An Introduction to Type Theory

Part 3

Tallinn, September 2003

with cartoons by Conor McBride

`http://www.cs.nott.ac.uk/~txa/tallinn/`

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University of Nottingham

The established social order

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:

terms

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types

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- Polymorphic types can be use to represent square matrices

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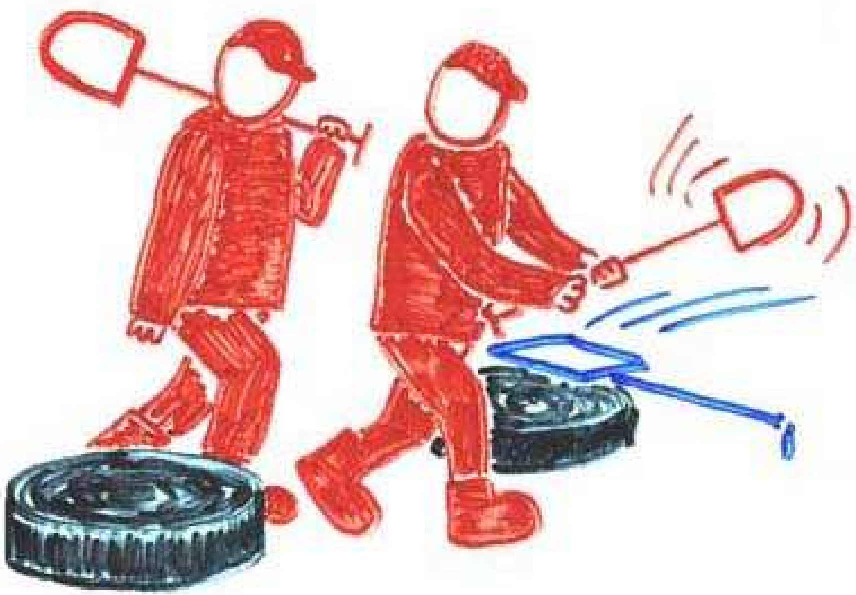

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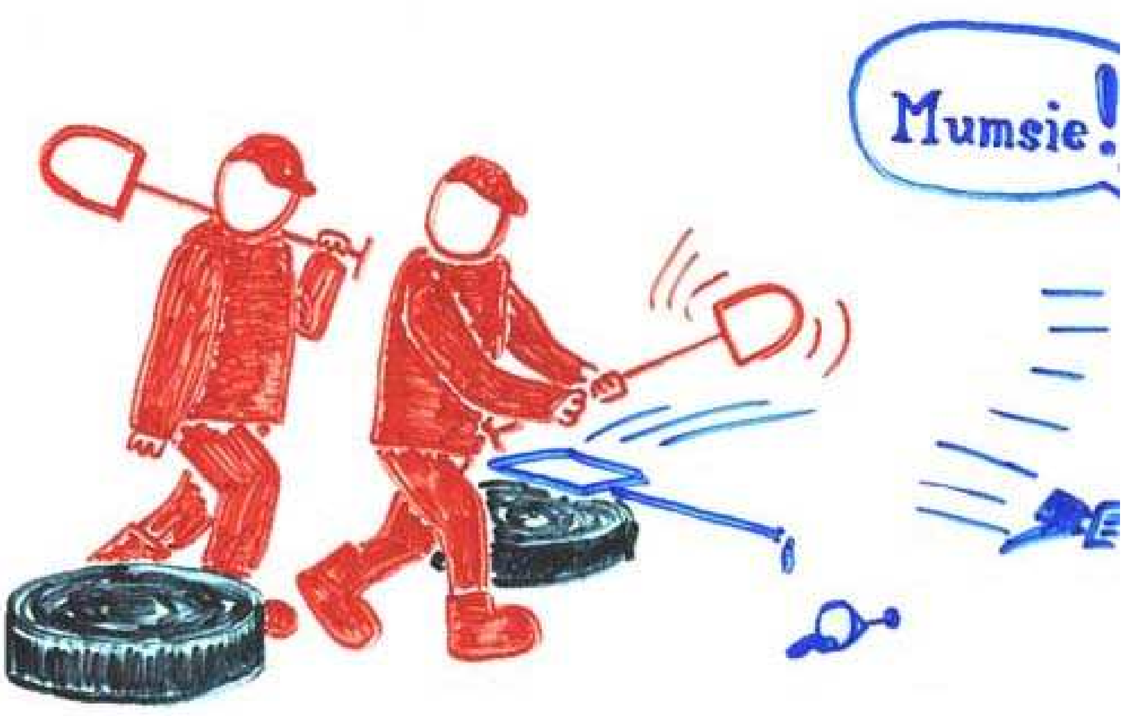

