

# Simple proofs of strong cut elimination

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## Goal

1. logic: intuitionistic implication  $\supset$

2.  $SCE = SN + \epsilon$

3. More precisely:

$$SN \text{ for } \lambda^G = (\text{properties of } \lambda) + \epsilon$$

where

$$\begin{aligned}\lambda^G &= \text{neat } \lambda\text{-calculus for sequent calculus} \\ \epsilon &= \text{small combinatorial overhead}\end{aligned}$$

4. Plan:

(a) indicate properties of  $\lambda$

(b) advertise  $\lambda^G$

(c) do  $\epsilon$

## Properties of $\lambda$

1. Strong normalisation

$$M \text{ typable} \Rightarrow M \in SN_{\beta}.$$

2. Fundamental lemma of perpetuity

$$\left. \begin{array}{l} (\lambda x.M)N \notin SN_{\beta} \\ x \notin FV(M) \Rightarrow N \in SN_{\beta} \end{array} \right\} \Rightarrow [N/x]M \notin SN_{\beta}$$

Stronger version of 2.

$$x \in FV(M) \Rightarrow \|(\lambda x.M)N\|_{\beta} \leq \|[N/x]M\|_{\beta} + 1$$

$$x \notin FV(M) \Rightarrow \|(\lambda x.M)N\|_{\beta} \leq \|M\|_{\beta} + \|N\|_{\beta} + 1$$

## Sequent calculus/ $\lambda^G$ -calculus (1)

### Expressions

$$\begin{array}{ll} \text{(Terms)} & t, u, v ::= x \mid \lambda x. t \mid tk \\ \text{(Contexts)} & k ::= (x)v \mid u :: k \end{array}$$

Sequents     $\Gamma \vdash t : A$     and     $\Gamma ; A \vdash k : B$

### Typing / Inference rules

$$\frac{}{\Gamma, x : A \vdash x : A} Axiom \quad \frac{\Gamma, x : A \vdash v : B}{\Gamma ; A \vdash (x)v : B} Selection$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \supset B} Right$$

$$\frac{\Gamma \vdash u : A \quad \Gamma ; B \vdash k : C}{\Gamma ; A \supset B \vdash u :: k : C} Left$$

$$\frac{\Gamma \vdash t : A \quad \Gamma ; A \vdash k : B}{\Gamma \vdash tk : B} Cut$$

## Sequent calculus/ $\lambda^G$ -calculus (2)

Cuts:  $tk = t(u_1 :: \dots :: u_m(x)v)$ ,  $m \geq 0$

$m = 1$     $t(u :: (x)v)$    generalized application  
 $m = 0$                $t(x)v$    “explicit” substitution

N.B.  $t(x)v$  is written  $\langle t/x \rangle v$

### Reduction rules

$$\begin{array}{lll} (\sigma) & \langle t/x \rangle v & \rightarrow \text{subst}(t, x, v) \\ (\pi) & (tk)k' & \rightarrow t \text{ append}(k, k') \\ (\beta) & (\lambda x.t)(u :: k) & \rightarrow \langle u/x \rangle (tk) \end{array}$$

### Logical reading (cut elimination)

- ( $\sigma$ ) complete right permutation
- ( $\pi$ ) complete left permutation
- ( $\beta$ ) key step

## Doing $\epsilon$ (1)

First step: reduce SN for  $\lambda^G$  to SN for a version of  $\lambda x$

1. Recall the terms of  $\lambda x$ :

$$M, N, P ::= x \mid \lambda x.M \mid MN \mid \langle N/x \rangle M$$

2. Equip this set with

$$\begin{array}{lll} (\beta) & (\lambda x.M)N & \rightarrow \langle N/x \rangle M \\ (\sigma) & \langle N/x \rangle M & \rightarrow [N/x]M \\ (\pi_1) & (\langle P/x \rangle M)N & \rightarrow \langle P/x \rangle (MN) \\ (\pi_2) & \langle \langle P/y \rangle N/x \rangle M & \rightarrow \langle P/y \rangle \langle N/x \rangle M \end{array}$$

and put  $\pi = \pi_1 \cup \pi_2$

3. Define  $Q : \lambda^G \rightarrow \lambda x$  as follows:

$$t(u_1 :: \dots :: u_m(x)v) \mapsto \langle MN_1 \dots N_m/x \rangle P$$

4. SN for  $\lambda x \Rightarrow$  SN for  $\lambda^G$

## Doing $\epsilon$ (2)

Second step: enrich  $\lambda$

1. Equip the set of  $\lambda$ -terms with

$$\begin{array}{lll} (\pi_1) & (\lambda x.M)NP & \rightarrow (\lambda x.MP)N \\ (\pi_2) & M((\lambda y.P)N) & \rightarrow (\lambda y.MP)N \end{array}$$

and put  $\pi = \pi_1 \cup \pi_2$

2. In  $\lambda, \rightarrow_\pi$  terminates

3. In  $\lambda, \rightarrow_\pi$  does not increase  $||_-||_\beta$

(fund. lemma perpetuality used here)

### Doing $\epsilon$ (3)

Third step: conclude

1. Define  $(\_)^{\bullet} : \lambda x \rightarrow \lambda$  by

$$(\langle N/x \rangle M)^{\bullet} = (\lambda x. M^{\bullet})N^{\bullet}$$

2. In  $\lambda x, \rightarrow_{\beta\pi}$  terminates

3. For all  $M \in \lambda x, M^{\bullet} \in SN_{\beta} \Rightarrow M \in SN_{\beta\pi\sigma}$

4. Corollary (SN for  $\lambda x$ ):

For all  $M \in \lambda x, M$  typable  $\Rightarrow M \in SN_{\beta\pi\sigma}$

(SN for  $\lambda$  used here)

5. Theorem (SN for  $\lambda^G$ ):

For all  $t \in \lambda^G, t$  typable  $\Rightarrow t \in SN_{\beta\pi\sigma}$

## Conclusions

1. Consequences:

- (a) SN for variants of  $\lambda$  with generalised application
- (b) SN for  $LJ$  equipped with various “protocols”

2. Simple proof:

- (a) reuse a previous result instead of repeating a previous proof
- (b) substitution is suspended, but not explicitly executed
- (c) simple  $\Rightarrow$  stronger