COMP2012/G52LAC
Languages and Computation
Lecture 18
Decidability and the Halting Problem

Henrik Nilsson

University of Nottingham

Recap: Turing Machines (1)

- A Turing Machine (TM) is a mathematical model of a general-purpose computer.

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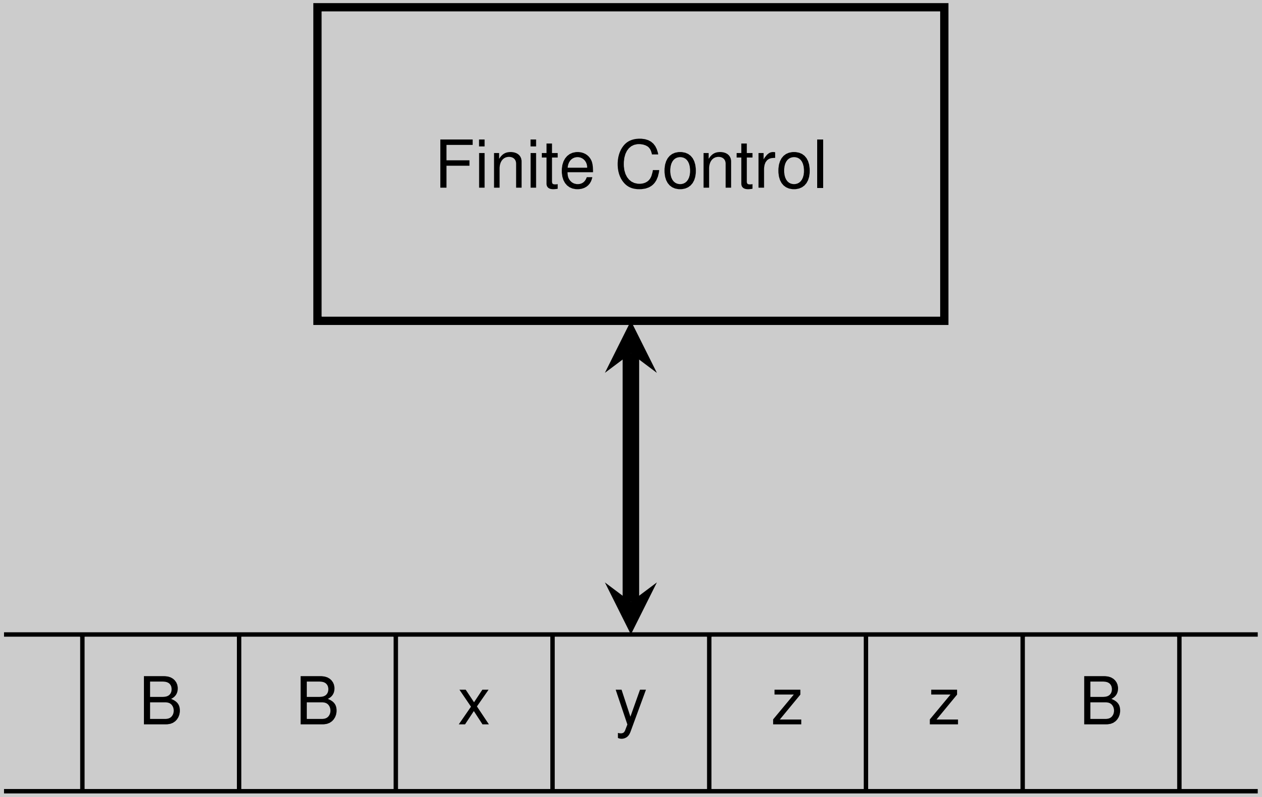
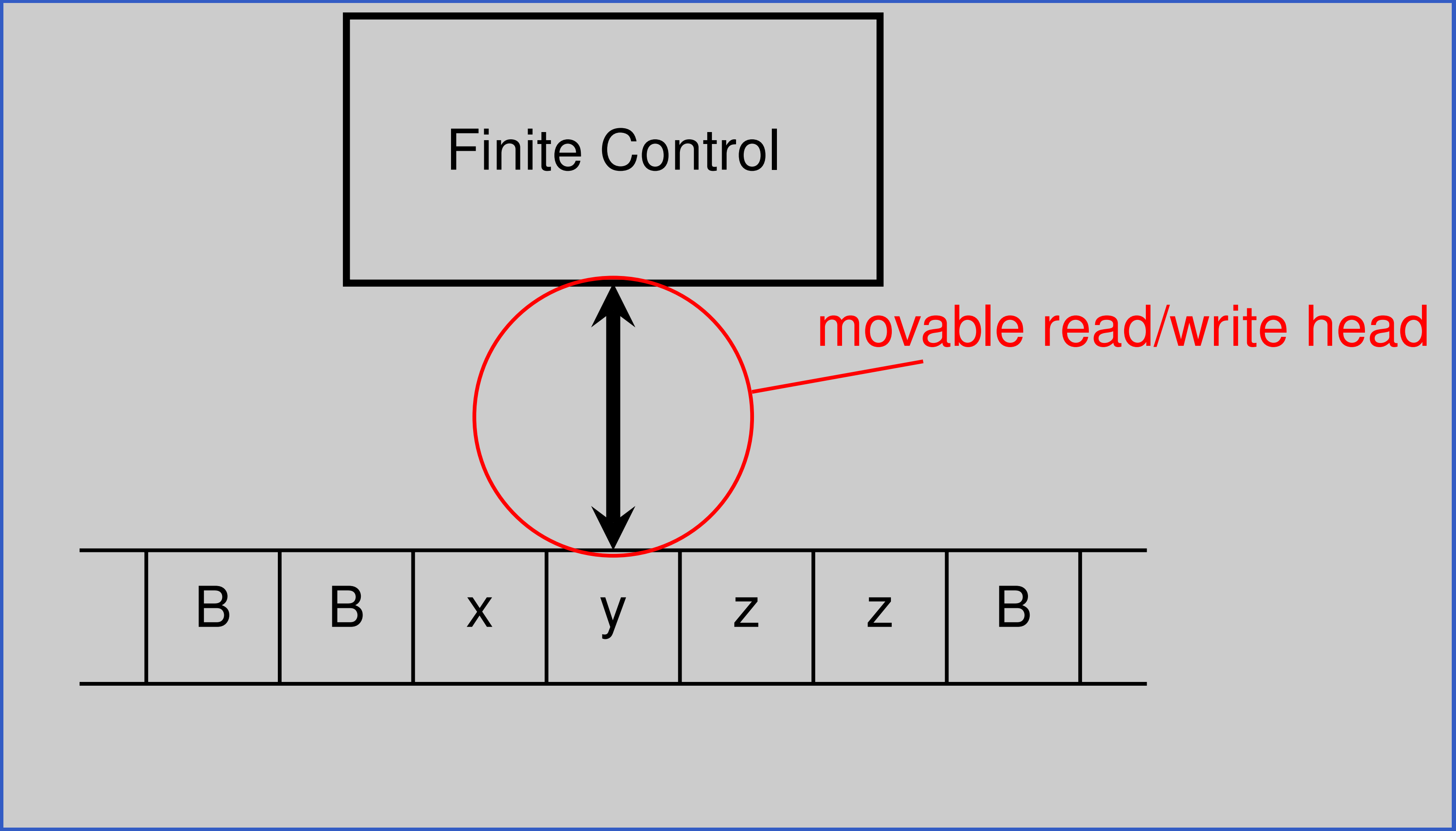
- A TM is a generalisation of a PDA: TM = FA + infinite tape