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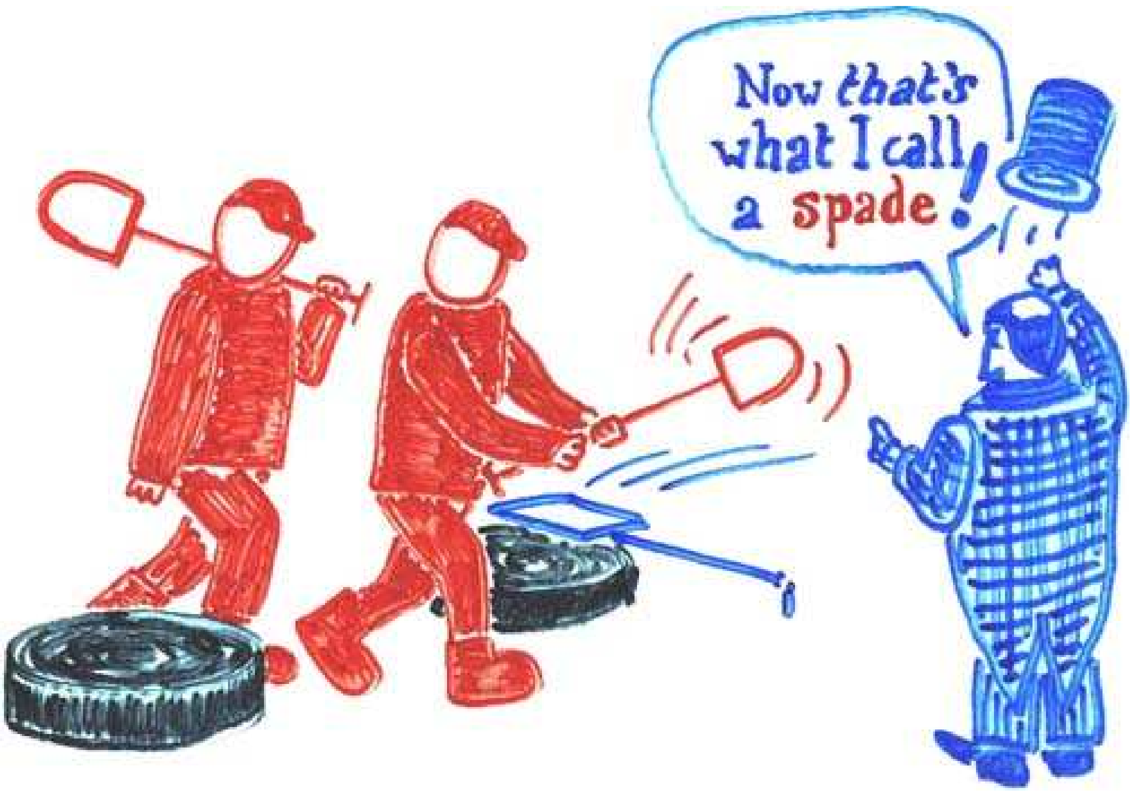

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$$\text{nth}\ A\ l\ n \mapsto ?$$

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- The function nth is partial

$$\mathrm{nth}\ 3\ [1, 2]$$

leads to a runtime error.

