A games semantics for reductive logic and proof-search

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Joint work with David Pym

Introduction

Proofs are constructed from hypothesis to conclusions (top down)

In sequent calculus LK, have

$$\frac{}{\sigma, \phi \supset \psi \vdash \sigma, \tau}\,Ax \qquad \frac{}{\sigma, \phi \supset \psi, \tau \vdash \tau}\,Ax$$

Introduction

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In sequent calculus LK, have

$$\dfrac{\dfrac{}{\sigma, \phi \supset \psi \vdash \sigma, \tau}\,Ax \qquad \dfrac{}{\sigma, \phi \supset \psi, \tau \vdash \tau}\,Ax}{\sigma, \phi \supset \psi, \sigma \supset \tau \vdash \tau}\supset L$$

Reasoning about proofs via propositions-as-types analogy

$$
\begin{tikzcd}
\begin{array}{c}\Phi \Rightarrow \Gamma \vdash \phi \\ \text{Proofs}\end{array} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \begin{array}{c}[\Gamma] \vdash [\Phi] : [\phi] \\ \lambda\text{-terms}\end{array} \arrow[dl, leftrightarrow] \\
& \begin{array}{c}[\![\Gamma]\!] \xrightarrow{[\![\Phi]\!]} [\![\phi]\!] \\ \text{semantics(eg CCC)}\end{array} &
\end{tikzcd}
$$

$\Rightarrow$ Simplifying proofs $\cong$ computation in functional programming

Have very different situation in Theorem Proving:

- Start with conclusion (Theorem to be proved)
- Try to apply inference rules backwards to obtain proof
  use term Reduction to describe such attempts

Example

$$\frac{}{\sigma, \phi \supset \psi, \sigma \supset \tau \text{ ?- } \tau}$$

Example

$$\cfrac{\cfrac{}{\sigma, \phi \supset \psi \mathrel{?\text{-}} \sigma, \tau} \qquad \cfrac{}{\sigma, \phi \supset \psi, \tau \mathrel{?\text{-}} \tau}}{\sigma, \phi \supset \psi, \sigma \supset \tau \mathrel{?\text{-}} \tau} \supset L$$

Example

$$\dfrac{\dfrac{}{\sigma, \phi \supset \psi \;?\text{-}\; \sigma, \tau}\,Ax \qquad \dfrac{}{\sigma, \phi \supset \psi, \tau \;?\text{-}\; \tau}\,Ax}{\sigma, \phi \supset \psi, \sigma \supset \tau \;?\text{-}\; \tau} \supset L$$

Example revisited

However, this is not only possible proof attempt:

$$\dfrac{}{\sigma, \phi \supset \psi, \sigma \supset \tau \text{ ?- } \tau}$$

Example revisited

However, this is not only possible proof attempt:

$$\dfrac{\dfrac{}{\sigma, \sigma \supset \tau \mathrel{?\text{-}} \phi, \tau} \qquad \dfrac{}{\sigma, \sigma \supset \tau, \psi \mathrel{?\text{-}} \tau}}{\sigma, \phi \supset \psi, \sigma \supset \tau \mathrel{?\text{-}} \tau} \supset L$$

Example revisited

However, this is not only possible proof attempt:

$$\dfrac{\dfrac{\dfrac{}{\sigma \;?\text{-}\; \sigma, \tau, \phi} \qquad \dfrac{}{\sigma, \tau \;?\text{-}\; \tau, \phi}}{\sigma, \sigma \supset \tau \;?\text{-}\; \tau, \phi} \supset L \qquad \dfrac{\vdots}{\sigma, \sigma \supset \tau, \psi \;?\text{-}\; \tau} \supset L}{\sigma, \phi \supset \psi, \sigma \supset \tau \;?\text{-}\; \tau} \supset L$$

Example revisited

We can complete this proof:

$$\dfrac{\dfrac{\dfrac{}{\sigma \text{ ?- } \sigma, \tau, \phi}\,Ax}{\sigma, \sigma \supset \tau \text{ ?- } \tau, \phi} \supset L \qquad \dfrac{\vdots}{\sigma, \sigma \supset \tau, \psi \text{ ?- } \tau} \supset L}{\sigma, \phi \supset \psi, \sigma \supset \tau \text{ ?- } \tau} \supset L$$

Proofs differ by choice of principal formula (disjunctive choices)

What about Semantics?

