Category Theory
as
Coherently Constructive Lattice Theory

$\boxed{\text{Working Document}}$

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Chapter 1

Introduction

This document is about programming. It is also about type theory, lattice theory and category theory, but these are only the means and programming is the end. The document contains lots of programs all to do with transforming one datatype to another. The programs are not written in a conventional programming language but in the language of mathematics. To the uninitiated, therefore, it may seem that the document contains no programs and has nothing to do with programming.

We cannot offer any solace to the uninitiated. This is a working document used by the authors to collate their ongoing research and make it available to anyone who might be interested in reading it. As a working document, it contains unrefereed material and is subject to ongoing revision. It is not a document for the faint-hearted since one of its main purposes is as a repository for the complete proofs of properties included in articles submitted to conferences where space constraints force us to omit the proofs. Ultimately the intention is to write a document that does emphasise the relevance to programming and is accessible to the uninitiated.

The work reported here has its origins in the first-named author's interest in constructive type theory. In [2] the importance and relevance of constructive type theory to program design was argued —we shall not reiterate the arguments here— . The paper concluded as follows:

> Finally, the relationship between the work reported here and categorical accounts of type structures is one that we have only just hinted at. We have not discussed it in depth because we ourselves are not capable of doing so at this point in time. Nevertheless it is a topic that we believe will receive particular attention in the future.

The current document does give particular attention to a categorical formulation of type structure and its relationship to constructive type theory.

Category theoreticians view a preordered set as a particular sort of category in which there is at most one arrow between any pair of objects. According to this view, several concepts of lattice theory are instances of concepts of category theory as shown in table 1.1.

Lattice theory concept	is an instance of the category theory concept
preorder	category
monotonic function	functor
(pointwise) ordering between functions	natural transformation between functors
supremum	colimit
least	initial
Galois connection	adjunction
prefix point	algebra
closure operator	monad

Table 1.1: Lattice theory versus category theory

An alternative viewpoint, advocated by Lambek [13], is that lattice theory is a valuable source of inspiration for novel results in category theory. Indeed, it is the thesis of this document that for the purposes of advancing programming methodology category theory may profitably be regarded as "coherently constructive lattice theory[1]". That is to say, arrows between objects of a category may be seen as "witnesses" to a preordering between the objects. Category theory is thus "constructive" because it is a theory about how to construct such witnesses rather than a theory solely about their existence. Category theory is "coherently constructive" because it is also a theory about the relations between such witnesses (i.e. the existence of commuting diagrams and naturality properties). Adopting this view of category theory, the theory's contribution to programming methodology can be likened to the contribution of constructive type theory, viz. the emphasis on program construction as a by-product of the manipulation of types.

This idea is not new. Apart from Lambek cited earlier, in Scott's work [25] there is a clear progression from lattice theory to category theory, and Smyth and Plotkin [26] acknowledge "the well-known analogy between partial orders and categories" as the basis of their generalisation of the solution of fixed point equations to the construction of initial fixed points. In the textbook by Rydeheard and Burstall [23], on the other hand, the idea that category theory is constructive is the explicit theme although they do not make the link with lattice theory. Most recently, Pratt [22] has observed the relationship between the Curry-Howard isomorphism between propositions and types, the well-known focal point of constructive type theory, and residuated lattices. The lattice-theoretic properties we are studying and the organisation of those properties are documented in [1].

There is something more to be gained from looking at category theory in this way. It turns out to be very beneficial in improving our own understanding of category theory. If we are able to explain category theory as an extension of lattice theory, with as basic difference

[1]It has been remarked that we should say "preorder" theory rather than "lattice" theory. From the point of view of computing science, however, a category without sums and products has little relevance. Thus, it is indeed lattice theory that is our source of inspiration.

the introduction of witnesses, then category theory could become more understandable for a wider audience.

Those familiar with publications on category theory will be surprised that this paper contains very few diagrams. We don't reject the use of diagrams. For small problems it can be illuminating and save some writing. In particular, the nodes in a diagram are only written down once, while in normal calculations they are repeated several times. However, we want to avoid the use of diagrams, because a diagram doesn't make clear in which order the arrows are drawn and why the construction is indeed correct. Furthermore, a diagram is drawn within one particular category. We frequently deal with more than one category at once. We have no idea how in such situations we can effectively use diagrams.

