Monads and More: Part 2

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Monads from adjunctions (Huber)

- For any pair of adjoint functors $L : C \to D$, $R : D \to C$, $L \dashv R$ with unit $\eta : \text{Id}_C \to RL$ and counit $\varepsilon : LR \to \text{Id}_D$, the functor $RL$ carries a monad structure defined by
  - $\eta^{RL} = \text{df } \text{Id} \xrightarrow{\eta} RL$,
  - $\mu^{RL} = \text{df } RLRL \xrightarrow{R\varepsilon L} RL$.

- The Kleisli and Eilenberg-Moore adjunctions witness that any monad on $C$ admits a factorization like this.
Examples

- Side-effect monad:
  \[ L, R : C \to C, \quad LA =_{df} A \times S, \quad RB =_{df} S \Rightarrow B, \]
  \[
  \frac{A \times S \to B}{A \to S \Rightarrow B}
  \]
  \[ RLA = S \Rightarrow A \times S, \]

- An exotic one:
  \[ L, R : C \to C, \quad LA =_{df} \mu X.A + X \times S \cong A \times \text{List}S, \]
  \[ RB =_{df} \nu Y.B \times (S \Rightarrow Y), \]
  \[
  \frac{\mu X.A + X \times S \to B}{A \to \nu Y.B \times (S \Rightarrow Y)}
  \]
  \[ RLA = \nu Y.(\mu X.A + X \times S) \times (S \Rightarrow Y) \cong \]
  \[ \nu Y.A \times \text{List}S \times (S \Rightarrow Y). \]
  \[ \text{What notion of computation does this correspond to?} \]
Continuations monad:

- \( L : C \rightarrow C^{\text{op}} \), \( LA = \text{df} \ A \Rightarrow E \),
- \( R : C^{\text{op}} \rightarrow C \), \( RB = \text{df} \ B \Rightarrow E \),

\[
\begin{align*}
A \Rightarrow E & \leftarrow B \\
E & \leftarrow B \times A \\
A \times B & \rightarrow E \\
A & \rightarrow B \Rightarrow E
\end{align*}
\]

- \( RLA = (A \Rightarrow E) \Rightarrow E \).
Monads from adjunctions ctd.

- Given two functors \( L : C \to D \) and \( R : D \to C \), \( L \dashv R \) and a monad \( T \) on \( D \), we obtain that \( RTL \) is a monad on \( C \).
- This is because \( T \) factorizes as \( UJ \) where \( J \vdash U \) is the Kleisli adjunction.
  That means an adjoint situation \( JL \vdash RU \) implying that \( RUJL = RTL \) is a monad.
- The monad structure is
  \[
  \eta^{RTL} = \text{df} \quad \text{Id} \xrightarrow{\eta} RL \xrightarrow{R\eta^T_L} RTL,
  \]
  \[
  \mu^{RTL} = \text{df} \quad RTLRTL \xrightarrow{RT\varepsilon_T L} RTTL \xrightarrow{\mu^T} RTL.
  \]
Examples

- **State monad transformer:**
  - \( L, R : C \rightarrow C, \; LA =_{df} A \times S, \; RB =_{df} S \Rightarrow B, \)
  - \( T \) – a monad on \( C, \)
  - \( RTLA = S \Rightarrow T(A \times S), \)
  - In particular, for \( T \) the exceptions monad we get \( RTLA = S \Rightarrow (A \times S) + E. \)

- **Continuations monad transformer:**
  - \( L : C \rightarrow C^\text{op}, \; LA =_{df} A \Rightarrow E, \)
    \( R : C^\text{op} \rightarrow C, \; RB =_{df} B \Rightarrow E, \)
  - \( T \) – a monad on \( C^\text{op}, \) i.e., a comonad on \( C, \)
  - \( RTLA =_{df} T(A \Rightarrow E) \rightarrow E. \)
Free algebras

- Given an endofunctor $H$ on a category $C$, the initial algebra of $(H^*A, [\eta_A, \tau_A])$ of $A + H$—(if it exists)—is the type of wellfounded $H$-trees with mutable leaves from $A$, i.e., $H$-terms over variables from $A$.
- $((H^*A, \tau_A), \eta_A))$ is the free $H$-algebra on $A$.
- $(H^*, \eta, \mu)$ is a monad where $\mu$ flattens a tree whose mutable leaves are trees into a tree, i.e., a term over terms into a term.
- $((H^*, \eta, \mu), \tau)$ is the free monad on $H$.
- The final coalgebras $H^\infty A$ of $A + H$—for each $A$ also a give a monad.
Monads from parameterized monads via initial algebras / final coalgebras (U.)

- A *parameterized monad* on $\mathcal{C}$ is a functor $F : \mathcal{C} \rightarrow \text{Monad}(\mathcal{C})$.

- If $F$ is a parameterized monad then the functors $T, T^\infty : \mathcal{C} \rightarrow \mathcal{C}$ defined by $TA =_{\text{df}} \mu X. FXA$ and $T^\infty A =_{\text{df}} \nu X. FXA$ carry a monad structure.

- In fact more can be said about them, but here we won’t.
Examples

- Free monads:
  - \( FXA =_{df} A + HX \) where \( H : C \to C \),
  - \( TA =_{df} \mu X.A + HX, \ T \infty A =_{df} \nu X.A + HX. \)
  - These are the types of wellfounded/nonwellfounded \( H \)-trees with mutable leaves from \( A \).

