Towards Type Theory with Continuity

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Setting

- Extensional Type Theory with W and Quotient types
- In category speak: LCC pretopos with W predicative topos
- **Prop** = sets with at most one inhabitant.
- Define $\exists a : A.P\,a = [\Sigma a : A.P\,a]$ (bracket types).

$$\frac{A : \mathbf{Set}}{[A] : \mathbf{Prop}} \qquad [A] = A/(\lambda x, y.\mathrm{true})$$

- Logic, Set Theory and Programming Language
- As a Programming Language: purely functional, **total !**
- How to capture *real world* programming, i.e. *computational effects*?

Effects as Monads

Functional Programming

Effects = Monads (Moggi, Wadler)

Monad

$$M : \mathbf{Set} \to \mathbf{Set}$$

$$\frac{a : A}{\eta\, a : M\, A} \qquad \frac{m : M\, A \quad f : A \to M\, B}{a \gg\!= f : MB}$$

+ equations ($A \to M\, B$ is a category)

Examples of computational effects

Monads

Error	$M_E\, X = 1 + X$
State	$M_S\, X = S \to S \times X$
Cont.	$M_C\, X = (X \to R) \to R$

Kleisli category

$A \to M\, B$ = effectful computations.

Partiality as an effect

Delay M_D

Partial $M_P\,X = (M_D\,X)/\sim$

Idea

Partial functions from A to B = $A \to M_P\,B$.

Based on published work by Venanzio Capretta and unpublished work with Tarmo Uuustalu and Venanzio.

Delay

$$\textbf{codata}\ \frac{A : \mathbf{Set}}{M_D\, A : \mathbf{Set}} \quad \text{where} \quad \frac{a : A}{\mathrm{N}\, a : M_D\, A} \quad \frac{d : M_D\, A}{\mathrm{L}\, d : M_D\, A}$$

Divergent computation

$$\begin{aligned} \bot &: M_D\, A \\ \bot &= \mathrm{L}\, \bot \end{aligned}$$

Monad structure

$$\eta_D\, a = \mathrm{N}\, a$$

$$\begin{aligned} \mathrm{N}\, a \gg\!= f &= f\, a \\ \mathrm{L}\, d \gg\!= f &= \mathrm{L}(d \gg\!= f) \end{aligned}$$

Termination

$$\textbf{data}\ \frac{d : M_D\, A \quad a : A}{d \downarrow a : \textbf{Prop}} \quad \text{where} \quad \frac{}{\mathrm{N}\, a \downarrow a} \quad \frac{d \downarrow a}{\mathrm{L}\, d \downarrow a}$$

Termination order

$$\begin{aligned} d \sqsubseteq d' &= \Pi a : A. d \downarrow a \rightarrow d' \downarrow a \\ d \sim d' &= d \sqsubseteq d' \wedge d' \sqsubseteq d \end{aligned}$$

Partiality

- $M_P\, X = (M_D\, X) / \sim$
- Classically: $M_P\, X = X + \{\bot\}$
- Inherits monad structure ($>\!\!>\!\!=_D$ stable under $\sim$).
- Lift order: $\sqsubseteq$: $M_D\, A \to M_D\, A \to \textbf{Prop}$

Recursion (call by value)

How to construct ?

$$\frac{f : (A \rightarrow M_P\, B) \rightarrow A \rightarrow M_P\, B}{\mathrm{fix}\, f : A \rightarrow M_P\, B}$$

$$\mathrm{fix}\, f = \bigsqcup(\lambda n. f^n \bot)$$

ω-CPO structure

directed completeness

$$\mathrm{Chain} = \{\vec{d} : \mathbb{N} \to M_D\,A \mid \Pi n : \mathbb{N}.\vec{d}\,n \sqsubseteq \vec{d}\,(n+1)\}$$

$$\frac{\vec{d} : \mathrm{Chain}}{\bigsqcup \vec{d} : M_D\,A} \qquad \bigsqcup \vec{d} = \vec{d}\,0 \sqcup \mathrm{L} \bigsqcup \vec{d} \circ (+1)$$

race

$$\begin{aligned} \mathrm{N}\,a \sqcup d' &= \mathrm{N}\,a \\ \mathrm{L}\,d \sqcup \mathrm{N}\,b &= b \\ \mathrm{L}\,d \sqcup \mathrm{L}\,d' &= \mathrm{L}\,(d \sqcup d') \end{aligned}$$

Continuity?

- $\bigsqcup$ works on M_P (but not $\sqcup$).
- ω-CPO structure lifts pointwise to $A \to M_P\, B$.
- We need that $f : (A \to M_P\, B) \to A \to M_P\, B$ is ω-continuous, i.e.

$$f(\bigsqcup \vec{d}) = \bigsqcup \lambda i.f(\vec{d}\, i)$$

- We have to prove ω-continuity again and again.
- We cannot define a non-continuous f!