This paper contains a large number of detailed proofs. This is done for two reasons. First, the authors have limited experience with category theory. So, we wanted to verify all the minute details, which in a publication are normally omitted. Second, for category theoreticians the construction of witnesses to a proof is a by-product not relevant for further calculations, their existence being all that's needed. For them the knowledge that two elements are isomorphic is enough and what the witnesses are is not considered to be relevant. However, if your interest is in the construction of programs, as ours is, then the witnesses are relevant. When stating the theorem, we have tried to separate these specific details from the theorem itself by putting them under the name "Specifics".

This paper is oriented to the derivation of categorical fixed point rules corresponding to four basic fixed point rules in lattice theory — the abstraction theorem, the fusion rule, the rolling rule and the diagonal rule. Unavoidable preliminaries are first an introduction to the basic definitions of category theory and some elementary properties, all explained from the point of view that important categorical concepts correspond to lattice-theoretical concepts. This is the content of chapter 2 which is immediately followed in chapter 3 by two substantial examples of the methodology of deriving category theoretic results by generalising from lattice theory. In chapter 5 the notion of an algebra is introduced, which corresponds in lattice theory to the notion of a prefix point. Chapter 4 is about one of the most important concepts in category theory viz. an adjunction, which corresponds in lattice theory to a Galois connection. In chapter 6 we derive the four basic categorical fixed point rules. In that same chapter we'll also immediately use two of them to derive yet another categorical fixed point rule, that turns out to be useful later. Finally in chapter 8 the categorical fixed point rules are used to establish isomorphisms between certain list structures and at the same time construct the corresponding witnesses.

Chapter 2

Basic Definitions

In this chapter we summarise the most elementary notions of category theory — category, functor and natural transformation. We emphasise the viewpoint that a category is a "constructive" preorder such that the "witnesses" to orderings are "coherent". Chapter 3 presents two substantial illustrations of this viewpoint. Definitions are taken from [14] with some notational adaptations.

2.1 Categories

We begin with the definition of a category via the notions of a graph and a deductive system.

A *graph* consists of two classes, the class of *arrows* and the class of *objects*, and two mappings from the class of arrows to the class of objects, called the *codomain* and *domain* mappings. The codomain and domain mappings are denoted by cod and dom, respectively. We denote the collection of all arrows to x from y by $x \leftarrow y$. Thus, $f \in x \leftarrow y$ is synonymous with $\mathsf{cod}.f = x \wedge \mathsf{dom}.f = y$. We also often say that f is *to* x and *from* y.

A *deductive system* is a graph in which to each object x there is associated an arrow id_x with codomain and domain both equal to x, i.e.

(2.1) $\quad \mathsf{id}_x \in x \leftarrow x \quad ,$

and to each pair of arrows $f \in x \leftarrow y$ and $g \in y \leftarrow z$ there is associated an arrow $f \circ g$ with codomain x and domain z. That is,

(2.2) $\quad f \in x \leftarrow y \wedge g \in y \leftarrow z \Rightarrow f \circ g \in x \leftarrow z \quad .$

The arrow id_x is called the *identity* arrow on x, and $f \circ g$ is called the *composition* of f with g.

A *category* is a deductive system in which the following equations hold for all arrows $f \in w \leftarrow x$, $g \in x \leftarrow y$ and $h \in y \leftarrow z$:

$$\mathsf{id}_w \circ f = f = f \circ \mathsf{id}_x \wedge (f \circ g) \circ h = f \circ (g \circ h) \quad .$$

We use the notation $x{\in}\mathcal{C}$ to denote that x is an object of the category $\mathcal{C}$.

A graph is said to be *small* if the classes of objects and arrows are sets, and *locally small* if for each pair of objects x and y the class of arrows to x from y is a set. Similarly, we talk of a *small deductive system* and a *small category* if, in each case, the underlying graph is small.

Note that the use of the membership symbol in the above definitions should not be taken to imply that we only consider small categories.

Rule (2.2) can be stated solely in terms of the domain and codomain mappings. Specifically,

$$\mathsf{dom}.f = \mathsf{cod}.g \;\Rightarrow\; \mathsf{cod}.(f\circ g) = \mathsf{cod}.f \wedge \mathsf{dom}.(f\circ g) = \mathsf{dom}.g \quad .$$

We sometimes refer to this (and other elementary rules about the domain and codomain mappings) as *(co)domain calculus*.