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  leads to a runtime error.
- Reason: The type of nth is not informative enough.

data types

$$\text{data } \frac{}{\mathrm{Nat} \in *} \text{ where } \quad \frac{}{0 \in \mathrm{Nat}} \quad \frac{n \in \mathrm{Nat}}{\mathrm{s}\ n \in \mathrm{Nat}}$$

$$\text{data } \frac{\mathrm{A} \in *}{\mathrm{List\ A} \in *} \text{ where}$$

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- nth is a total function.

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- nth is a total function.
- $\mathrm{nth}\ 3\ [1, 2]$ is not well-typed.

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\mathrm{verify}\ (\mathrm{s}\ n)\ 0 & \mapsto & \mathrm{just}\ 0' & & \\
\mathrm{verify}\ (\mathrm{s}\ n)\ (\mathrm{s}\ i) & \| & \mathrm{verify}\ n\ i & & \\
 & | & \mathrm{nothing} & \mapsto & \mathrm{nothing} \\
 & | & \mathrm{just}\ i & \mapsto & ?
\end{array}$$

Verify

- How can we use nth on lists of unknown length user input,...?

$$\text{let}\ \frac{n, i \in \mathrm{Nat}}{\mathrm{verify}\ n\ i \in \mathrm{Maybe}\ (\mathrm{Fin}\ n)}$$

$$\begin{array}{lcll}
\mathrm{verify}\ 0\ i & \mapsto & \mathrm{nothing} & \\
\mathrm{verify}\ (\mathrm{s}\ n)\ 0 & \mapsto & \mathrm{just}\ 0' & \\
\mathrm{verify}\ (\mathrm{s}\ n)\ (\mathrm{s}\ i) & \| & \mathrm{verify}\ n\ i & \\
 & | & \mathrm{nothing} & \mapsto\ \mathrm{nothing} \\
 & | & \mathrm{just}\ i & \mapsto\ \mathrm{just}\ (\mathrm{s}'\ i)
\end{array}$$

Going further

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- How is $\mathrm{just}\ j \equiv \mathrm{verify}\ n\ i$ related to j?

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- The type of verify is not informative enough for some of its potential applications.
- How is $\text{just } j \equiv \text{verify } n \; i$ related to j?
- When does verify return nothing?

verify **improved.**

$$\text{let}\ \frac{i \in \mathrm{Fin}\ \ n}{\mathrm{val}\ i \in \mathrm{Nat}} \qquad \begin{array}{lcl} \mathrm{val}\ 0' & \mapsto & 0 \\ \mathrm{val}\ (\mathrm{s}'\ i) & \mapsto & \mathrm{s}\ (\mathrm{val}\ i) \end{array}$$

verify **improved.**

let $\dfrac{i \in \mathrm{Fin}\ \ n}{\mathrm{val}\ i \in \mathrm{Nat}}$ $\quad \mathrm{val}\ 0' \mapsto 0$

$\qquad\qquad\qquad\qquad\quad \mathrm{val}\ (\mathrm{s}'\ i) \mapsto \mathrm{s}\ (\mathrm{val}\ i)$

data $\dfrac{n, i \in \mathrm{Nat}}{\mathrm{Bound}\ n\ i \in *}$ where $\dfrac{i \in \mathrm{Fin}\ n}{\mathrm{bound}\ n\ i \in \mathrm{Bound}\ n\ (\mathrm{val}\ i)}$ $\quad \dfrac{i, n \in \mathrm{Nat}}{\mathrm{tooBig}\ n\ i \in \mathrm{Bound}\ n\ (n{+}i)}$

verify **improved.**

$$\text{let}\ \frac{i \in \mathrm{Fin}\ n}{\mathrm{val}\ i \in \mathrm{Nat}} \qquad \begin{array}{lll} \mathrm{val}\ 0' & \mapsto & 0 \\ \mathrm{val}\ (\mathrm{s}'\ i) & \mapsto & \mathrm{s}\ (\mathrm{val}\ i) \end{array}$$

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$$\text{let}\ \frac{n, i \in \mathrm{Nat}}{\mathrm{verify}\ n\ i \in \mathrm{Bound}\ n\ i}$$

verify **improved.**

$$\text{let}\ \frac{i \in \mathrm{Fin}\ n}{\mathrm{val}\ i \in \mathrm{Nat}} \qquad \begin{array}{lcl} \mathrm{val}\ 0' & \mapsto & 0 \\ \mathrm{val}\ (\mathrm{s}'\ i) & \mapsto & \mathrm{s}\ (\mathrm{val}\ i) \end{array}$$

