G53CLP
Constraint Logic Programming

Dr Rong Qu
Search Orders in CSP
Variable and Value Ordering

- The order of the variables labelled and the values assigned has significant effect on the effectiveness of backtrack

Aims
- Minimise the depth of branches explored
- Minimise the number of branches explored
- Minimise the size of search tree explored
Variable Ordering

Order variables before the search

- Heuristics
  - choose the variable with smallest domain size
  - choose the most constrained variables
  - choose the variable with smallest domain
Variable Ordering

- Order variables before the search

  - Minimal width ordering
    - Reduce the backtracking

  - Minimal bandwidth ordering
    - Reduce the number of re-assignment when backtracking

  - Max-cardinality ordering
    - Approximation of minimal bandwidth ordering
Minimal Width Ordering

- Label the variables that are constrained by fewer others to the last
  - Based on constraint graph
  - Reduce the need of backtracking

- Find a total ordering for the variables
  - With the minimal width
- Label the variables by the ordering
Minimal Width Ordering

- Label the variables that are constrained by fewer others to the last

- Use topology in graph theory
  - Total ordering of a minimal width

- Let’s look at
  - Total ordering
  - Minimal width
Minimal Width Ordering

Total ordering

- Every two elements in a set $S$ are ordered
- $<$

- for all $a$, $b$ and $c$ in set $S$
  - if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry)
  - if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)
  - $a \leq b$ or $b \leq a$ (totality)
Minimal Width Ordering

- **Minimal width**
  - Given a total ordering $<$ on the nodes of a graph
    - **Width of a node** $\nu$
      - the number of nodes before and adjacent to $\nu$
    - **Width of an ordering**
      - the maximum width of all nodes
    - **Width of the graph**
      - the minimal width of all possible orderings
Minimal Width Ordering

- Constraint graph of map coloring
- An ordering of the nodes in the graph
  - \( A B D C E F \)

Width of ordering: 3
Minimal Width Ordering

Another ordering of the nodes in the graph

\[ C B D A E F \]

Width of ordering: 3
Minimal Width Ordering

Ordering: A, B, D, C, E, F
Minimal Width Ordering

Ordering: C, B, D, A, E, F
Minimal Width Ordering

- The smaller the width of an ordering of variables, the more chance of backtracking reduced.

- Variables at the front of ordering are in general more constrained. Labelling them earlier leaves less trouble at later stage.
Minimal Width Ordering

- Finding the minimal width of a graph*
  
  REPEAT
  - Pick the node $n$ with the least degree
  - Put $n$ at the beginning of the ordering
  - Remove $n$ and all adjacent edges to $n$
  UNTILL all nodes are in the ordering

- Complexity of this algorithm
  - $O(n^2)$
  
  * From Freuder (1982)
Minimal Width Ordering vs. k-Consistency

- Finding the minimal width of a graph
  - Complexity of this algorithm $O(n^2)$
  - Not too expensive to find the minimal width of a graph in practice

- What benefit can this offer?

- The complexity of finding strong $k$-consistency
  - Exponential

- Help reducing the $k$-consistency calculations
  - Backtrack free search!
Minimal Width Ordering vs. k-Consistency

Theorem

- A depth first search is backtrack-free if the level of strong k-consistency is greater than the width of the ordered constraint graph

- Freuder, 1982

Minimal Width Ordering vs. k-Consistency

- **k-consistency**
  - For values of (k-1) variables
  - At least one value in the k^{th} variable
  - Consistent with the k-1 assignment

- k-consistency doesn’t mean k-1 consistency

- **Strong k-consistency**
  - All j < k-1, j-consistency
  - Computation time: exponential
Minimal Width Ordering vs. k-Consistency

- This indicates that if a constraint graph has a width $w$
  - Then we never need to achieve strong $k$-consistency for $k > w + 1$
  - The smaller $(w - k)$ is, the less backtracking is needed
Minimal Bandwidth Ordering

- Based on constraint graph
- Pre-process: ordering of variables
- The closer the constrained variables in the ordering, the less distance one has to backtrack
Minimal Bandwidth Ordering

- Find a total ordering for the variables
  - With the minimal bandwidth
- Label the variables by the ordering

- Let’s look at
  - Bandwidth
Minimal Bandwidth Ordering

- **Minimal bandwidth**
  - Given a total ordering \(<\) on the nodes of a graph
    - Bandwidth of a node \(v\)
      - the maximum distance between any other adjacent node and \(v\)
    - Bandwidth of an ordering
      - the maximum bandwidth of all nodes
    - Bandwidth of the graph
      - the minimal bandwidth of all possible orderings
Minimal Bandwidth Ordering

- Constraint graph of map coloring

- An ordering of the nodes in the graph

A B D C E F

bandwidth of ordering: 3
Minimal Bandwidth Ordering

bandwidth of ordering: 5
Minimal Bandwidth Ordering

Ordering: A, B, D, C, E, F

Ordering: C, B, D, A, E, F

Observe the distance the search has to backtrack
Minimal Bandwidth Ordering

- Finding minimal bandwidth ordering is time inefficient
  - $O(n^k)$; $k$: bandwidth

- Max-cardinality ordering can be seen as an approximation of minimal bandwidth ordering

*Gurari & Sudbough (1984)*
Max-cardinality Ordering

- Finding the max-cardinality ordering
  
  - Randomly pick one node
  
  REPEAT
  
  - Choose a node $n$ in the remaining nodes
    
    - with the maximum number of adjacent edges to those already picked
  
  - Put $n$ at the beginning of the ordering
  
  UNTILL all nodes are in the ordering
Max-cardinality Ordering

- Finding the max-cardinality ordering

![Graph Diagram]

Bandwidth of ordering: 5

G53CLP – Constraint Logic Programming
Finding the minimal width of a graph

REPEAT

- Pick the node \( n \) with the least degree
- Put \( n \) at the beginning of the ordering
- Remove \( n \) and all adjacent edges to \( n \)

UNTIL all nodes are in the ordering

\( C D B E F A \)

\( C B D A E F \)
Summary

- CP Techniques
  - Variable ordering
    - Heuristics
    - Minimum width ordering
    - Minimum bandwidth ordering
    - Max-cardinality Ordering
  - Value ordering
    - Heuristics