Type Theory with Continuity ?

- How to add continuity to Type Theory?
- Consistent with extensionality.
- Computational (BHK).
- Explained by translation?

1st order continuity

- Consider $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$
- What are the possible computations of this type?

Eating games (Hancock et al)

$$\textbf{data}\ G : \textbf{Set} \quad \text{where} \quad \frac{n : \mathbb{N}}{\mathrm{R}\, n : G} \qquad \frac{g : \mathbb{N} \to G}{\mathrm{G}\, g : G}$$

$$\frac{g : G}{[\![G]\!] : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}} \qquad \begin{array}{rcl} [\![\mathrm{R}\, n]\!] h & = & n \\ [\![\mathrm{G}\, g]\!] h & = & [\![g\,(h\,0)]\!] h \circ (+1) \end{array}$$

$$\frac{f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}}{q\, f : G} \qquad \begin{array}{rcl} [\![q\, f]\!] & = & f \\ q\, [\![g]\!] & = & g \end{array}$$

Too intensional!

Extensional games

$$g \sim g' = (\llbracket g \rrbracket = \llbracket g' \rrbracket)$$

$$\frac{f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}}{q\, f : G/\sim} \qquad \begin{array}{rcl} \llbracket q\, f \rrbracket & = & f \\ q\, \llbracket g \rrbracket & = & g \end{array}$$

Local continuity

$$\frac{f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \quad h : \mathbb{N} \to \mathbb{N}}{\mathrm{lc}\, f\, h : \exists n : \mathbb{N}.\Pi h' : \mathbb{N} \to \mathbb{N}.(\Pi i < n.h\, n = h'\, n) \to f\, h = f\, h'}$$

Derive lc using lc':

$$\frac{g : G \quad h : \mathbb{N} \to \mathbb{N}}{\mathrm{lc}'\, g\, h : \Sigma n : \mathbb{N}.\Pi h' : \mathbb{N} \to \mathbb{N}.(\Pi i < n.h\, n = h'\, n) \to f\, h = f\, h'}$$

Higher order continuity?

What are games for:

$$((\mathbb{N} \to \mathbb{N}) \to \mathbb{N}) \to \mathbb{N}?$$

$$\begin{aligned}
&((\mathbb{N} \to \mathbb{N}) \to \mathbb{N}) \to \mathbb{N} \\
&\quad \simeq \quad G/\sim \to \mathbb{N} \\
&\quad \simeq \quad \{f : G \to \mathbb{N} \mid \Pi g, g' : G.g \sim g' \to f\,g = f\,g'\}
\end{aligned}$$

We need games for:

$$G \to \mathbb{N}$$

Higher order games ($G \to \mathbb{N}$)

$$\begin{array}{rcl} S & : & \textbf{Set} \\ Q & : & S \to \textbf{Set} \\ R & : & \textbf{Set} \\ n & : & \Pi s : S, q : Q\, s . R \to S \end{array}$$

Synek-Petersson trees: $T : S \to \textbf{Set}$

$$\textbf{data}\; S : \textbf{Set} \quad \textbf{where} \quad \frac{n : \mathbb{N}}{\mathrm{R}\, n : S} \quad \frac{\vec{s} : S^*}{\mathrm{N}\, \vec{s} : S}$$

$$\textbf{data}\; \frac{s : S}{Q\, s : \textbf{Set}} \quad \textbf{where} \quad \frac{}{\mathrm{H} : Q\,(\mathrm{N}\, \vec{s})} \quad \frac{q : Q\, \vec{s}_i}{\mathrm{D}\, i\, q : Q\,(\mathrm{N}\, \vec{s})}$$

$$\textbf{data}\; \frac{}{R : \textbf{Set}} \quad \textbf{where} \quad \frac{n : \mathbb{N}}{\mathrm{R}\, n : R} \quad \frac{}{\mathrm{N} : R}$$

Interpreting T

$$\frac{t : T\, s}{[\![t]\!] : \{g : G \mid s < g\} \to \mathbb{N}}$$

$$t \sim t' = ([\![t]\!] = [\![t']\!])$$

$$\frac{f : \{g : G \mid s < g\} \to \mathbb{N}}{q\, f : (T\, s)/\sim}$$

We can use T to give an interpretation of a 3rd order type by 1st order games.

Loose ends

- Can we interpret all arithmetic types by 1st order games? (using Synek-Petersson trees).
- Such a construction should give rise to a translation justifying continuity in Type Theory.
- Has this been done in intuitionistic logic?
- Applications to the elimination of extensionality in Observational Type Theory.