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$$\text{let}\ \frac{n, i \in \mathrm{Nat}}{\mathrm{verify}\ n\ i \in \mathrm{Bound}\ n\ i}$$

$$\mathrm{verify}\ n\ i \quad \mapsto \quad ?$$

verify **improved.**

$$\text{let}\ \frac{i \in \text{Fin}\ n}{\text{val}\ i \in \text{Nat}} \qquad \begin{array}{lcl} \text{val}\ 0' & \mapsto & 0 \\ \text{val}\ (\text{s}'\ i) & \mapsto & \text{s}\ (\text{val}\ i) \end{array}$$

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$$\begin{array}{lcl} \text{verify}\ 0\ i & \mapsto & ? \\ \text{verify}\ (\text{s}\ n)\ i & \mapsto & ? \end{array}$$

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$$\text{let}\ \frac{n, i \in \mathrm{Nat}}{\mathrm{verify}\ n\ i \in \mathrm{Bound}\ n\ i}$$

$$\begin{array}{lcl} \mathrm{verify}\ 0\ i & \mapsto & \mathrm{tooBig}\ 0\ i \\ \mathrm{verify}\ (\mathrm{s}\ n)\ 0 & \mapsto & \mathrm{bound}\ n\ 0' \\ \mathrm{verify}\ (\mathrm{s}\ n)\ (\mathrm{s}\ i) & \mapsto & ? \end{array}$$

verify **improved.**

$$\text{let}\ \frac{i \in \mathrm{Fin}\ n}{\mathrm{val}\ i \in \mathrm{Nat}} \qquad \begin{array}{lcl} \mathrm{val}\ 0' & \mapsto & 0 \\ \mathrm{val}\ (\mathrm{s}'\ i) & \mapsto & \mathrm{s}\ (\mathrm{val}\ i) \end{array}$$

$$\text{data}\ \frac{n, i \in \mathrm{Nat}}{\mathrm{Bound}\ n\ i \in *}\ \text{where}\ \frac{i \in \mathrm{Fin}\ n}{\mathrm{bound}\ n\ i \in \mathrm{Bound}\ n\ (\mathrm{val}\ i)} \quad \frac{i, n \in \mathrm{Nat}}{\mathrm{tooBig}\ n\ i \in \mathrm{Bound}\ n\ (n{+}i)}$$

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$$\begin{array}{lclcl} \mathrm{verify}\ 0\ i & \mapsto & \mathrm{tooBig}\ 0\ i & & \\ \mathrm{verify}\ (\mathrm{s}\ n)\ 0 & \mapsto & \mathrm{bound}\ n\ 0' & & \\ \mathrm{verify}\ (\mathrm{s}\ n)\ (\mathrm{s}\ i) & \| & \mathrm{verify}\ n\ i & \mapsto & ? \end{array}$$

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$$\text{let}\ \frac{n, i \in \text{Nat}}{\text{verify}\ n\ i \in \text{Bound}\ n\ i}$$

$$\begin{array}{lclcl} \text{verify}\ 0\ i & \mapsto & \text{tooBig}\ 0\ i & & \\ \text{verify}\ (\text{s}\ n)\ 0 & \mapsto & \text{bound}\ n\ 0' & & \\ \text{verify}\ (\text{s}\ n)\ (\text{s}\ i) & \| & \text{verify}\ n\ i & & \\ \text{verify}\ (\text{s}\ n)\ (\text{s}\ (n{+}i)) & | & \text{tooBig}\ n\ i & \mapsto & ? \\ \text{verify}\ (\text{s}\ n)\ (\text{val}\ i) & | & \text{bound}\ n\ i & \mapsto & ? \end{array}$$

verify **improved.**

$$\text{let}\ \frac{i \in \text{Fin}\ n}{\text{val}\ i \in \text{Nat}} \qquad \begin{array}{lcl} \text{val}\ 0' & \mapsto & 0 \\ \text{val}\ (\text{s}'\ i) & \mapsto & \text{s}\ (\text{val}\ i) \end{array}$$