It is often the case that one category is defined in terms of another. For example, given a category $\mathcal{C}$, we can define objects of a category $\mathcal{D}$ to be some subclass of the arrows of $\mathcal{C}$. So, in category $\mathcal{D}$ we might have arrows to arrows of $\mathcal{C}$ from arrows of $\mathcal{C}$. To prevent confusion, we sometimes mention explicitly to which category an arrow belongs. For example, if f and g are arrows in $\mathcal{C}$, then we denote the arrow φ in $\mathcal{D}$ to f from g by

$$\varphi \in f \overset{\mathcal{D}}{\longleftarrow} g \quad .$$

In general, if we have an expression and it may not be clear in which category we are working then we'll tag that expression in some way with the intended category.

Objects x and y in the same category are said to be *isomorphic* if there are two arrows $f\in x{\leftarrow}y$ and $g\in y{\leftarrow}x$ such that $f{\circ}g = \mathsf{id}_x$ and $g{\circ}f = \mathsf{id}_y$, i.e. they are each other's inverses. If this is the case we write $x \cong y$ or, if we want to be explicit about the arrows, we write $f\in x{\cong}y \ni g$. We also say that x and y are *equal up to isomorphism*. We sometimes use the notion $f^{\cup}$ to denote the inverse of an arrow f. As a result, we often write $f\in x \cong y$. This means that x and y are isomorphic, $f\in x{\leftarrow}y$ and there exists an arrow $f^{\cup}\in y{\leftarrow}x$ that is its inverse. Finally, $\cong$ is transitive: if $f\in x\cong y$ and $g\in y\cong z$ then $f{\circ}g\in x\cong z$.

Suppose we define the relation $\sqsupseteq$ on objects in a deductive system by

$$x \sqsupseteq y \equiv \exists(f{::}\ f\in x{\leftarrow}y) \quad .$$

Then $\sqsupseteq$ is reflexive (by (2.1)) and transitive (by (2.2)). Thus the objects of a (small) deductive system form a preordered set under this relation. Borrowing jargon from constructive type theory we can read "$f\in x{\leftarrow}y$" as "f *witnesses* the ordering $x \sqsupseteq y$". In this sense, category theory is constructive: it is a theory about how to construct such witnesses rather than a theory about the existence of the witnesses.

The axioms of a category say that witnesses must be *coherent*. Consider the identity axiom: the requirement that $\mathsf{id}_w \circ f = f = f\circ \mathsf{id}_x$ whenever $f\in w{\leftarrow}x$. Assuming $w \sqsupseteq x$

there are (at least) three ways of concluding that $w \sqsupseteq x$: immediately, by arguing that $w \sqsupseteq w$ (by reflexivity) and $w \sqsupseteq x$ (by assumption) and hence by transitivity $w \sqsupseteq x$, or by arguing that $w \sqsupseteq x$ (by assumption) and $x \sqsupseteq x$ (by reflexivity) and hence by transitivity $w \sqsupseteq x$. If we augment these arguments with the construction of witnesses to the ordering we obtain f, $\mathsf{id}_w \circ f$ and $f \circ \mathsf{id}_x$. The identity axiom says that these must be the same. A similar argument applies to the associativity axiom: the requirement that $(f{\circ}g){\circ}h = f{\circ}(g{\circ}h)$ whenever $f \in w{\leftarrow}x$, $g \in x{\leftarrow}y$ and $h \in y{\leftarrow}z$. Specifically, given that $w \sqsupseteq x$, $x \sqsupseteq y$ and $y \sqsupseteq z$ there are two different ways we can combine the orderings into the ordering $w \sqsupseteq z$: we may first combine $w \sqsupseteq x$ and $x \sqsupseteq y$ by transitivity to obtain $w \sqsupseteq y$ and then combine the latter with $y \sqsupseteq z$, or we may begin by combining $x \sqsupseteq y$ and $y \sqsupseteq z$ to obtain $x \sqsupseteq z$ and then combine this with $w \sqsupseteq x$. If we augment these arguments with the construction of witnesses to the ordering we obtain $(f{\circ}g){\circ}h$ and $f{\circ}(g{\circ}h)$, respectively.

