Monadic containers and universes

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Container (polynomial functors)

- A container (polynomial functor) $S \lhd P$ is given by

  $$\text{Shapes}\ \ S : \mathbf{Set}$$
  $$\text{Positions}\ \ P : \to \mathbf{Set}$$

  and gives rise to $[\![S \lhd P]\!] : \mathbf{Set} \to \mathbf{Set}$:

  $$[\![S \lhd P]\!]\, X = \Sigma s : S.P\, s \to X$$

- Example $\mathrm{List} = \mathbb{N} \lhd \mathrm{Fin}$
  with $\mathrm{Fin}\, n = \{0, 1, \ldots, n-1\}$
- Containers are functors, given $f : A \to B$:

  $$[\![S \lhd P]\!]\, f : [\![S \lhd P]\!]\, A \to [\![S \lhd P]\!]\, B$$
  $$[\![S \lhd P]\!]\, f\, (s, g) = (s, f \circ g)$$

Constructions on containers

Products

$$(S \lhd P) \times (T \lhd Q) = (S \times T) \lhd (\lambda(s,t).P\,s + Q\,t)$$

Sums

$$(S \lhd P) + (T \lhd Q) = (S + T) \lhd (\lambda \begin{array}{lcl} \text{left}\,s & . & P\,s \\ \text{right}\,t & . & Q\,t \end{array})$$

Initial algebras

$$\mu(S \lhd P) = W\,S\,P$$

Application of containers

Generic constructions on inductive (and coinductive) types

Container morphisms

- Given container $S \lhd P$, $T \lhd Q$ we define a container morphism $f \lhd g : \mathrm{Cont}\,(S \lhd P)\,(T \lhd Q)$ given by

$$\begin{aligned} f &: S \to T \\ g &: \Pi s : S.Q\,(f\,s) \to P\,s \end{aligned}$$

- A container morphism gives rise to a natural transformation given by

$$\begin{aligned} [\![f \lhd g]\!] : \Pi X : \mathbf{Set}.[\![S \lhd P]\!]\,X &\to [\![T \lhd Q]\!]\,X \\ [\![f \lhd g]\!]\,X\,(s, h) &= (f\,s, \lambda s.h \circ g\,s) \end{aligned}$$

Completeness of morphisms

There is a one-to-one correspondence (an isomorphism) between container morphisms and natural transformation between containers.

Container are a 2-category

Identity

$$I = 1 \lhd 1$$

Composition

$$(S \lhd P) \circ (T \lhd Q) = (\Sigma s : S.P\, s \to T) \lhd (\lambda(s, f).\Sigma p : P\, s.Q\, (f\ p))$$

Monads

- A monad on **Set** is given by:
  Function on sets $M : \mathbf{Set} \to \mathbf{Set}$
  unit $\eta : \Pi_{X:\mathbf{Set}} X \to M\, X$
  bind $\_ \gg= \_ : \Pi_{X,Y:\mathbf{Set}} M\, X \to (X \to M\, Y) \to M\, Y$
  such that

$$\begin{aligned}(\eta\, x) \gg= f &= f\, x \\ m \gg= \eta &= m \\ (m \gg= f) \gg= g &= m \gg= (\lambda x. f\, x \gg= g)\end{aligned}$$

- Every monad is a functor with $M\, f = \lambda m. m \gg= \eta \circ f$
- Example List with

$$\begin{aligned}\eta\, x &= [x] \\ [] \gg= f &= [] \\ (a :: l) \gg= f &= (f\, a) +\!\!+ (l \gg= f)\end{aligned}$$

Monad (alternative definition)

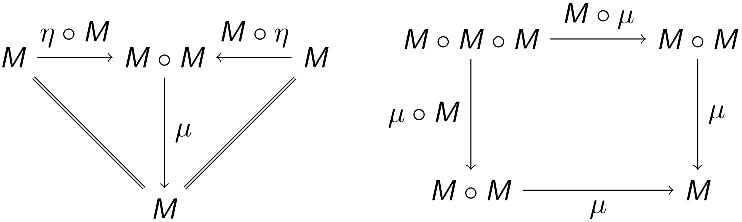
- A monad is given by:

  a functor $M : \mathbf{Set} \to \mathbf{Set}$,

  unit a natural transformation: $\eta : I \to M$

  join a natural transformation $\mu : M \circ M \to M$

  such that the following diagrams commute:



