The case of the smart case
How to implement conditional convertibility?

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Based on work with Andreas Abel, Thomas Anberre, Nils Anders Danielsson and Shin-Cheng Mu
ΠΣ in a nutshell

- Partial core language for DTP.
- Ingredients:
  - Type : Type
  - Finite enumerations, eg \( \textbf{Bool} = \{ \text{true}, \text{false} \} \).
  - \( \Pi \)-types
  - \( \Sigma \)-types
  - Flexible mutual recursive definitions
  - Lifted types to control recursive unfolding.
  - Extended \( \alpha \) conversion for recursive definitions.

ΠΣ: *Dependent Types Without the Sugar*
T.A., Nils Anders Danielsson, Andres Löh and Nicolas Oury
FLOPS 2010
How to implement eliminators for datatypes?
For the moment we consider just \textbf{Bool}
Rule for the simply typed eliminator:

\[
\begin{align*}
\Gamma \vdash & \quad t_0, t_1 : \sigma \\
\Gamma \vdash & \quad u : \text{Bool} \\
\hline
\Gamma \vdash & \quad \text{case } u \text{ of } \{ \text{true } \rightarrow t_0 \mid \text{false } \rightarrow t_1 \} : \sigma
\end{align*}
\]

- Pattern matching is reduced to case.
- Local case expressions.
Dependently typed eliminator with motive

\[ \Gamma, x : \text{Bool} \vdash \sigma \]
\[ \Gamma \vdash u : \text{Bool} \]
\[ \Gamma \vdash t_0 : \sigma[x := \text{true}] \]
\[ \Gamma \vdash t_1 : \sigma[x := \text{false}] \]

\[ \Gamma \vdash \text{elim}_{x.\sigma}^\text{Bool} u \ \text{of} \ \{ \text{true} \rightarrow t_0 \mid \text{false} \rightarrow t_1 \} : \sigma[x := u] \]

- Not syntax directed!
- We have to come up with the motive \( x.\sigma \).
- Local case expressions?
- Can be (partially) simulated using auxiliary definitions (with).
\[ \Gamma \vdash x : \text{Bool} \]
\[ \Gamma \vdash t_0 : \sigma[x := \text{true}] \]
\[ \Gamma, \vdash t_1 : \sigma[x := \text{false}] \]

\[ \Gamma \vdash \text{case } x \text{ of } \{ \text{true } \rightarrow t_0 \mid \text{false } \rightarrow t_1 \} : \sigma \]

- Syntax directed.
- Eliminator can be easily derived.
- No need for motives.
- Variable restriction leads to failure of subject reduction.
- Also no local case analysis.
\[
\Gamma \vdash u : \textbf{Bool} \\
\Gamma, u = \text{true} \vdash t_0 : \sigma \\
\Gamma, u = \text{false} \vdash t_1 : \sigma \\
\hline
\Gamma \vdash \text{case } u \text{ of } \{ \text{true } \rightarrow t_0 \mid \text{false } \rightarrow t_1 \} : \sigma
\]

- Addresses issue with Subject Reduction
- Local case expressions (more general than with)
- Need equational assumptions in contexts.
- Need to decide convertibility with assumptions.
We allow equational assumptions of the form $t = b$ in the context.
We add the rule
$$\Gamma, t = b \vdash t = b$$
and weakening rules.
Here $b$ has to be a constructor (e.g. true, false)
The remaining rules remain unchanged, e.g.
$$\text{case true of } \{ \text{true }\rightarrow t_0 | \text{false }\rightarrow t_1 \} = t_0$$

We do not consider (for the moment):
$$\Gamma, u = \text{true} \vdash t_0 = v$$
$$\Gamma, u = \text{false} \vdash t_1 = v$$
$$\Gamma \vdash \text{case } u \text{ of } \{ \text{true }\rightarrow t_0 | \text{false }\rightarrow t_1 \} = v$$
Equational assumptions can be inconsistent. E.g. the context

\[ x : \textbf{Bool}, x = \text{true}, x = \text{false} \]

is inconsistent, i.e. every equation is derivable.

\[
t = \text{case true of } \{\text{true }\rightarrow t \mid \text{false }\rightarrow u\} \\
= \text{case } x \text{ of } \{\text{true }\rightarrow t \mid \text{false }\rightarrow u\} \\
= \text{case false of } \{\text{true }\rightarrow t \mid \text{false }\rightarrow u\} \\
= u
\]
How to implement conditional $\beta$-equality (for boolean pattern equations)?
We define (mutually):

**Constraint sets** \( C \)

**Normalisation with constraints** \( C \vdash t \Downarrow v \)

**Convertibility with constraints** \( C \vdash t \sim u \)

**Creation of constraint sets** \( \Gamma \Downarrow C \)

**Merging of constraint sets** \( C + D \Downarrow E \)
A constraint set $\mathcal{C}$ is either

**INCONSISTENT**

or

$$n_0 = b_0, n_1 = b_1, \ldots, n_m = b_m$$

where

$$b_i \in \{\text{true, false}\}$$

$n_i$ is a neutral term

such that for all $i$:

$$\mathcal{C} - n_i = b_i \vdash n_i \Downarrow n_i$$
Reduction  We add the rule

\[ n = b \in C \]
\[ C \vdash n \downarrow b \]

Convertibility

\[ \text{INCONSISTENT} \vdash t \sim u \]
\[ C \vdash t \downarrow v \quad C \vdash u \downarrow v \]
\[ C \vdash t \sim u \]

Creation of constraint sets

\[ \Gamma \downarrow C \quad C \vdash t \downarrow n \quad n = b \uplus C \downarrow D \]
\[ \Gamma, t = b \downarrow D \]
Merging Constraint sets

\[ C \uplus \epsilon \downarrow C \]

\[ C \vdash n \downarrow b \quad C \uplus D \downarrow E \]

\[ C \uplus n = b, D \downarrow E \]

\[ C \vdash n \downarrow \neg b \]

\[ C \uplus n = b, D \downarrow \text{INCONSISTENT} \]

\[ C \vdash n \downarrow n \quad C, n = b \uplus D \downarrow E \]

\[ C \uplus n = b, D \downarrow E \]

\[ C \vdash n \downarrow n' \quad n' = b \uplus C, D \downarrow E \]

\[ C \uplus n = b, D \downarrow E \]
Soundness and completeness

soundness

\[ \Gamma \Downarrow C \quad C \vdash t \sim u \]
\[ \Gamma \vdash t = u \]

completeness

\[ \Gamma \Downarrow C \quad \Gamma \vdash t = u \]
\[ C \vdash t \sim u \]

relies on the key lemma:

\[ n = b \quad + \quad C \Downarrow D \]
\[ D \vdash n \Downarrow b \]

termination

Have shown termination for a simply typed variant. Goal: Modular termination.
Extensions

- Arbitrary equations for booleans (congruence closure).
  Extensional equality for booleans.
- Extend to all first order datatypes
  All finite types and $\Sigma$-types.
- Conditional equality on higher order types seems undecidable.