Categorical views on bottom-up tree transducers

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Three types of bottom-up tree transducers, ordered by generality:

- relabelling (branching-preserving) = purely synthesized attribute grammars
- rebranching (layering-preserving)
- 1-n relayering (= the classical notion)

For each type, we have a triangular picture: transducers (modulo bisimilarity) are the same as (co/bi)-Kleisli maps of a comonad/distributive law and a subclass of tree functions.
RELABELLING BOTTOM-UP TREE TRANSDUCERS
RELABELLING BOTTOM-UP TREE TRANSDUCERS: THE TRIANGLE

- $F$ — a fixed endofunctor on the base category
  $A, B, C$ — typical objects of the base category
- $LT_{tree} A = \mu Z.A \times F Z$ — $A$-labelled $F$-branching trees
- $DA = \mu Z A \times F Z$ — "subtrees"; $D$ is a comonad on the base category!

realizations
relabelling BU tree trans-s (mod bisim)

\[(X, d : A \times FX \rightarrow B \times X)\]

c-co-Kleisli behaviors
c-co-Kleisli maps of $D$
\[k : DA \rightarrow B,\]
\[i.e. \quad k : LT_{tree} A \rightarrow B\]

tree function behaviors
relabelling BU tree fun-s
\[f : LT_{tree} A \rightarrow LT_{tree} B\]
• We have three different constructed categories on the objects of the base category.

• The three categories are equivalent: the maps are in a 1-1 correspondence, and typing, the identities and composition agree.

• Moreover, for each of the three categories, we have an identity-on-objects inclusion functor from the base category, which preserves products (i.e., an “arrow” and more).

• The “arrows” are equivalent too: the inclusion functors agree as well.
Relabelling bottom-up tree transducers

- Relabelling bottom-up tree transducers for a fixed branching type $F$ are pairs $(X, d : A \times FX \to B \times X)$ ($X$ — state space, $d$ — transition function)

- Identity on $A$:

  $$(1, A \times F1 \to A \to A \times 1)$$

- Composition of $(X, d : A \times FX \to B \times X)$ and $(X', e : B \times FX' \to C \times X')$:

  $$(X \times X', A \times F(X \times X') \to A \times FX \times FX' \to B \times X \times FX' \to C \times (X \times X'))$$
Bisimilarity of relabelling BU tree trans-s

- \((X_0, d_0 : A \times FX_0 \to B \times X_1)\) and \((X_1, d_1 : A \times FX_1 \to B \times X_1)\) are defined to be bisimilar, if there exist a span \((R, r_0, r_1)\) (a bisimulation) and a map \(s : A \times FR \to B \times R\) (its bisimulationhood witness) such that

\[
\begin{array}{ccc}
A \times FX_0 & \xrightarrow{d_0} & B \times X \\
\downarrow{\text{id} \times Fr_0} & & \downarrow{\text{id} \times r_0} \\
A \times FR & \xrightarrow{s} & B \times R \\
\downarrow{\text{id} \times Fr_1} & & \downarrow{\text{id} \times r_1} \\
A \times FX_1 & \xrightarrow{d_1} & B \times X_1 \\
\end{array}
\]
The comonad for relabelling BU tree trans-s is \((D, \varepsilon, \delta)\) where

- \(DA = \text{df} \ LTree A = \text{df} \ \mu Z.A \times FZ\) — (sub)trees
- \(\varepsilon_A = \text{df} \ DA \xrightarrow{\cong} A \times F(DA) \xrightarrow{\text{fst}} A\) — extraction of the root label
- \(\delta_A = \text{df} \ DA \xrightarrow{\delta'_A} DA \times D(DA) \xrightarrow{\text{snd}} D(DA)\) — replacement of the label with the subtree at every node

The map \(\delta'_A = \langle \text{id}, \delta_A \rangle\) is given by initiality:
• Co-Kleisli maps are maps \( k : DA \to B \), the identity on \( A \) is

\[
DA \xrightarrow{\varepsilon_A} A
\]

the composition of \( k : DA \to B \), \( \ell : DB \to C \) is

\[
DA \xrightarrow{\delta_A} D(DA) \xrightarrow{Dk} DB \xrightarrow{\ell} C
\]

(by the general definition of a co-Kleisli category of a comonad)
**Relabelling BU Tree Functions**

- Tree functions are maps \( f : LTree A \rightarrow LTree B \), the identity and composition are taken from the base category.

- A tree function \( f \) is defined to be bottom-up relabelling if

\[
\begin{array}{c}
LTree A \xrightarrow{f} LTree B \\
\cong \\
A \times F(LTree A) \xrightarrow{\mathsf{snd}} B \times F(LTree B) \\
\cong \\
F(LTree A) \xrightarrow{Ff} F(LTree B)
\end{array}
\]

- The identity tree functions are BU relabelling and the composition of two BU relabelling tree functions is BU relabelling.
REBRANCHING BOTTOM-UP TREE TRANSDUCERS

\[
\begin{array}{c}
Y_0 \\
Y_1 \\
Y_2 \\
x
\end{array}
\]
REBRANCHING BOTTOM-UP TREE TRANSDUCERS: THE TRIANGLE

- $G, H, K$ — typical endofunctors on the base category
- $\text{Tree } G = \text{df } \mu Z. GZ$ — $G$-branching trees
- $G\# Y = \text{df } G(Y \times \text{Tree } G)$ – “child-position aware subtrees”;
  $(-)\#$ is a comonad on the endofunctor category!

