Naturality for Free

The category interpretation of directed type theory

Thorsten Altenkirch

jww Filippo Sestini

Functional Programming Laboratory
School of Computer Science
University of Nottingham

March 28, 2019

The first commandment of Type Theory

Thou shalt not inspect a type!

Consequences of the 1st commandment

Parametricity
: *Polymorphic functions preserve all logical relations.*

Univalence
: *Isomorphic types are equal.*

How are these related?

reverse is natural

$$\begin{aligned} \mathrm{List} &: \underline{\mathbf{Set}} \to \underline{\mathbf{Set}} \\ \mathrm{rev} &: \Pi_{A:\mathbf{Set}} \mathrm{List}\, A \to \mathrm{List}\, A \\ f &: A \to B \end{aligned}$$

$$\begin{array}{ccc} \mathrm{List}\, A & \xrightarrow{\mathrm{rev}_A} & \mathrm{List}\, A \\ {\scriptstyle \mathrm{List}\, f}\downarrow & & \downarrow{\scriptstyle \mathrm{List}\, f} \\ \mathrm{List}\, B & \xrightarrow[\mathrm{rev}_B]{} & \mathrm{List}\, B \end{array}$$

Proof by . . . -induction

$$\mathrm{List}\, f\, [a_0, a_1, \ldots, a_{n-1}] = [f\ a_0, f\ a_1, \ldots, f\ a_{n-1}]$$

$$\mathrm{rev}_A\, [a_0, a_1, \ldots, a_{n-1}] = [a_{n-1}, \ldots, a_1, a_0]$$

$$\begin{aligned}
(\mathrm{rev}_B \circ \mathrm{List}\, f)\, [a_0, a_1, \ldots, a_{n-1}] &= \mathrm{rev}_B\, (\mathrm{List}\, f\, [a_0, a_1, \ldots, a_{n-1}]) \\
&= \mathrm{rev}_B\, [f\ a_0, f\ a_1, \ldots, f\ a_{n-1}]) \\
&= [f\ a_{n-1}, \ldots, f\ a_1, f\ a_0]) \\
&= \mathrm{List}\, f\, [a_{n-1}, \ldots, a_1, a_0] \\
&= \mathrm{List}\, f\, (\mathrm{rev}_A\, [a_0, a_1, \ldots, a_{n-1}]) \\
&= (\mathrm{List}\, f \circ \mathrm{rev}_A)\, [a_0, a_1, \ldots, a_{n-1}]
\end{aligned}$$

Everything is natural ...

$$F, G : \underline{\mathbf{Set}} \to \underline{\mathbf{Set}}$$
$$\alpha : \Pi_{A:\mathbf{Set}} F\,A \to G\,A$$
$$f : A \to B$$

$$\begin{array}{ccc} F\,A & \xrightarrow{\alpha_A} & G\,A \\ {\scriptstyle F\,f}\downarrow & & \downarrow{\scriptstyle G\,f} \\ F\,B & \xrightarrow[\alpha_B]{} & G\,B \end{array}$$

. . . but we can't prove it.

- We know that all families of functions are natural.
- But we cannot prove it.
- It should be a *free theorem*.

The hint (HoTT)

$$
\begin{aligned}
F, G &: \underline{\mathbf{Set}} \to \underline{\mathbf{Set}} \\
\alpha &: \Pi_{A:\mathbf{Set}} F\,A \simeq G\,A \\
f &: A \simeq B
\end{aligned}
$$

$$
\begin{array}{ccc}
F\,A & \xrightarrow{\alpha_A} & G\,A \\
{\scriptstyle F\,f}\big\downarrow & & \big\downarrow{\scriptstyle G\,f} \\
F\,B & \xrightarrow[\alpha_B]{} & G\,B
\end{array}
$$

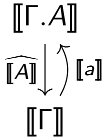
- $A \simeq B$ means isomorphism (for sets).
- This is provable in HoTT.
- It follows from univalence + J.

Summary

- The set-level fragment of HoTT can be interpreted using the groupoid model (Hofmann & Streicher).
- This interpretation also gives rise to a univalent, truncated universe of sets (but it doesn't classify hsets).
- Can we replace groupoids by categories?
- Yes, but we need to take care of polarities.
- And some places we do need groupoids, hence we need an operation calculating the groupoid associated to a category (the core).
- I am going to derive a type theory guided by the semantics.

The category with families of categories

Contexts	$\mathrm{Con} : \mathbf{Set}$	$\Gamma : \mathrm{Con}$	$[\![\Gamma]\!] : \mathbf{Cat}$
Types	$\mathrm{Ty} : \mathrm{Con} \to \mathbf{Set}$	$A : \mathrm{Ty}\,\Gamma$	$[\![A]\!] : [\![\Gamma]\!] \to \mathbf{Cat}$
Terms	$\mathrm{Tm} : (\Gamma : \mathrm{Con}) \to \mathrm{Ty}\,\Gamma \to \mathbf{Set}$	$a : \mathrm{Tm}\,\Gamma\,A$	$[\![a]\!] : \ldots$
Subst	$\mathrm{Tms} : \mathrm{Con} \to \mathrm{Con} \to \mathbf{Set}$	$\gamma : \mathrm{Tms}\,\Gamma\,\Delta$	$[\![\gamma]\!] : [\![\Gamma]\!] \to [\![\Delta]\!]$



We write $\widehat{[\![A]\!]}$ for the associated opfibration.