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$$\text{let}\ \frac{n, i \in \mathrm{Nat}}{\mathrm{verify}\ n\ i \in \mathrm{Bound}\ n\ i}$$

$$\begin{array}{lclcl}
\mathrm{verify}\ 0\ i & \mapsto & \mathrm{tooBig}\ 0\ i & & \\
\mathrm{verify}\ (\mathrm{s}\ n)\ 0 & \mapsto & \mathrm{bound}\ n\ 0' & & \\
\mathrm{verify}\ (\mathrm{s}\ n)\ (\mathrm{s}\ i) & \| & \mathrm{verify}\ n\ i & & \\
\mathrm{verify}\ (\mathrm{s}\ n)\ (\mathrm{s}\ (n+i)) & | & \mathrm{tooBig}\ n\ i & \mapsto & \mathrm{tooBig}\ (\mathrm{s}\ n)\ i \\
\mathrm{verify}\ (\mathrm{s}\ n)\ (\mathrm{val}\ i) & | & \mathrm{bound}\ n\ i & \mapsto & \mathrm{bound}\ (\mathrm{s}\ n)\ (\mathrm{s}'\ i)
\end{array}$$

Definitional equality

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- The typing of verify depends on the equations:

$$\begin{aligned} 0+n &\equiv n \\ (\mathrm{s}\ m)+n &\equiv \mathrm{s}\ (m+n) \end{aligned}$$

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- This equations need to be true *definitionally*.

Definitional equality

- The typing of verify depends on the equations:

$$\begin{aligned} 0+n &\equiv n \\ (\mathrm{s}\ m)+n &\equiv \mathrm{s}\ (m+n) \end{aligned}$$

- This equations need to be true *definitionally*.
- If we need $n+0 = n$ we have to use propositional equality.

Propositional equality

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data $\dfrac{A \in * \quad a, b \in A}{a{=}b \in *}$ where $\dfrac{a \in A}{\mathrm{refl}\ a \in a{=}a}$

Propositional equality

$$\text{data}\ \frac{\mathrm{A} \in * \quad a, b \in \mathrm{A}}{a{=}b \in *}\ \text{where}\ \frac{a \in \mathrm{A}}{\mathrm{refl}\ a \in a{=}a}$$

$$\text{let}\ \frac{n \in \mathrm{Nat}}{\mathrm{add}_0\ n \in n{+}0{=}n}$$

$$\begin{array}{lllllll}
\mathrm{add}_0\ 0 & \mapsto & \mathrm{refl}\ 0 & & & & \\
\mathrm{add}_0\ (\mathrm{s}\ n) & \| & n + 0 & & & & \\
 & | & m & \| & \mathrm{add}_0\ n & & \\
 & | & n & | & \mathrm{refl} & \mapsto & \mathrm{refl}
\end{array}$$

Propositional equality

$$\text{data}\ \frac{\mathrm{A} \in * \quad a, b \in \mathrm{A}}{a = b \in *}\ \text{where}\ \frac{a \in \mathrm{A}}{\mathrm{refl}\ a \in a = a}$$

$$\text{let}\ \frac{n \in \mathrm{Nat}}{\mathrm{add}_0\ n \in n + 0 = n}$$

$$\begin{array}{llllll}
\mathrm{add}_0\ 0 & \mapsto & \mathrm{refl}\ 0 & & & \\
\mathrm{add}_0\ (\mathrm{s}\ n) & \| & n + 0 & & & \\
& | & m & \| & \mathrm{add}_0\ n & \\
& | & n & | & \mathrm{refl} & \mapsto\ \mathrm{refl}
\end{array}$$

$$\text{let}\ \frac{q \in a = b \quad \mathrm{P} \in \mathrm{A} \to * \quad x \in \mathrm{P}\ a}{\mathrm{subst}\ q\ \mathrm{P}\ x \in \mathrm{P}\ b} \qquad \mathrm{subst}\ (\mathrm{refl}\ a)\ \mathrm{P}\ x \mapsto x$$

Problems with $=$

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- Programs cluttered with coercions.
- Programming requires theorem proving.
- Equality on functions is not extensional, i.e.

$$\text{let } \frac{f, g \in \mathrm{A} \to \mathrm{B} \quad p \in \forall x \in \mathrm{A}. f\ x = g\ x}{\mathrm{ext}\ p \in f = g}$$

cannot be derived.