**realizations**
rebranching BU tree trans-s (mod bisim)

$$(X, (d_Y : G(Y \times X) \rightarrow H Y \times X)_Y)$$

c**co-Kleisli behaviors**
c**co-Kleisli maps of $()\#$

$k : G\# \rightarrow H$,

i.e., $(k_Y : G(Y \times \text{Tree } G) \rightarrow H Y)_Y$

**tree function behaviors**
rebranching BU tree fun-s

$f : \text{Tree } G \rightarrow \text{Tree } H$
**Rebranching BU Tree Functions**

- Tree functions are maps $f : \text{Tree } G \to \text{Tree } H$, the identity and composition are taken from the base category.

- A tree function $f$ as above is defined to be rebranching BU if there is a natural transformation $(k_Y : G(Y \times \text{Tree } G) \to HY)_Y$ (its rebranching BU witness) such that

\[
\begin{array}{ccc}
\text{Tree } G & \xrightarrow{f} & \text{Tree } H \\
\downarrow \cong & & \downarrow \cong \\
G(\text{Tree } G) & \xrightarrow{G\Delta} & G(\text{Tree } G \times \text{Tree } G) \\
\downarrow G\Delta & & \downarrow k_{\text{Tree } G} \\
G(\text{Tree } G \times \text{Tree } G) & \xrightarrow{k_{\text{Tree } G}} & H(\text{Tree } G) \\
& & \xrightarrow{Hf} H(\text{Tree } H)
\end{array}
\]

- $k$ determines $f$.

- The identity tree functions are relabelling BU and the composition of two relabelling BU tree functions is relabelling BU.
CLASSICAL (1-N RELAYERING) BOTTOM-UP TREE TRANSDUCERS

\[ \begin{array}{c}
\text{x} \\
Y_0 \quad \text{x}_0 \quad \text{x}_1 \quad \text{x}_2 \\
Y_1 \\
\end{array} \]
CLASSICAL (1-N RELAYERING) BOTTOM-UP TREE TRANSDUCERS: THE TRIANGLE

- \( G, H, K \) — typical endofunctors on the base category
  
  \( \text{Tree } G =_{\text{df}} \mu Z.GZ \) — \( G \)-branching trees

- \( G^\# Y =_{\text{df}} G(Y \times \text{Tree } G) \) — “child-position aware subtrees”;  
  \((-)^\# \) is a comonad on the endofunctor category!

- \( G^* Y =_{\text{df}} \mu Z.Y + GZ \) — \( G \)-branching trees with \( Y \)-leaves;  
  \( G^* Y \) is a monad on the base category (the free monad of \( G \));  
  \((-)^* \) is a monad on the endofunctor category!

- \( \text{Tree } G \cong G^\# 0 \)

- The comonad \((-)^\# \) distributes over the monad \((-)^* \)!
realizations

BU tree trans-s (mod bisim)

\((X, (d_Y : G(Y \times X) \rightarrow H^*Y \times X)_Y)\)

bi-Kleisli behaviors

bi-Kleisli maps of \((-)^\sharp, (-)^*\)

\(k : G^\sharp \rightarrow H^*,\)

i.e., \((k_Y : G(Y \times \text{Tree } G) \rightarrow H^*Y)_Y\)

tree function behaviors

BU tree fun-s

\(f : \text{Tree } G \rightarrow \text{Tree } H\)
As before, tree functions are maps \( f : \text{Tree } G \to \text{Tree } H \), the identity and composition are taken from the base category.

A tree function \( f \) as above is defined to be 1-n relayering BU if there is a natural transformation \( (k_Y : G(Y \times \text{Tree } G) \to H^*(Y))_Y \) (its rebranching BU witness) such that

\[
\begin{array}{ccc}
\text{Tree } G & \xrightarrow{f} & \text{Tree } H \\
\xrightarrow{\cong} & & \cong \\
G(\text{Tree } G') & & H^*0 \\
G\Delta & \downarrow & \\
G(\text{Tree } G \times \text{Tree } G) & \xrightarrow{k_{\text{Tree } G}} & H^*(\text{Tree } G) \\
& \xrightarrow{Hf} & H^*(\text{Tree } H) \\
& \cong & H^*(H^*0)
\end{array}
\]

\( k \) determines \( f \).

The identity tree functions are 1-n relayering BU and the composition of two 1-n relayering BU tree functions is 1-n relayering BU.
The same types of tree transducers are possible, to represent top-down tree functions of these types.

- relabelling TD TTs:

\[(X, q_I : 1 \rightarrow X, d : A \times X \rightarrow B \times (F'1 \Rightarrow X))\]

\[(F'1 \Rightarrow X — assignments of a state to every child of the current node in the input tree)\]

- rebranching TD TTs:

\[(X, q_I : 1 \rightarrow X, (d_Y : GY \times X \rightarrow H(Y \times X))_Y)\]

- 1-n relayering TD TTs:

\[(X, q_I : 1 \rightarrow X, (d_Y : GY \times X \rightarrow H^*(Y \times X))_Y)\]
Variations: Relabelling tree transducers with lookahead

- Relabelling transducers can be augmented with lookahead, so they can represent functions using information from both below and above any given node.
  - relabelling BU TTs with lookahead:
    \[
    (X, d : A \times (\mu Z.1 + A \times F'1) \times FX \to B \times X)
    \]
  - relabelling TD TTs with lookahead:
    \[
    (X, q_I : 1 \to X, d : LTree A \times X \to B \times (F'1 \Rightarrow X))
    \]