Separability in classical lambda calculi

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Plan

- Böhm's theorem in lambda calculus
- Böhm's theorem in classical setting
- Failure of separability in $\lambda\mu$
- Restoration of separability in $\Lambda\mu$
- Extension of Parigot's $\lambda\mu$
- Discussion on separability in symmetric $\overline{\lambda}\mu\widetilde{\mu}$

Böhm's theorem in lambda calculus

BT1 *If* M *and* N *are two different* $\beta\eta$ *normal forms, then there is a context* $C[\ ]$ *such that*

* $C[M]$ *reduces to* x
* $C[N]$ *reduces to* y

$$C_1[\ ] = (\lambda xy.C[\ ])I\Omega$$

$$C_1[M] = I \qquad\qquad C_1[N] = \Omega$$

BT2 *If* M *and* N *are two different* $\beta\eta$ *normal forms, then there is a context* $C[\ ]$ *such that* $C[M]$ *reduces to a normal form, whereas* $C[N]$ *is nonterminating.*

Consequences of BT

Maximality of consistent equality

P and Q having different $\beta\eta$ nf ($P = Q$ cannot be proved in $\lambda_{\beta\eta}$)

$*$ By BT, $\lambda_{\beta\eta} + P = Q$ is inconsistent

$*$ By BT, $\lambda_{\beta\eta}$ is maximal consistent for normalisable terms

Observational equivalence

$M,\ N$ observationally equiv. *iff* $C[M]$has a nf $\Leftrightarrow$ $C[N]$has a nf

$*$ By BT, observational equivalence for normalisable terms coincides with $\beta\eta$-equivalence

$*$ Proof of BT is a refutation procedure for observational equivalence

Failure of separability in Parigot's $\lambda\mu$

$$M \quad ::= \quad x \parallel \lambda x.M \parallel M\,M \parallel \mu\alpha.c$$ (unnamed terms)

$$c \quad ::= \quad [\alpha]M$$ (named terms, or commands)

Failure of separability in Parigot's $\lambda\mu$

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Reduction rules

$$(\beta) \quad (\lambda x.M)\,N \;\rightarrow\; M[N/x]$$

$$(\mu_{app}) \quad (\mu\alpha.c)\,N \;\rightarrow\; \mu\alpha.c[[\alpha](\Box\,N))/\alpha]$$

$$(\mu_{var}) \quad [\beta]\mu\alpha.c \;\rightarrow\; c[\beta/\alpha]$$

$$(\eta_\mu) \quad \mu\alpha.[\alpha]M \;\rightarrow\; M \qquad \text{if } \alpha \text{ not free in } M$$

$$(\eta) \quad \lambda x.(M\,x) \;\rightarrow\; M \qquad \text{if } x \text{ not free in } M$$

David and Py [2001]: failure of separability in $\lambda\mu$

Restoration of separability in $\Lambda\mu$

de Groote, Ong, Selinger, Saurin[2005] - alternative syntax

$\Lambda\mu$-calculus

$$M, c \quad ::= \quad x \parallel \lambda x.M \parallel M\ M \parallel \mu\alpha.c \parallel [\alpha]M$$

Separability proposition

If M and N are not equal in $\Lambda\mu$-calculus their observational behaviour is separable, i.e., for distinct fresh variables x and y, there is a context $M_1\ \ldots\ M_n$, such that $M\ M_1\ \ldots\ M_n = x$ and $N\ M_1\ \ldots\ M_n = y$.

Reasons

The difference lies in the rule μ_{var} which in the case of Parigot's $\lambda\mu$-calculus can only occur in a configuration of the form

$$M(\mu\gamma.[\beta]\mu\alpha.c) \quad \rightarrow \quad M(\mu\gamma.c[\beta/\alpha])$$

while in the case of Saurin's $\lambda\mu$-calculus, it can also occur in a configuration of the form

$$M\,([\beta]\mu\alpha.c) \quad \rightarrow \quad M\,(c[\beta/\alpha])$$

so that the computational effect of any $\mu\alpha.c$ can be cancelled if we succeed in putting it in a context of the form $[\beta]\Box$. This last property is actually the reason why Saurin's completeness theorem works.

