Proving termination using dependent types: the case of xor-terms

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Outline

Formal models of cryptographic systems

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- Protocols
- Security APIs

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Examples from a security API called CCA (Common Cryptographic Architecture):

$$x, y, \{z\}_{x \oplus KP \oplus KM} \mapsto \{z \oplus y\}_{x \oplus KP \oplus KM}$$
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Reasoning involves:

Commutativity:	$x \oplus y \simeq y \oplus x$
Associativity:	$(x \oplus y) \oplus z \simeq x \oplus (y \oplus z)$
Neutral element:	$x \oplus 0 \simeq x$
Involutivity:	$x \oplus x \simeq 0$

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General setting: quotiented first order-terms

We are given

- A type of terms $\mathcal{T}$ with constructors C_k:
  Inductive $\mathcal{T}$: *Set* :=
  $$\begin{array}{l} \mid C_1 : \mathcal{T} \\ \quad \vdots \\ \mid C_k : \ldots \to \mathcal{T} \ldots \to \mathcal{T} \ldots \to \mathcal{T} \\ \quad \vdots \end{array}$$

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- A congruence $\simeq : \mathcal{T} \to \mathcal{T} \to Prop$
  - For each constructor C_k
    $\forall a, \ldots x_1, y_1, b, \ldots x_2, y_2, \ldots c,$
    $x_1 \simeq y_1 \to x_2 \simeq y_2 \to$
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We want to reason on $\mathcal{T}$ up to $\simeq$

Already well-known examples

- finite bags represented by finite lists

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- algebra of formal arithmetic expressions
  $+$ is associative, commutative, 0 is neutral
  $\times$ is associative, commutative, 1 is neutral
  $\times$ distributes over $+$
- (mobile) process calculi, chemical abstract machines
  parallel composition and choice operators are AC

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- Explicit approach :
  - Define a normalization function N on $\mathcal{T}$
  - Compare terms using syntactic equality on their norms : $x \simeq y$ iff $N\,x = N\,y$

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Cryptographic systems need more

Reasoning on such systems involves

- comparing terms up to AC + involutivity of $\oplus$:

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\text{Commutativity:} & x \oplus y & \simeq & y \oplus x \\
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\begin{array}{ll}
x \preceq y & \text{if } x \simeq y \\
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$\longrightarrow$ normalization is needed!

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- AC matching: a non trivial matter

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trivial forgetful morphism $\mathcal{T}' \to \mathcal{T}$.

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Interesting part $= \mathcal{T} \to \mathcal{T}'$

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Lunch time!

Lasagnas reveal the truth

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- normalizing sauce = rearranging + removing duplicates

Outline

$\mathcal{T}$ as a lasagna

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Inductive $\mathcal{T}$: Set :=

| *Zero*: $\mathcal{T}$

| *PC*: *public_const* $\rightarrow \mathcal{T}$ | *SC*: *secret_const* $\rightarrow \mathcal{T}$

| *E*: $\mathcal{T} \rightarrow \mathcal{T} \rightarrow \mathcal{T}$

| *Xor*: $\mathcal{T} \rightarrow \mathcal{T} \rightarrow \mathcal{T}$

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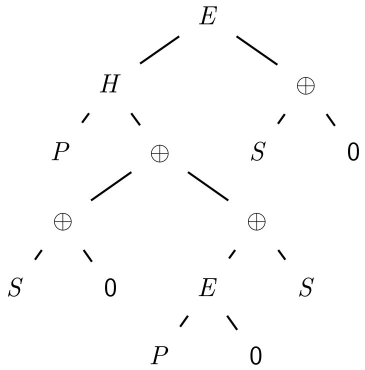

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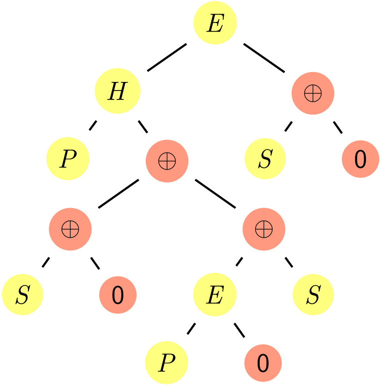

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Decomposing $\mathcal{T}$

Inductive $\mathcal{T}_x$:*Set* :=
| *X_Zero* : $\mathcal{T}_x$
| *X_Xor* : $\mathcal{T}_x \to \mathcal{T}_x \to \mathcal{T}_x$

Inductive $\mathcal{T}_n$: *Set* :=
| *NX_PC* : *public_const* $\to \mathcal{T}_n$
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| NX_sum : $A \to \mathcal{T}_n$

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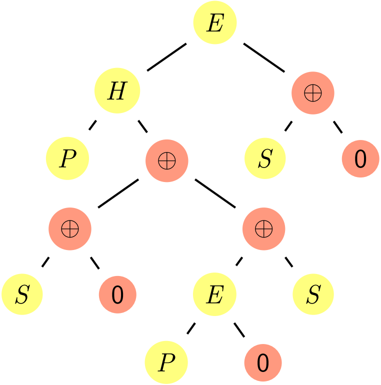

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  - heavy encoding of $m - n$ or $n - m$
  - need to lift $\mathcal{L}_x\, n$ and $\mathcal{L}_x\, m$ to $\mathcal{L}_x\, (\max n\, m)$

Lifting lasagna

$$\mathcal{L}_x\, k \stackrel{\text{def}}{=} \underbrace{\mathcal{T}_x(\mathcal{T}_n(\mathcal{T}_x(\ldots(\emptyset))))}_{k \text{ layers}} \text{ for } k \text{ large enough.}$$

- What is k?
- The number of layers on the left subterm and on the right subterm are different in general.