- Rose tree types:
  - \( FXA =_{df} A \times HX \) where \( H : C \to \text{Monoid}(C) \),
  - \( TA =_{df} \mu X.A \times HX, \ T \infty A =_{df} \nu X.A + HX. \)
  - If \( HX =_{df} \text{List}X \), these are the types of wellfounded/nonwellfounded \( A \)-labelled rose trees.
- Types of hyperfunctions with a fixed domain:
  - \( FXA = \text{df} \ HX \Rightarrow A \) where \( H : C \rightarrow C^{\text{op}} \),
  - \( TA = \text{df} \ \mu X. HX \Rightarrow A \), \( T^\infty A = \text{df} \ \nu X. HX \Rightarrow A \).
  - If \( HX = \text{df} \ X \Rightarrow E \), these are the types of
    wellfounded/nonwellfounded hyperfunctions from \( E \) to
    \( A \). (Of course these types do no exist in \textbf{Set}.)
Distributive laws

- If $T$, $S$ are monads on $C$, it is not generally the case that $ST$ is a monad. But sometimes it is.
- A **distributive law** of a monad $T$ over a monad $S$ is a natural transformation $\lambda : TS \to ST$ satisfying

\[
\begin{align*}
T & \xrightarrow{\eta^S} TS \\
T^S & \xrightarrow{\mu^S} TST
\end{align*}
\]

\[
\begin{align*}
TS & \xrightarrow{\lambda} ST \\
S & \xrightarrow{\eta^T} TS
\end{align*}
\]

\[
\begin{align*}
TSS & \xrightarrow{\lambda_S} STS \\
TTS & \xrightarrow{T\lambda} TST
\end{align*}
\]

\[
\begin{align*}
S & \xrightarrow{\eta^T} TS \\
S & \xrightarrow{\mu^T} ST
\end{align*}
\]

\[
\begin{align*}
ST & \xrightarrow{\mu^S} SST \\
ST & \xrightarrow{\lambda_T} S TT
\end{align*}
\]
A distributive law $\lambda$ of $T$ over $S$ gives a monad structure on the endofunctor $ST$:

- $\eta^{ST} = \text{df} \text{ Id } \xrightarrow{\eta^S \eta^T} ST$,
- $\mu^{ST} = \text{df} \text{ STST } \xrightarrow{S \lambda T} SSTT \xrightarrow{\mu S \mu T} ST$.
Examples

- The exceptions monad distributes over any monad.
  - \(S – a\) monad,
  - \(TA =_{df} A + E\) where \(E\) is an object,
  - \(\lambda =_{df} SA + E \xrightarrow{id+\eta^S} SA + SE \xrightarrow{[\text{Sinl},\text{Sinr}]} S(A + E),\)
  - \(STA = S(A + E).\)
  - For \(T\) the state monad, this gives
    \(ST = S \Rightarrow (A + E) \times S\), which is a different combination of exceptions and state than we saw before.

- The output monad distributes over any \((1, \times)\) strong monad.
  - \((S, sl) – a\) strong monad,
  - \(TA =_{df} A \times E\) where \(E\) is a monoid,
  - \(\lambda =_{df} SA \times E \xrightarrow{sr} S(A \times E),\)
  - \(STA = S(A \times E).\)
Any \((1, \times)\) strong monad distributes over the environment monad.

- \((T, sl)\) – a strong monad,
- \(SA =_{df} E \Rightarrow A\) where \(E\) is an object,
- \(\lambda =_{df} \Lambda(T(E \Rightarrow A) \times A \xrightarrow{sr} T((E \Rightarrow A) \times A) \xrightarrow{T_{ev}} E)\),
- \(STA = E \Rightarrow TA\).
An interesting way to combine monads is the coproduct of monads.

A coproduct of two monads $T_0$ and $T_1$ on $C$ is their coproduct in $\text{Monad}(C)$.

I.e., it is a monad $T_0 +^m T_1$ together with two monad maps $\text{inl}^m : T_0 \to^m T_0 +^m T_1$, $\text{inr}^m : T_0 \to^m T_0 +^m T_1$ such that for any monad $S$ and monad maps $\tau_0 : T_0 \to^m S$, $\tau_1 : T_1 \to^m S$ there exists a unique map $T_0 +^m T_1 \to^m S$ satisfying

\[
\begin{array}{ccc}
T_0 & \xrightarrow{\text{inl}^m} & T_0 +^m T_1 & \xleftarrow{\text{inr}^m} & T_1 \\
\downarrow \tau_0 & & \downarrow \text{unique map} & & \downarrow \tau_1 \\
S & & & & S
\end{array}
\]
Coproduct of free monads

- The coproduct of the free monads of functors $F$, $G$ is the free monad of their coproduct:

$$F^* +^m G^* = (F + G)^*$$

(obvious, since the free monad delivering functor has a left adjoint and hence preserves colimits).

- More generally, the coproduct of a free monad $F^*$ with an arbitrary monad $S$ is this (if $(FS)^*$ exists):

$$F^* +^m S = S(FS)^*$$

i.e.,

$$(F^* +^m S)A = S(\mu X. A + FSX) = \mu X. S(A + FX)$$
An *ideal monad* on $\mathcal{C}$ is a monad $(T, \eta, \mu)$ together with an endofunctor $T'$ on $\mathcal{C}$ and a natural transformation $\mu' : T'T \to T'$ such that

- $T = \text{Id} + T'$,
- $\eta = \text{inl}$,
- $\mu = [id, \text{inr} \circ \mu']$. 
Given two ideal monads $R = \text{Id} + R'$ and $S = \text{Id} + S'$, their coproduct is an ideal monad $T = \text{Id} + T_0 + T_1$ where

$$(T_0 A, T_1 A) = \text{df} \mu(X, Y). (R'(A + Y)), S'(A + X))$$