Translating $\Lambda\mu$ into Parigot's $\lambda\mu$

$\lambda\mu_{\mathsf{tp}}$ extension of $\lambda\mu$:

$*$ continuation constant tp

$*$ dynamic binder $\widehat{\mu}\mathsf{tp}$

Naive interpretation $\Lambda\mu$ in $\lambda\mu$

$*$ $\mu\alpha.M$ as a $\lambda\mu$-term $\mu\alpha.[\mathsf{tp}]M$

$*$ $[\alpha]M$ as a $\lambda\mu$-term $\widehat{\mu}\mathsf{tp}.[\alpha]M$

Formal embedding: $\Pi : \Lambda\mu \mapsto \lambda\mu_{\mathsf{tp}}$

$$
\begin{array}{lcl}
\Pi(x) & = & x \\
\Pi(\lambda x.M) & = & \lambda x.\Pi(M) \\
\Pi(M\,N) & = & \Pi(M)\,\Pi(N) \\
\Pi(\mu\alpha.M) & = & \mu\alpha.[\mathsf{tp}]\Pi(M) \\
\Pi([\alpha]M) & = & \widehat{\mu}\mathsf{tp}.[\alpha]\Pi(M)
\end{array}
$$

Rules of $\lambda\mu_{\mathsf{tp}}$

$$
\begin{array}{lll}
(\mu_{\mathsf{tp}}) & [\mathsf{tp}]\widehat{\mu}\mathsf{tp}.c & \longrightarrow \quad c \\
(\eta_{\mathsf{tp}}) & \widehat{\mu}\mathsf{tp}.[\mathsf{tp}]M & \longrightarrow \quad M
\end{array}
$$

If $M = N$ in $\Lambda\mu$ then $\Pi(M) = \Pi(N)$ in $\lambda\mu_{\mathsf{tp}}$-calculus.

Translating $\lambda\mu_{\mathsf{tp}}$ back into $\Lambda\mu$

In order to show the equivalence of $\lambda\mu_{\mathsf{tp}}$ and $\Lambda\mu$

Formal embedding $\Sigma : \lambda\mu_{\mathsf{tp}} \mapsto \Lambda\mu$

$$
\begin{array}{lcll}
\Sigma(x) & = & x & \\
\Sigma(\lambda x.M) & = & \lambda x.\Sigma(M) & \\
\Sigma(M\,N) & = & \Sigma(M)\,\Sigma(N) & \\
\Sigma(\mu\alpha.[\beta]M) & = & \mu\alpha.([\beta]\Sigma(M)) & \text{if } \beta \text{ distinct of tp} \\
\Sigma(\mu\alpha.[\mathsf{tp}]M) & = & \mu\alpha.(\Sigma(M)) & \\
\Sigma(\widehat{\mu}\mathsf{tp}.[\alpha]M) & = & [\alpha]\Sigma(M) & \text{if } \alpha \text{ distinct of tp} \\
\Sigma(\widehat{\mu}\mathsf{tp}.[\mathsf{tp}]M) & = & \Sigma(M) &
\end{array}
$$

Separability in $\lambda\mu_{\mathbf{tp}}$

If $M = N$ in $\lambda\mu_{\mathbf{tp}}$ then $\Sigma(M) = \Sigma(N)$ in $\Lambda\mu$-calculus.

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Separability in $\lambda\mu_{\mathbf{tp}}$

If $M = N$ in $\lambda\mu_{\mathbf{tp}}$ then $\Sigma(M) = \Sigma(N)$ in $\Lambda\mu$-calculus.

If $M = N$ in $\Lambda\mu$ then $\Pi(M) = \Pi(N)$ in $\lambda\mu_{\mathbf{tp}}$-calculus.

$\lambda\mu_{\mathbf{tp}}$ is observationally complete

for any M and N not equal there exists a context $M_1 \ldots M_n$, such that $M\, M_1 \ldots M_n = x$ and $N\, M_1 \ldots M_n = y$ for x and y being arbitrary fresh variables.