Solutions ?

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- In many cases the need for propositional equality can be avoided.
- DML shows that many equalities needed in programming can be proven automatically.
  Proposal: Integrate an extensible constraint prover into the elaboration process.
- The problem with extensional equality can be overcome using a different approach to equality.

Eval

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- Implement an evaluator for
  a simply typed object language.

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  a simply typed object language.
- We use type-checking to avoid run-time errors.
- First we implement a simply typed version.
- Then a dependently typed version,
  exploiting the *verify pattern*.

The object language

data $\dfrac{}{\mathrm{Val} \in *}$ where $\dfrac{n \in \mathrm{Nat}}{\mathrm{vnat}\ n \in \mathrm{Val}}$ $\dfrac{b \in \mathrm{Bool}}{\mathrm{vbool}\ b \in \mathrm{Val}}$

The object language

$$\text{data } \frac{}{\mathrm{Val} \in *} \text{ where } \quad \frac{n \in \mathrm{Nat}}{\mathrm{vnat}\ n \in \mathrm{Val}} \quad \frac{b \in \mathrm{Bool}}{\mathrm{vbool}\ b \in \mathrm{Val}}$$

$$\text{data } \frac{}{\mathrm{Tm} \in *} \text{ where}$$

$$\frac{v \in \mathrm{Val}}{\mathrm{tval}\ v \in \mathrm{Tm}} \quad \frac{t, u_0, u_1 \in \mathrm{Tm}}{\mathrm{tif}\ t\ u_0\ u_1 \in \mathrm{Tm}} \quad \frac{t, u, \in \mathrm{Tm}}{\mathrm{tadd}\ t\ u \in \mathrm{Tm}}$$

Object types

data $\overline{\mathrm{Ty} \in *}$ where $\quad \overline{\mathrm{nat} \in \mathrm{Ty}} \quad \overline{\mathrm{bool} \in \mathrm{Ty}}$

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let $\dfrac{t \in \mathrm{Tm}}{\mathrm{verify}\ t \in \mathrm{Maybe}\ \ \mathrm{Ty}}$ $\ldots$

Eval — simply typed

$$\text{let } \frac{t \in \mathrm{Tm}}{\mathrm{eval}\ t \in \mathrm{Val}}$$

Eval — simply typed

$$\text{let}\ \frac{t \in \mathrm{Tm}}{\mathrm{eval}\ t \in \mathrm{Val}}$$

$$
\begin{array}{llllllll}
\mathrm{eval}\ (\mathrm{tval}\ v) & \mapsto & v \\
\mathrm{eval}\ (\mathrm{tif}\ t\ u_0\ u_1) & \| & \mathrm{eval}\ t \\
 & | & \mathrm{vnat}\ n & \mapsto & ? \\
 & | & \mathrm{vbool\ true} & \mapsto & \mathrm{eval}\ u_0 \\
 & | & \mathrm{vbool\ false} & \mapsto & \mathrm{eval}\ u_1 \\
\mathrm{eval}\ (\mathrm{tadd}\ t\ u) & \| & \mathrm{eval}\ t \\
 & | & \mathrm{vnat}\ m & \| & \mathrm{eval}\ u \\
 & & & | & \mathrm{vnat}\ n & \mapsto & \mathrm{vnat}(m+n) \\
 & & & | & \mathrm{vbool}\ b & \mapsto & ? \\
 & | & \mathrm{vbool}\ b & \mapsto & ?
\end{array}
$$

Safe eval ?

