Truth values algebras and normalization

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Models for constructive logic

$0, 1\}$-models: in a given model the continuum hypothesis is either valid or not, excluded middle valid

Replace $\{0, 1\}$ by an arbitrary boolean algebra, e.g. $\mathcal{P}(\{\pi, e\})$

Better: the continuum hypothesis may have an intermediate truth value but excluded middle still valid

Replace $\{0, 1\}$ it by a Heyting algebra: Completeness

What is a Heyting algebra?

Like a boolean algebra: an ordered set

with *lub* (for $\vee$, $\exists$ and $\top$) and *glb* (for $\wedge$, $\forall$ and $\bot$)

But no complement ($a \cap \overline{a} = Min$, $a \cup \overline{a} = Max$)

Instead a weak complement

$$a \leq \overline{b} \text{ iff } a \cap b = Min$$

verifies $a \cap \overline{a} = Min$ but not always $a \cup \overline{a} = Max$

Relative weak complement: $a \leq \overline{b}^{c}$ iff $a \cap b \leq c$

What are the key features of Heyting algebras?

Order? Not really

Soundness:

Definition of $\tilde{\wedge}$, $\tilde{\Rightarrow}$, ...

$(a \;\tilde{\wedge}\; b) \;\tilde{\Rightarrow}\; a$ always in $\{Max\}$

... (closure by deduction rules)

Thus provable formulae valid (induction over proof structure)

Completeness:

A theory Γ, can build a model that validates exactly $Thm(\Gamma)$

Truth values algebras

$\mathcal{B}, \mathcal{B}^+, \tilde{\wedge}, \tilde{\Rightarrow}, ...\rangle$

$a \mathbin{\tilde{\wedge}} b) \mathbin{\tilde{\Rightarrow}} a$ always in $\mathcal{B}^+$, ...

Generalizes Heyting algebras

Completeness for free

Soundness: the closure conditions on $\mathcal{B}^+$ are (the weakest) sufficient conditions

An alternative presentation (suggested by Thierry Coquand)

rom a truth value algebra, we can define a relation

$$a \leq b \text{ iff } a \mathrel{\tilde{\Rightarrow}} b \in \mathcal{B}^+$$

erifies all the properties of Heyting algebras except one:
ntisymmetry

 simple remark: antisymmetry is useless in the definition of
eyting algebras, it can be dropped

nly antisymmetry can: otherwise no closure by deduction rule

A drawback of Heyting algebras

Due to antisymmetry, in a Heyting algebra

$$A \Leftrightarrow B \text{ valid}$$

$$\text{iff}$$

$$[\![A]\!] = [\![B]\!]$$

Truth values are denotations not meanings

n deduction modulo and in type theories: no semantic difference between $A \Leftrightarrow B$ and $A \equiv B$

Not in truth values algebras

Complete truth values algebras

Add an order $\sqsubseteq$ (need not be $a \mathrel{\tilde{\Rightarrow}} b \in \mathcal{B}^+$)

$\mathcal{B}^+$ is upward closed,

Connectors and quantifiers are (anti)-monotonous

Every subset of $\mathcal{B}$ has a least upper bound

Soundness: for free

Completeness: complete Heyting algebras

Soundness

$\mathcal{B}$-model $\mathcal{M} = \langle \mathcal{M}, \mathcal{B}, \hat{f}_i, \hat{P}_j \rangle$

f $\mathcal{T}$ has a $\mathcal{B}$-model for some $\mathcal{B}$ then $\mathcal{T}$ consistent

Extends to deduction modulo:

$A \longrightarrow B$ valid in $\mathcal{M}$ iff $[\![A]\!] = [\![B]\!]$

Super-consistency

f $\mathcal{R}$ has a $\mathcal{B}$-model for some $\mathcal{B}$ then it is consistent

f $\mathcal{R}$ has a $\mathcal{B}$-model for all $\mathcal{B}$ then it is called super-consistent

Examples

Arithmetic, simple type theory, ... are super-consistent

an arbitrary truth values algebra $\mathcal{B}$

$$\mathcal{M}_\iota = \{0\}$$

$$\mathcal{M}_o = \mathcal{B}$$

$$\mathcal{M}_{T \to U} = \mathcal{M}_U^{\mathcal{M}_T}$$

$$[\![\alpha]\!](a, b) = a(b) \quad [\![\varepsilon]\!](a) = a$$

$$[\![S]\!]_{T,U,V} = a \mapsto (b \mapsto (c \mapsto a(c)(b(c)))) \quad [\![K]\!]_{T,U} = a \mapsto (b \mapsto a)$$

$$[\![\dot{\top}]\!] = \tilde{\top} \quad [\![\dot{\bot}]\!] = \tilde{\bot} \quad [\![\dot{\Rightarrow}]\!] = \tilde{\Rightarrow} \quad [\![\dot{\wedge}]\!] = \tilde{\wedge} \quad [\![\dot{\vee}]\!] = \tilde{\vee}$$

$$[\![\dot{\forall}_T]\!] = a \mapsto \tilde{\forall}_T(Range(a)) \quad [\![\dot{\exists}_T]\!] = a \mapsto \tilde{\exists}_T(Range(a))$$

The theorem

f $\mathcal{T}, \mathcal{R}$ super-consistent theory in deduction modulo

hen all proofs in $\mathcal{T}, \mathcal{R}$ strongly normalize

The truth values algebra $\mathcal{C}$ of reducibility candidates

Reducibility candidates are sets of proofs (with some conditions)

$a \tilde{\Rightarrow} b$: set of terminating proof-terms π s.t. if π reduces to $\lambda\alpha\ \pi_1$ then for every π' in a, $(\pi'/\alpha)\pi_1$ in b

$\tilde{\forall}\ A$: set of terminating proof-terms π s.t. if π reduces to $\lambda x\ \pi_1$ then for every term t and every element a of A, $(t/x)\pi_1$ in a

...

$\mathcal{C}^+$: set of candidates containing a closed proof-term

$a \leq b$ if $a \tilde{\Rightarrow} b$ contains a closed term

$a \sqsubseteq b$ if a subset of b

Not a Heyting algebra

$a \leq b$ if $a \tilde{\Rightarrow} b$ contains a closed term

Not antisymmetric

$\tilde{\top} \leq (\tilde{\top} \tilde{\Rightarrow} \tilde{\top})$ $\qquad$ $\tilde{\top} \tilde{\Rightarrow} (\tilde{\top} \tilde{\Rightarrow} \tilde{\top})$ contains a closed term

$(\tilde{\top} \tilde{\Rightarrow} \tilde{\top}) \leq \tilde{\top}$ $\qquad$ $(\tilde{\top} \tilde{\Rightarrow} \tilde{\top}) \tilde{\Rightarrow} \tilde{\top}$ contains a closed term

But $(\tilde{\top} \tilde{\Rightarrow} \tilde{\top}) \neq \tilde{\top}$

Prove normalization
without knowing what a reducibility candidate is

To prove normalization: prove super-consistency

No need to understand the notion of reducibility candidates

Candidates hidden in the proof that super-consistency implies normalization

Explains why the the flavor of candidates does not matter

Candidates can be abstracted away