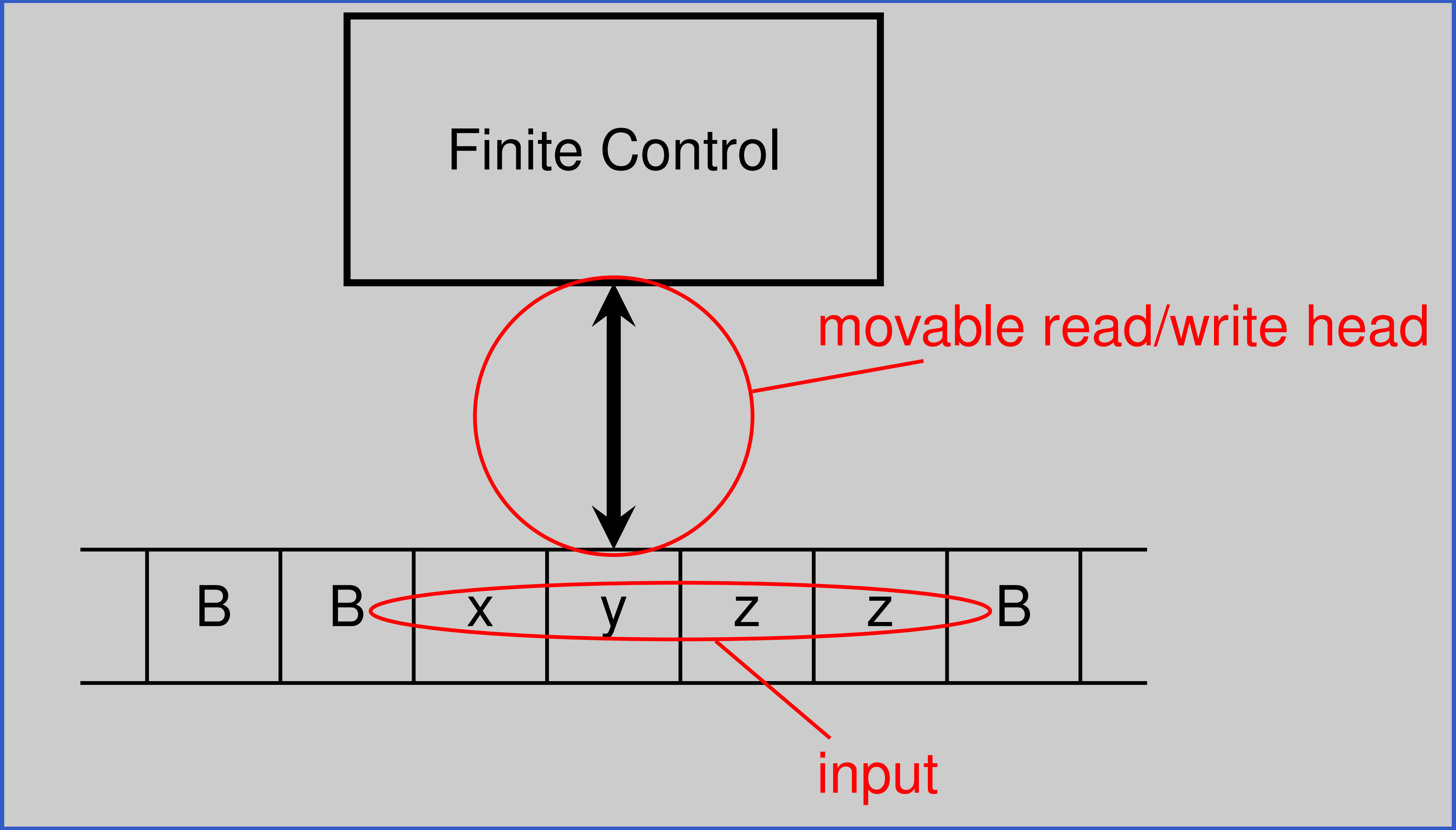
Recap: Turing Machines (2)