$$\text{let } \frac{t \in \mathrm{Tm}}{\mathrm{seval}\ t \in \mathrm{Maybe}\ \ \mathrm{Val}}$$

Safe eval ?

$$\text{let } \frac{t \in \mathrm{Tm}}{\mathrm{seval}\ t \in \mathrm{Maybe\ \ Val}}$$

$$\begin{array}{lllll} \mathrm{seval}\ t & \| & \mathrm{verify}\ t & & \\ & | & \mathrm{just}\ u & \mapsto & \mathrm{just}\ (\mathrm{eval}\ u) \\ & | & \mathrm{nothing} & \mapsto & \mathrm{nothing} \end{array}$$

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- seval is inefficient:
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  - eval checks the tags

Object language using dependent types

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$$\text{data } \frac{a \in \mathrm{Ty}}{\mathrm{TVal}\ a \in *} \text{ where } \qquad \frac{n \in \mathrm{Nat}}{\mathrm{vnat}\ n \in \mathrm{TVal}\ \mathrm{nat}} \qquad \frac{b \in \mathrm{Bool}}{\mathrm{vbool}\ b \in \mathrm{TVal}\ \mathrm{bool}}$$

Object language using dependent types

$$\text{data}\ \frac{a \in \mathrm{Ty}}{\mathrm{TVal}\ a \in *}\ \text{where}\qquad \frac{n \in \mathrm{Nat}}{\mathrm{vnat}\ n \in \mathrm{TVal}\ \mathrm{nat}}\qquad \frac{b \in \mathrm{Bool}}{\mathrm{vbool}\ b \in \mathrm{TVal}\ \mathrm{bool}}$$

$$\text{data}\ \frac{a \in *}{\mathrm{TTm}\ a \in *}\ \text{where}$$

Object language using dependent types

$$\text{data}\ \frac{a \in \mathrm{Ty}}{\mathrm{TVal}\ a \in *}\ \text{where} \qquad \frac{n \in \mathrm{Nat}}{\mathrm{vnat}\ n \in \mathrm{TVal}\ \mathrm{nat}} \qquad \frac{b \in \mathrm{Bool}}{\mathrm{vbool}\ b \in \mathrm{TVal}\ \mathrm{bool}}$$

$$\text{data}\ \frac{a \in *}{\mathrm{TTm}\ a \in *}\ \text{where}$$

$$\frac{v \in \mathrm{TVal}\ a}{\mathrm{tval}\ v \in \mathrm{TTm}\ a} \qquad \frac{t \in \mathrm{TTm}\ \mathrm{bool} \quad u_0, u_1 \in \mathrm{TTm}\ a}{\mathrm{tif}\ t\ u_0\ u_1 \in \mathrm{TTm}\ a} \qquad \frac{t, u, \in \mathrm{TTm}\ \mathrm{nat}}{\mathrm{tadd}\ t\ u \in \mathrm{TTm}\ \mathrm{nat}}$$

Eval — dependently

$$\text{let}\ \frac{t \in \mathrm{TTm}\ a}{\mathrm{eval}\ t \in \mathrm{Val}\ a}$$

Eval — dependently

$$\text{let } \frac{t \in \mathrm{TTm}\ a}{\mathrm{eval}\ t \in \mathrm{Val}\ a}$$

$$
\begin{array}{llllllll}
\mathrm{eval}\ (\mathrm{tval}\ v) & \mapsto & v \\
\mathrm{eval}\ (\mathrm{tif}\ t\ u_0\ u_1) & \| & \mathrm{eval}\ t \\
 & | & \mathrm{vbool\ true} & \mapsto & \mathrm{eval}\ u_0 \\
 & | & \mathrm{vbool\ false} & \mapsto & \mathrm{eval}\ u_1 \\
\mathrm{eval}\ (\mathrm{tadd}\ t\ u) & \| & \mathrm{eval}\ t \\
 & | & \mathrm{vnat}\ m & \| & \mathrm{eval}\ u \\
 & & & | & \mathrm{vnat}\ n & \mapsto & \mathrm{vnat}(m+n)
\end{array}
$$

Safe eval !