Operations on contexts

$$\frac{}{\bullet : \mathrm{Con}} \qquad \frac{A : \mathrm{Ty}\,\Gamma}{\Gamma.A : \mathrm{Con}}$$

$$\begin{aligned} |[\![\bullet]\!]| &= \mathbf{1} \\ [\![\bullet]\!](x, y) &= \mathbf{1} \end{aligned}$$

$$\begin{aligned} |[\![\Gamma.A]\!]| &= (x : |[\![\Gamma]\!]|) \times |[\![A]\!]\, x| \\ [\![\Gamma.A]\!]((x, a), (y, b)) &= (f : [\![\Gamma]\!](x, y)) \times ([\![A]\!]\, y)([\![A]\!]\, f\, a, b) \end{aligned}$$

Grothendieck construction

Opposites

$$\frac{\Gamma : \mathrm{Con}}{\Gamma^{\mathrm{op}} : \mathrm{Con}} \qquad \frac{A : \mathrm{Ty}\,\Gamma}{A^{\mathrm{op}} : \mathrm{Ty}\,\Gamma}$$

$$[\![\Gamma^{\mathrm{op}}]\!] = [\![\Gamma]\!]^{\mathrm{op}}$$
$$[\![A^{\mathrm{op}}]\!]\,x = ([\![A]\!]\,x)^{\mathrm{op}}$$

- Note that $\_^{\mathrm{op}} : \mathbf{Cat} \to \mathbf{Cat}$ is covariant!
- But what is $(\Gamma.A)^{\mathrm{op}}$?
- It cannot be $\Gamma^{\mathrm{op}}.A^{\mathrm{op}}$

Fibrations

$$\frac{A : \mathrm{Ty}\, \Gamma^{\mathrm{op}}}{\Gamma .^{\mathrm{op}} A : \mathrm{Con}}$$

$$|[\![\Gamma .^{\mathrm{op}} A]\!]| = (x : |[\![\Gamma|]\!]) \times |[\![A]\!]\, x|$$
$$[\![\Gamma .^{\mathrm{op}} A]\!]((x, a), (y, b)) = (f : [\![\Gamma]\!](x, y)) \times ([\![A]\!]\, x)(a, [\![A]\!]\, f\, b)$$

$$(\Gamma . A)^{\mathrm{op}} = \Gamma^{\mathrm{op}} .^{\mathrm{op}} A^{\mathrm{op}}$$

Σ-types, undirected

$$\frac{A : \mathrm{Ty}\,\Gamma \quad B : \mathrm{Ty}\,\Gamma.A}{\Sigma\,A\,B : \mathrm{Ty}\,\Gamma}$$

$$\Gamma.A.B \cong \Gamma.(\Sigma\,A\,B)$$

On objects:

$$(\Sigma\,A\,B)\,x = (A\,x).(B\,x)$$

Σ-types with polarities

$$\frac{A : \mathrm{Ty}\,\Gamma \quad B : \mathrm{Ty}\,\Gamma.A^s}{\Sigma^s\,A\,B : \mathrm{Ty}\,\Gamma}$$

$$\Gamma.A^s.B \cong \Gamma.(\Sigma^s\,A\,B)$$

On objects:

$$(\Sigma^s\,A\,B)\,x = (A\,x).^s(B\,x)$$

$$(\Sigma^s\,A\,B)^{\mathrm{op}} = \Sigma^{s\,\mathrm{op}}\,A^{\mathrm{op}}\,B^{\mathrm{op}}$$

Π-types, undirected

$$\frac{A : \mathrm{Ty}\,\Gamma \quad B : \mathrm{Ty}\,\Gamma.A}{\Pi\,A\,B : \mathrm{Ty}\,\Gamma}$$

$$\mathrm{Tm}\,\Gamma.A\,B \cong \mathrm{Tm}\,\Gamma\,(\Pi\,A\,B)$$

On objects:

$$|[\![\Pi\,A\,B]\!]\,x| = \mathrm{Tm}\,(A\,x)\,(B\,x)$$

Π-types with polarities

$$\frac{A : \mathrm{Ty}\,\Gamma^{\mathrm{op}} \quad B : \mathrm{Ty}\,\Gamma.^{\mathrm{op}}A^s}{\Pi^s\,A\,B : \mathrm{Ty}\,\Gamma}$$

$$\mathrm{Tm}\,\Gamma.^{\mathrm{op}}A^s\,B \cong \mathrm{Tm}\,\Gamma\,(\Pi^s\,A\,B)$$

$$(\Pi\,A\,B)^{\mathrm{op}} = \Pi^{\mathrm{op}}\,A^{\mathrm{op}}\,B^{\mathrm{op}}$$