Recap: Turing Machines (2)



Recap: Turing Machines (2)



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- There are other notions of computation, such that the λ***-calculus***.
- All suggested notions of computation have so far proved to be equivalent.
- ***The Church-Turing Thesis***: "Every function which would naturally be regarded as 'computable' can be computed by a TM".

The Language of a TM (1)

$$L(M) = \{w \in \Sigma^* \mid (\epsilon, q_0, w) \underset{M}{\overset{*}{\vdash}} (\gamma_L, q, \gamma_R) \ \wedge \ q \in F\}$$

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A TM stops in a non-accepting state if the transition function returns stop for that state and current tape input.

However, it may also ***never*** stop!

This is unlike the machines lige DFAs, NFAs, PDAa.

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What may come as a surprise is that there are languages for which a TM ***necessarily*** cannot decide membership; i.e., will loop on some inputs.

Recursive Language

L is ***recursive*** if $L = L(M)$ for a TM M such that

1. if $w \in L$, then M accepts w (and thus halts)
2. if $w \notin L$, then M eventually halts without ever entering an accepting state.

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We also say that M ***decides*** L.

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Why “recursively enumerable”?

Because it is possible to construct a TM that enumerates all strings in such a language in some order. (But it will necessarily keep trying to enumerate strings forever.)

Decidable and Undecidable

There are even languages that have no TM! The non-RE languages.

- Decidable: a language or problem (encoded as a language) that is recursive.
- Undecidable: a language or problem that is RE but not recursive, or non-RE.

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Example of non-RE language: The set of all Turing machines accepting exactly 3 words.

(In fact, a simple cardinality argument shows that most languages are non-RE: there are “many more” languages than there are TMs.)

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Famous example of a RE language that is not recursive; i.e. an undecidable language.

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Proof sketch on whiteboard.

Other Undecidable Problems

- Whether two programs (computable functions) are equal
- Whether a CFG is ambiguous
- Whether two CFGs are equivalent
- Rice's Theorem: Whether the language of a given TM has some particular ***non-trivial*** property. (Non-trivial: holds for some but not all languages.)

Rice's Theorem (1)

(After Henry Gordon Rice; also known as the Rice-Myhill-Shapiro theorem.)

Let C be a set of languages. Define

$$L_C = \{ \ M \mid L(M) \in C \ \}$$

where M ranges over all TMs. Then either L_C is empty, or it contains all TMs, or it is undecidable.

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For example, C might be the set of regular languages. As there are some TMs that recognise regular languages, but not all do, L_C is undecidable in this case.

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Caveat: Rice's theorem is concerned with properties of the ***language*** accepted by a TM, not about properties of the TM (code) itself. E.g., it is certainly decidable if a TM has at most 10 states.

`http://www.eecs.berkeley.edu/~luca/cs172/noterice.pdf`