

# Number Conversion

**Roland Backhouse**

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## Outline

We discuss the mathematical properties of converting real numbers to integers.

In particular, we show how to implement rounding up (of rational numbers) in terms of rounding down. This provides a mechanism for implementing rounding up in Java (which defines integer division as rounding towards zero).

The discussion illustrates the use of equivalences in defining important mathematical notions.

# Casting

*Casting* is the name given in languages like C and Java for the operation of converting a value of one type to another.

The cast from real numbers to integers occurs when evaluating an integer division.

The text `1/2` (for example) means an integer division and evaluates to zero. So, if the real value is intended, one has to write, say, `1.0/2.0`.

Adding decimal points doesn't help, however, when the values being divided are expressions. So if, for example, `m` and `n` have been declared to be integers then we have to write something like `(real)m/(real)n` to force the compiler to compute the real value of `m/n`.

# The Floor Function

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On the right side of the equivalence, the at-most relation (“ $\leq$ ”) is between reals whereas on the left side it is between integers.

Making all conversions explicit, temporarily adopting a Java-like notation, is illuminating. Doing so the definition becomes that, for all real  $x$ , `(floor)x` is an integer such that for all integers  $n$ ,

$$n \leq \text{(floor)}x \equiv \text{(real)}n \leq x .$$

So the floor of  $x$  is defined by connecting it to the conversion from integers to reals in a simple equivalence.

## Properties of Floor

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1.  $\lfloor x \rfloor$  is by definition an integer. So we can instantiate  $n$  to  $\lfloor x \rfloor$ .

We get

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3. Instantiate  $x$  in 1. to  $n$ . We get

$$\lfloor n \rfloor \leq n .$$

4. Combining 2. and 3., for all integers  $n$

$$\lfloor n \rfloor = n .$$

## Properties of Floor — Summary

For all reals  $x$ ,

$$\lfloor x \rfloor \leq x .$$

For all integers  $n$ ,

$$\lfloor n \rfloor = n .$$

## Properties of Floor (Continued)

Recall the rule of contraposition:

$$p \equiv q \equiv \neg p \equiv \neg q .$$

The *contrapositive* of the definition of the floor function is, for all integers  $n$  and real  $x$ ,

$$\neg(n \leq \lfloor x \rfloor) \equiv \neg(n \leq x) .$$

But  $\neg(n \leq m) \equiv m < n$ . So

$$\lfloor x \rfloor < n \equiv x < n .$$

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$$\lfloor x \rfloor < n \equiv x < n \text{ .}$$

Instantiating  $n$  with  $\lfloor x \rfloor + 1$  and simplifying we deduce:

$$x < \lfloor x \rfloor + 1 \text{ .}$$

Recalling that  $\lfloor x \rfloor \leq x$ , we have established

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \text{ .}$$

## Indirect Equality

More complex properties use the rule of *indirect equality*:

$$x = y \equiv \langle \forall z \mid : z \leq x \equiv z \leq y \rangle .$$

Example:

$$\begin{aligned} & n \leq \lfloor x \rfloor + m \\ = & \quad \{ \text{arithmetic} \} \\ & n - m \leq \lfloor x \rfloor \\ = & \quad \{ \text{definition of floor} \} \\ & n - m \leq x \\ = & \quad \{ \text{arithmetic} \} \\ & n \leq x + m \\ = & \quad \{ \text{definition of floor} \} \\ & n \leq \lfloor x + m \rfloor . \end{aligned}$$

Hence,

$$\lfloor x \rfloor + m = \lfloor x + m \rfloor .$$

# Rounding Off

## Ceiling Function

**Definition:** For all real  $x$ ,  $\lceil x \rceil$  is an integer such that, for all integers  $n$ ,

$$\lceil x \rceil \leq n \equiv x \leq n .$$

Exercise: derive properties of the ceiling function dual to the properties of the floor function.

# The Problem

Rounding down an integer division of positive numbers  $m$  and  $n$  is expressed by

$$\left\lfloor \frac{m}{n} \right\rfloor$$

where  $\frac{m}{n}$  is the real division of  $m$  and  $n$ . Dually, rounding up is expressed by

$$\left\lceil \frac{m}{n} \right\rceil .$$

Implementing rounding up given an implementation of rounding down amounts to finding suitable values  $p$  and  $q$  so that

$$\left\lfloor \frac{p}{q} \right\rfloor = \left\lceil \frac{m}{n} \right\rceil .$$

The values  $p$  and  $q$  should be expressed as arithmetic functions of  $m$  and  $n$  (that is, functions involving addition and multiplication, but not involving the floor or ceiling functions).



# The Strategy

We can *derive* suitable expressions for  $p$  and  $q$  using the rule of indirect equality. For arbitrary integer  $k$ , we aim to eliminate the ceiling function from the inequality

$$k \leq \left\lceil \frac{m}{n} \right\rceil$$

obtaining an inequality of the form

$$k \leq e$$

where  $e$  is an arithmetic expression in  $m$  and  $n$ . We may then conclude that

$$\lfloor e \rfloor = \left\lceil \frac{m}{n} \right\rceil .$$

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 & k-1 < \frac{m}{n} \\
 = & \quad \{ \text{arithmetic, } n > 0 \} \\
 & n \times (k-1) < m \\
 = & \quad \{ \text{integer inequalities} \} \\
 & n \times (k-1) + 1 \leq m \\
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 & k \leq \frac{m+n-1}{n}
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 & n \times (k-1) + 1 \leq m \\
 = & \quad \{ \text{arithmetic, } n > 0 \} \\
 & k \leq \frac{m+n-1}{n} \\
 = & \quad \{ \text{definition of floor function} \} \\
 & k \leq \left\lfloor \frac{m+n-1}{n} \right\rfloor .
 \end{aligned}$$

We have derived:

$$k \leq \left\lceil \frac{m}{n} \right\rceil \equiv k \leq \left\lfloor \frac{m+n-1}{n} \right\rfloor .$$

Here  $k$  is arbitrary. So, by indirect equality, we get

$$\left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor .$$

In Java, therefore, if it is required to round up the result of dividing  $m$  by  $n$  one should compute  $(m+n-1)/n$  .