0$?
CSP - Constraints

- Redundant constraints
  - One constraint $C_1$ implies another constraint $C_2$, $C_1 \rightarrow C_2$
  - Solutions of $C_1$ are a subset of the solutions of $C_2$
    - $x < 3 \& x < 4$
  - $C_1$ and $C_1 \land C_2$ are equivalent

- Equivalent constraints
  - Two constraints $C_1$ and $C_2$ are equivalent if they have the same set of solutions
CSP - Constraints

- Simplification of constraints
  - Replace a constraint by an equivalent constraint which has a simpler form
  - Problem is easier to understand
  - Information of original form is more apparent
CSP - Solutions

- A solution to a CSP is an assignment of each variable with a value (within its domain), such that all constraints C are satisfied

  - if a CSP has a *feasible* solution
  - find one (any) solution
  - all solutions
Often we not only want a CSP solution but also the best one (optimal solution)

- Constraint Optimisation Problems (COP)
- Objective function: to measure how good a solution is
CSP – Other Definitions

- **Label**
  - a variable-value pair representing assigning the value to the variable

- **Compound label**
  - the finite set of labels representing simultaneous assignments
CSP – Other Definitions

- **Projection** \((N, M)\)
  - \(m\) and \(n\) are integers, \(m \leq n\)
  - \(n\)-compound label \(N\)
  - \(m\)-compound label \(M\)
  - \(M\) is a projection of \(N\) if labels in \(M\) all appear in \(N\)

- \((<a,1><c,3>)\) is a projection of \((<a,1><b,2><c,3>)\)
- Projection \(((<a,1><b,2><c,3>), (<a,1><c,3>))\) is true
CSP – Other Definitions

- Satisfiable
  - A CSP \((V, D, C)\) is satisfiable iff there is a compound label assigning values to all variables \(V\) that satisfy all constraints \(C\)

- A solution can then be defined as
  - A compound label for all variables which satisfy all the constraints
CSP – Other Definitions

- Binary CSP
  - With only unary and binary constraints

- Other definitions may be introduced in the context of the module
CSP – Example I

- Variables
  - \(\{WA, NT, Q, SA, NSW, V, T\}\)

- Domains
  - \{red, green, blue\}

- Constraints
  - \{not_same(WA, NT), not_same(WA, SA), not_same(NT, SA), \ldots\}\
The above formulate the map coloring problem into a CSP

Solution

\{ \text{WA}=\text{red}, \text{NT}=\text{green}, \text{SA}=\text{blue}, \text{Q}=\text{red}, \text{NSW}=\text{green}, \text{V}=\text{red}, \text{T}=\text{red} \}
CSP – Example I

- $WA = \text{red is a label}$
- The problem is satisfiable.
The above solution using 3 colors
- Is actually an optimal solution (using the least colors)

There are more than one solution
- For example, given 7 colors
- We’ll have 7! solutions for the problem
Graph coloring is NP-Hard (Karp, 1972)
- Optimal solutions of least colors cannot be obtained for 90-vertex graph (Johnson, 1991)
- Constraint based techniques are the mostly studied early approaches
CSP – Example II

- Scene labeling problem
  - The first CSP studied
  - In vision, scene are captured as images, and processed as lines to represent images
  - Lines need to be interpreted into types
CSP – Example II

- Scene labeling problem
  - Types of lines
    - Convex edges +
    - Concave edges -
    - Occluding edges →←
CSP – Example II

- Scene labeling problem defined as CSP
  - Variables
  - Domains

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CSP – Example II

- Scene labeling problem defined as CSP
  - Constraints

![Diagram of constraints](image)

Figure 1.8 Legal labels for junctions (from Huffman, 1971)
CSP – Example III

- **Sudoku**
  - Variables
  - Domain
  - Constraints

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CSP – Real World Applications

- Graph layout
- Natural language processing
- Molecular biology / genomics
- Optimization
- Machine vision
- Scheduling
- Resource allocation
- Theorem proving
- Building design
- Planning
Summary

- Constraint satisfaction problems
  - Examples
    - map coloring, scene labeling ...
  - Defining a CSP
    - Variables
    - Domain
    - Constraints
  - Other definitions
    - Solutions, compound label, projection ...

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