Would like to have semantics to reason about search.

Issues:

- Proof is total; reduction is partial:
  Start from Putative Conclusions and construct Sufficient Premisses but
  Sufficient Premisses might not be provable!

- Need to model control aspects of searches (which reduction rule; which formula to operate on; when to backtrack)

Framework for Reductive Semantics

Can we obtain similar picture?

$\Phi \Rightarrow \Gamma$?- ϕ $\qquad\qquad$ $[\Gamma] \mathrel{|\!\sim} [\Phi] : [\phi]$

Reduction $\qquad\qquad$ Realisers for Reductions

$$[\![\Gamma]\!] \overset{[\![\Phi]\!]}{\leadsto} [\![\phi]\!]$$

Algebra

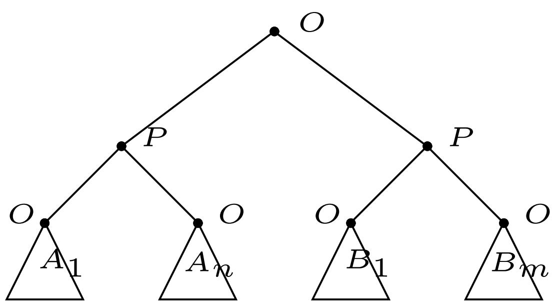
Here: describe games as realisers; omit the algebra (categorical semantics)

Games model of proof search

Have games in style of Hyland and Ong

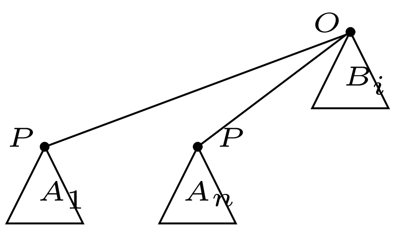
Arenas are given by

- $\top$: empty arena
- $\perp$: one node only
- $\phi \wedge \psi$: disjoint sum of arenas for ϕ and ψ;
- $\phi \vee \psi$: Assume $\mathcal{A}_i$ and $\mathcal{B}_j$ are trees for ϕ and ψ respectively;



Games semantics, continued

- $\phi \supset \psi$:



A play for arena is a sequence of moves such that

- Opponent always starts by asking initial question;
- Each player plays as many moves as he likes;

Games Semantics, continued

Strategies is a function from set of O-moves to set of P-moves
O-questions: challenges to provide evidence for conclusion (conjunctive choices)
P-moves: challenge to provide evidence for premiss (disjunctive choices)

Indeterminates captured by oracles: Proponent can make arbitrary moves in the arena corresponding to oracle

Substitution for indeterminates works by composing strategies

Uniform proofs provide simple algorithm for proof search

In uniform proofs, right reduction are preferred over left reductions
Hence, left rules applied only when RHS is atom

captured by the following restricted strategies:

- Opponent makes as many moves as possible;
- P makes moves L, R if possible;

obtain one-to-one correspondence between uniform proofs in LK and restricted strategies for LK

Conclusions

Semantics of proof search different from standard semantics:

- proofs constructed top-down,
  search constructed bottom-up
- have to consider partial, possibly non-completeable proofs
- need highly intensional semantics

Have shown: games semantics suitable

handles paradigmatic aspect of proof search: failure and restart

also deals with order of reduction rules

Further work

characterise further aspects of control:

- order on choices between rules
- which sequence to work on next

Will pursue domain-theoretic approach: consider ordering of reduction rules to capture preferences