Take the max

- Standard solution: {le n m} + {le m n}
  - interactive definition, large proof term
  - heavy encoding of $m - n$ or $n - m$
  - need to lift $\mathcal{L}_x\, n$ and $\mathcal{L}_x\, m$ to $\mathcal{L}_x\,(\max n\, m)$
- Lightweight approach: $\max n\, m \stackrel{\text{def}}{=} m + (n - m)$

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  - $lift_x : \mathcal{L}_x\, k \rightarrow \mathcal{L}_x\,(k + d)$, $lift_n : \mathcal{L}_n\, k \rightarrow \mathcal{L}_n\,(k + d)$

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- Lightweight approach: $\max\, n\, m \stackrel{\text{def}}{=} m + (n - m)$
  - $lift_x : \mathcal{L}_x\, k \to \mathcal{L}_x\, (k + d)$, $lift_n : \mathcal{L}_n\, k \to \mathcal{L}_n\, (k + d)$
  - No need to proof that max is the max.

Outline

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Well designed types help us to design programs

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Control them using alternating natural numbers

Inductive alt_{even}: Set :=
  | 0_e: alt_{even}
  | $S_{o \to e}$: $alt_{odd} \to alt_{even}$
with alt_{odd}: Set :=
  | $S_{e \to o}$: $alt_{even} \to alt_{odd}$

Outline

Forbid fake inclusions

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Inductive $\mathcal{T}_x$: Set :=
| X_Zero : $\mathcal{T}_x$
| X_ns : $A \rightarrow \mathcal{T}_x$
| X_Xor : $\mathcal{T}_x \rightarrow \mathcal{T}_x \rightarrow \mathcal{T}_x$

Inductive $\mathcal{T}_n$: Set :=
| NX_PC : $public\_const \rightarrow \mathcal{T}_n$
| NX_SC : $secret\_const \rightarrow \mathcal{T}_n$
| NX_sum : $A \rightarrow \mathcal{T}_n$
| NX_E : $\mathcal{T}_n \rightarrow \mathcal{T}_n \rightarrow \mathcal{T}_n$
| NX_Hash : $\mathcal{T}_n \rightarrow \mathcal{T}_n \rightarrow \mathcal{T}_n$

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$X\_ns\ (NX\_sum\ (\ X\_ns\ (NX\_sum\ (\ldots))))$

Forbid fake inclusions

Inductive $\mathcal{T}_x$: $bool \rightarrow Set :=$
| $X\_Zero : \forall\ b,\ \mathcal{T}_x\ b$
| $X\_ns : \forall\ b,\ Is\_true\ b \rightarrow A \rightarrow \mathcal{T}_x\ b$
| $X\_Xor : \forall\ b,\ \mathcal{T}_x\ true \rightarrow \mathcal{T}_x\ true \rightarrow \mathcal{T}_x\ b$

Inductive $\mathcal{T}_n$: $bool \rightarrow Set :=$
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Outline

Mutual induction

- Prefer fixpoints: built-in computation, no inversion

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Fixpoint *lift_lasagna_x* e_1 e_2 $\{struct\ e_1\}$:
  $\mathcal{L}_x\ e_1 \to \mathcal{L}_x\ (e_1 + e_2)$:=
  match e_1 return $\mathcal{L}_x\ e_1 \to \mathcal{L}_x\ (e_1 + e_2)$ with
  | $0_e \Rightarrow$ fun *emp* $\Rightarrow$ match *emp* with end
  | $S_{o\to e}\ o_1 \Rightarrow map_x\ (lift\_lasagna\_n\ o_1\ e_2)\ false$
  end
with *lift_lasagna_n* o_1 e_2 $\{struct\ o_1\}$:
  $\mathcal{L}_n\ o_1 \to \mathcal{L}_n\ (o_1 + e_2)$:=
  match o_1 return $\mathcal{L}_n\ o_1 \to \mathcal{L}_n\ (o_1 + e_2)$ with
  | $S_{e\to o}\ e_1 \Rightarrow map_n\ (lift\_lasagna\_x\ e_1\ e_2)\ false$
  end.

Outline

Conversion rule

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Used everywhere

Conversion rule

Used everywhere

Definition *bin_xor*

$(bin : \forall\ A\ b, \mathcal{T}_x\ A\ true \to \mathcal{T}_x\ A\ true \to \mathcal{T}_x\ A\ b)\ o_1\ o_2\ b$
$(l_1 : lasagna\_cand\_x\ o_1\ true)$
$(l_2 : lasagna\_cand\_x\ o_2\ true)$:
$lasagna\_cand\_x\ (max\_oo\ o_1\ o_2)\ b :=$

$bin\ (\mathcal{L}_n\ (max\_oo\ o_1\ o_2))\ b$
$(lift\_lasagna\_cand\_x\ true\ o_1\ (o_2 - o_1)\ l_1)$
$(coerce\_max\_comm$
$(lift\_lasagna\_cand\_x\ true\ o_2\ (o_1 - o_2)\ l_2))$.

Conclusion

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- Polymorphism

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Future work

- Study variants, compare and choose the best

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- Study variants, compare and choose the best
- application to CCA