On objects:

$$\begin{aligned} |(\Pi^s\,A\,B)\,x| &= \mathrm{Tm}^s\,(A\,x)\,(B\,x) \\ &= \mathrm{Tm}\,(A\,x)^s\,(B\,x)^s \end{aligned}$$

The universe of sets

$$\frac{}{\mathrm{U} : \mathrm{Ty}\,\Gamma} \qquad \frac{a : \mathrm{Tm}\,\Gamma\,\mathrm{U}}{\mathrm{El}\,a : \mathrm{Ty}\,\Gamma}$$

$$|[\![\mathrm{U}]\!]\,x| = \mathbf{Set}$$
$$([\![\mathrm{U}]\!]\,x)(A, B) = A \to B$$

$$|[\![\mathrm{El}\,a]\!]\,x| = [\![a]\!]\,x$$
$$([\![\mathrm{El}\,a]\!]\,x)(y, z) = (y = z)$$

The hom type

$$\frac{a : Tm\,\Gamma\,A^{\mathrm{op}} \quad b : \mathrm{Tm}\,\Gamma\,A}{a \sqsubseteq_A b : \mathrm{Ty}\,\Gamma}$$

$$[\![a \sqsubseteq_A b]\!]\,x = A\,x(a\,x, b\,x)$$

- But what about id (aka refl)?
- We would like to say

$$\frac{a : \mathrm{Tm}\,\Gamma\,A}{\mathrm{id}_a : a \sqsubseteq_A a}$$

  but this doesn't type check!

The core type

$$\frac{A : \mathrm{Ty}\,\Gamma}{\bar{A} : \mathrm{Ty}\,\Gamma} \quad \frac{a : \mathrm{Tm}\,\Gamma\,\bar{A}}{\underline{a} : \mathrm{Tm}\,\Gamma\,A^s}$$

$$\frac{a : \mathrm{Tm}\,\Gamma\,\bar{A}}{\mathrm{id}_a : \underline{a} \sqsubseteq_A \underline{a}}$$

$$\frac{\begin{array}{c} a, b : \mathrm{Tm}\,\Gamma\,\bar{A} \\ f : \mathrm{Tm}\,\Gamma\,(\underline{a} \sqsubseteq_A \underline{b}) \\ f^{\mathrm{op}} : \mathrm{Tm}\,\Gamma\,(\underline{b} \sqsubseteq_A \underline{a}) \\ l : \mathrm{Tm}\,\Gamma\,(f \circ f^{\mathrm{op}} \sqsubseteq \mathrm{id}_a) \\ r : \mathrm{Tm}\,\Gamma\,(f^{\mathrm{op}} \circ f \sqsubseteq \mathrm{id}_a) \end{array}}{\overline{f, f^{\mathrm{op}}, l, r} : \mathrm{Tm}\,\Gamma\,a \sqsubseteq_{\bar{A}} b)}$$

Directed Path induction (J)

$$\frac{\begin{array}{c} A : \mathrm{Ty}\,\Gamma \\ a : \mathrm{Tm}\,\Gamma\,\bar{A} \\ M : \mathrm{Ty}\,\Gamma, x : A, p : \underline{a} \sqsubseteq_{A^s} x \\ m : \mathrm{Tm}\,\Gamma\,M[x = \underline{a}, p = \mathrm{id}_a] \\ b : \mathrm{Tm}\,\Gamma\,A \\ q : \mathrm{Tm}\,\Gamma\,\underline{a} \sqsubseteq_{A^s} b \end{array}}{\mathrm{J}^s_a\,M\,m\,x\,p : M[x = b, p = q]}$$

The homtypes of sets

Homtypes of sets are symmetric

$$\frac{a : \mathrm{Tm}\,\Gamma\,A}{\overline{\mathrm{El}\,a} \simeq \mathrm{El}\,a}$$

Homtypes of sets are proof irrelevant

$$\frac{a : \mathrm{Tm}\,\Gamma\,A}{\mathrm{K}_a : \mathrm{Tm}\,\Gamma\,(\Pi a : \bar{A}, p : \underline{a} \sqsubseteq_A \underline{a}.p \sqsubseteq \mathrm{id}\,a)}$$

Directed univalence

$$a \sqsubseteq_A b \equiv \mathrm{El}\,a \to \mathrm{El}\,b$$

Everything is natural, provably!

$$\begin{aligned} F, G &: \underline{\mathbf{Set}} \to \underline{\mathbf{Set}} \\ \alpha &: \Pi_{A:\mathbf{Set}} F\,A \to G\,A \\ f &: A \to B \end{aligned}$$

$$\begin{array}{ccc} F\,A & \xrightarrow{\alpha_A} & G\,A \\ {\scriptstyle F\,f}\big\downarrow & & \big\downarrow{\scriptstyle G\,f} \\ F\,B & \xrightarrow[\alpha_B]{} & G\,B \end{array}$$

- It follows from directed univalence + directed J.

Further work

- Filippo is formalising the calculus and its semantics in Agda.
- What is the relation to logical relations?
- Can we do higher categories (full directed HoTT)?