Safe eval !

$$\text{let}\ \frac{t \in \mathrm{TTm}\ a}{\mathrm{strip}\ t \in \mathrm{Tm}} \qquad \ldots$$

Safe eval !

$$\text{let}\ \frac{t \in \text{TTm}\ a}{\text{strip}\ t \in \text{Tm}} \quad \cdots$$

$$\text{data}\ \frac{t \in \text{Tm}}{\text{Verify}\ t \in \text{Ty}}\ \text{where} \quad \frac{}{\text{error} \in \text{Verify}\ t} \quad \frac{t \in \text{TTm}\ a}{\text{ok}\ t \in \text{Verify}\ (\text{strip}\ t)}$$

$$\text{let}\ \frac{t \in \text{Tm}}{\text{verify}\ t \in \text{Verify}\ t} \quad \cdots$$

$$\text{let}\ \frac{t \in \text{Tm}}{\text{seval}\ t \in \text{Maybe}\ (\Sigma_{a \in \text{Ty}} \text{Val}\ a)}$$

$$\begin{array}{llcl}
\text{seval}\ t & \| & \text{verify}\ t & \\
 & | & \text{just}\ u & \mapsto \quad \text{just}\ (\text{eval}\ u) \\
 & | & \text{error} & \mapsto \quad \text{nothing}
\end{array}$$

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Safe eval !

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- seval is efficient:
  - No tags at runtime.
  - No checking.

Generic equality

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- Implement a generic equality function for non nested, concrete data types

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- Usually requires a language extension (Generic Haskell)

Generic equality

- Implement a generic equality function for non nested, concrete data types
- Usually requires a language extension (Generic Haskell)
- Topic developed further in our (Conor and me) paper: *Generic programming within dependently typed programming* Working Conference on Generic Programming 2002

Codes and data

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$$\text{data } \frac{}{\mathrm{Ty} \in *} \text{ where}$$

Codes and data

$$\text{data} \; \frac{}{\mathrm{Ty} \in *} \; \text{where}$$

$$\frac{}{\mathrm{unit} \in \mathrm{Ty}} \qquad \frac{a, b \in \mathrm{Ty}}{\mathrm{prod}\ a\ b \in \mathrm{Ty}} \qquad \frac{a, b \in \mathrm{Ty}}{\mathrm{sum}\ a\ b \in \mathrm{Ty}} \qquad \frac{}{\mathrm{rec} \in \mathrm{Ty}}$$

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$$\text{let } \frac{n \in \mathrm{Data\ nat}}{\mathrm{succ}\ n \in \mathrm{Data\ nat}} \quad \mathrm{succ}\ n \mapsto \mathrm{inr}\ (\mathrm{in}\ n)$$

Generic equality

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$$\begin{array}{lllll}
\mathrm{eq} & \mathrm{void} & \mathrm{void} & \mapsto & \mathrm{true} \\
\mathrm{eq} & (\mathrm{pair}\ x\ y) & (\mathrm{pair}\ x'\ y') & \mapsto & (\mathrm{eq}\ x\ x')\ \&\&\ (\mathrm{eq}\ y\ y') \\
\mathrm{eq} & (\mathrm{inl}\ x) & (\mathrm{inl}\ x') & \mapsto & \mathrm{eq}\ x\ x' \\
\mathrm{eq} & (\mathrm{inl}\ x) & (\mathrm{inr}\ y) & \mapsto & \mathrm{false} \\
\mathrm{eq} & (\mathrm{inr}\ y) & (\mathrm{inl}\ x) & \mapsto & \mathrm{false} \\
\mathrm{eq} & (\mathrm{inr}\ y) & (\mathrm{inr}\ y') & \mapsto & \mathrm{eq}\ y\ y' \\
\mathrm{eq} & (\mathrm{in}\ x) & (\mathrm{in}\ x') & \mapsto & \mathrm{eq}\ x\ x'
\end{array}$$

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- Easier to reason about

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- Inductive families have to be supported
- Type inference is generalized by elaboration
  Extensible elaboration ?
- Programs are constructed interactively,
  starting with the type as a partial specification