- $\mu : \Pi_{X:\mathbf{Set}} \mathrm{List}\,(\mathrm{List}\,X) \to \mathrm{List}\,X$

$$\begin{aligned} \mu\,[] &= [] \\ \mu\,l :: ll &= l +\!\!+ \mu\,ll \end{aligned}$$

Σ-universes

A Σ-universe is given by

$$\mathrm{U} : \textbf{Set} \qquad \iota : U$$
$$\mathrm{El} : \mathrm{U} \to \textbf{Set} \qquad \sigma : \Pi a : \mathrm{U}.(\mathrm{El}\, a \to \mathrm{U}) \to \mathrm{U}$$

such that

$$\begin{aligned} \mathrm{El}\,\iota &= 1 \\ \mathrm{El}\,(\sigma\, a\, b) &= \Sigma x : \mathrm{El}\, a.\mathrm{El}\,(b\, x) \end{aligned}$$

A Σ universe is univalent iff El is injective.

Example for a univalent Σ-universe

$$\mathrm{U} = \mathbb{N} \qquad \iota = 1$$
$$\mathrm{El} = \mathrm{Fin} \qquad \sigma\, n\, f = \Sigma_{i=0}^{n-1} f\, i$$

Monoidal structure

Given a Σ-universe we can define

$$a \otimes b = \sigma\, a\, (\lambda\_.b)$$

In any univalent Σ-universe the following holds:

$$\begin{aligned} a \otimes \iota &= a \\ \iota \otimes b &= b \\ \sigma\, a\, (\lambda x.\sigma\, (b\, x)\, (cx)) &= \sigma(\sigma\, a\, b)\, (\lambda x.c\, (\pi_0\, x)\, (\pi_1\, x)) \end{aligned}$$

. . . assuming a univalent metatheory.

From univalent Σ-universes to monadic containers

Proposition

Every univalent Σ-universe gives rise to a monadic container $\mathrm{U} \lhd \mathrm{El}$.

- ι provides the interpretation of η.
- σ provides the interpretation of μ.
- The monoidal structure is used to show that the diagrams commute.

Does every monadic container give rise to a univalent Σ-universe?

No. For example the reader monad $M\,X = \mathbb{N} \to X = 1 \lhd \mathbb{N}$ is a counterexample.

Lax Σ-universes

A lax Σ-universe is given by

$$\mathrm{U} : \mathbf{Set} \qquad \iota : U$$
$$\mathrm{El} : \mathrm{U} \to \mathbf{Set} \qquad \sigma : \Pi a : \mathrm{U}.(\mathrm{El}\, a \to \mathrm{U}) \to \mathrm{U}$$

with

$$\mathrm{un} : \mathrm{El}\,\iota \to 1$$
$$\mathrm{pr} : \mathrm{El}\,(\sigma\, a\, b) \to \Sigma x : \mathrm{El}\, a.\mathrm{El}\,(b\, x)$$

such that

$$a \otimes \iota = a$$
$$\iota \otimes b = b$$
$$\sigma\, a\,(\lambda x.\sigma\,(b\, x)\,(cx)) = \sigma(\sigma\, a\, b)\,(\lambda x.c\,(\pi_0\,(\mathrm{pr}\, x))\,(\pi_1\,(\mathrm{pr}\, x)))$$

Results

Proposition

Lax Σ-universes correspond exactly to monadic containers given by $\mathrm{U} \lhd \mathrm{El}$.

- Univalent Σ-universes correspond to cartesian monads.
- If we use a universe only closed under $\otimes$, we get applicative functors.
- The construction of a free monad over a container $S \lhd P$ is given by the free (univalent) Σ-universe over the family $P : S \to \mathbf{Set}$