On separability in $\overline{\lambda}\mu\widetilde{\mu}$

Equality in the non-deterministic version is inconsistent.

Focus on $\overline{\lambda}\mu_n$-calculus, the "canonical" CBN subsystem of $\overline{\lambda}\mu\widetilde{\mu}$.

$\overline{\lambda}\mu_n$-calculus

$$
\begin{array}{lcl}
M & ::= & x \parallel \lambda x.M \parallel \mu\alpha.c \\
E & ::= & \alpha \parallel M \bullet E \\
c & ::= & \langle\ M \ \parallel \ E\ \rangle
\end{array}
$$

Reduction rules (η-rules omitted)

$$
\begin{array}{llcl}
(\rightarrow^{\beta}) & \langle\ (\lambda x.M) \ \parallel \ N \bullet E\ \rangle & \rightarrow & \langle\ M[N/x] \ \parallel \ E\ \rangle \\
(\mu_n) & \langle\ \mu\alpha.c \ \parallel \ E\ \rangle & \rightarrow & c[E/\alpha]
\end{array}
$$

Mutual embedding of $\Lambda\mu$ and $\overline{\lambda}\mu_n$tp

Extend $\overline{\lambda}\mu_n$ to $\overline{\lambda}\mu_n$tp by adding tp, $\widehat{\mu}$tp.c and rules (μ_{tp}), (η_{tp})

Embedding $^{>}$: $\Lambda\mu \mapsto \overline{\lambda}\mu_n$tp

$$
\begin{array}{lcl}
(x)^{>} & = & x \\
(\lambda x.M)^{>} & = & \lambda x.(M)^{>} \\
(M\,N)^{>} & = & \mu\alpha.\langle\,(M)^{>} \;\|\; (N)^{>} \bullet \alpha\,\rangle \quad \alpha \text{ is fresh} \\
(\mu\alpha.M)^{>} & = & \mu\alpha.\langle\,(M)^{>} \;\|\; \text{tp}\,\rangle \\
([\alpha]M)^{>} & = & \widehat{\mu}\text{tp}\langle\,(M)^{>} \;\|\; \alpha\,\rangle
\end{array}
$$

Embedding $_>$: $\overline{\lambda}\mu_n\mathsf{tp} \mapsto \Lambda\mu$

$$
\begin{array}{lcl}
(x)_> & = & x \\
(\lambda x.M)_> & = & \lambda x.(M)_> \\
(\mu\alpha.c)_> & = & \mu\alpha.(c)_> \\
(\widehat{\mu}\mathsf{tp}.c)_> & = & (c)_> \\
(\alpha)_> \ \{O\} & = & [\alpha]O \\
(\mathsf{tp})_> \ \{O\} & = & O \\
(M \bullet E)_> \{O\} & = & (E)_> \{O(M)_>\} \\
(\langle \ M \ \ \| \ \ E \ \rangle)_> & = & (E)_> \{(M)_>\}
\end{array}
$$

Conjectures on separability in $\lambda\mu_{\mathbf{tp}}$

Conjecture 1 *If* $M = N$ *in* $\Lambda\mu$*-calculus then* $(M)^{>} = (N)^{>}$ *in* $\overline{\lambda}\mu_{ntp}$*-calculus.*

Conjecture 2 *If* $M = N$ *in* $\overline{\lambda}\mu_{ntp}$*-calculus then* $(M)_{>} = (N)_{>}$ *in* $\Lambda\mu$*-calculus.*

Ongoing and future work

* Investigation of Böhm's theorem in the simply typed $\lambda\mu$-calculi and $\overline{\lambda}\mu\widetilde{\mu}$-calculus based on the Böhm's theorem in the simply typed λ-calculs by Došen and Petrić[2001], Statman[1982], Simpson[1995].
* Investigation of the Böhm's theorem for CBV $\lambda\mu$-calculus?
* Relation of separability results of CBV $\lambda\mu$-calculus and Böhm's theorem in λ_v, CBV λ-calculus, which is a result by Paolini[2001].