The definition of a category is decomposed into three parts: a graph, a deductive system and, finally, a category. Our own understanding of category theory is that all concepts in the theory admit the same decomposition: Underlying the category-theory concept is a concept in lattice theory; to this is added a constructive element, a mechanism for building "witnesses" to orderings; finally, the construction of witnesses is required to be "coherent". In general, the coherence requirements state that the order in which the basic rules of category theory are applied has no effect on the obtained witnesses.

2.2 Initial Objects

The simplest illustration of the coherently constructive nature of category theory is the notion of an *initial* object in a category. An *initial object* in a category is an object a such that, for each object x in the category, there is a unique arrow to x from a. The corresponding concept in lattice theory is *least* element. The constructive element in the definition of initial object is that there *exists* an arrow from the object to each object in the category. The coherence requirement is the *uniqueness* of such arrows. Following Malcolm [17, 18] and Fokkinga [8], we denote the unique arrow in a category $\mathcal{C}$ to an object x from an initial object a by $(\!|\mathcal{C}\,;x =:\ a|\!)$. That is, a is initial in $\mathcal{C}$ if and only if, for all arrows f and all objects x in $\mathcal{C}$,

(2.3) $\quad f \in x \overset{\mathcal{C}}{\longleftarrow} a \equiv f = (\!|\mathcal{C}\,;x =:\ a|\!)$.

Most often we will drop the parameter $\mathcal{C}$. In the remainder of this section we present some elementary properties and theorems concerning initial objects. Section 3.1 presents a more substantial example.

There are three elementary consequences of the definition of an initial object. First: Let a be an initial object. By instantiating $f,x := \mathsf{id}_a,a$ in (2.3) we obtain:

(2.4) $\quad \mathsf{id}_a = (\!|a =:\ a|\!)$.

Second: We have the following *fusion* theorem. We give the theorem this name because it gives conditions under which an arrow can be "fused with" an arrow from an initial object.

(Alternatively, read as a right-to-left transformation, the theorem can be seen as a rule for "defusing" an arrow from an initial object into two arrows.)

Theorem 2.5 Let a be an initial object and let x and y be arbitrary objects in a category. Furthermore, suppose $f \in x \leftarrow y$, then

$$f \circ (\!|y =: a|\!) = (\!|x =: a|\!) \quad .$$

Proof

$$\begin{array}{ll} & f \circ (\!|y =: a|\!) = (\!|x =: a|\!) \\ \equiv & \{ \qquad (2.3) \quad \} \\ & f \circ (\!|y =: a|\!) \in x \leftarrow a \\ \Leftarrow & \{ \qquad (\!|y =: a|\!) \in y \leftarrow a \text{, composition} \quad \} \\ & f \in x \leftarrow y \quad . \end{array}$$

□

Third: In a partially ordered set, i.e. a pre-ordered set where the ordering is also anti-symmetric, least elements are unique. In a category initial objects are unique up to isomorphism.

Theorem 2.6 Initial objects in the same category are isomorphic.

□

The proof is left as an (easy) exercise for the reader.

We have taken the liberty of calling 2.5 and 2.6 "theorems" because of their relative importance. Theorem 2.5 forms the basis of a number of program transformation rules enabling programs to be made more efficient [9, 8]. Among these rules are ones for "loop fusion" which state when two repetitions ("loops") can be combined ("fused") into one. So our use of the word "fusion" is not without precedent.

Because the proofs of theorems 2.5 and 2.6 are so trivial, the hard work of applying them still has to be done. Their importance is conceptual. They emphasise the importance of identifying mathematical notions as initial objects in suitable categories.

2.3 Functors

The concept of a monotonic function in lattice theory is captured in category theory by the concept of a functor.

Definition 2.7 (Functor) Given are two categories $\mathcal{C}$ and $\mathcal{D}$. F is called a (*covariant*) *functor* to $\mathcal{C}$ from $\mathcal{D}$ if F maps objects of $\mathcal{D}$ to objects of $\mathcal{C}$ and arrows of $\mathcal{D}$ to arrows of $\mathcal{C}$ in such a way that: for all objects x and y in $\mathcal{D}$

$$F.f \in F.x \overset{\mathcal{C}}{\longleftarrow} F.y \Leftarrow f \in x \overset{\mathcal{D}}{\longleftarrow} y \quad .$$

Furthermore, F must also satisfy the following two coherence requirements. Let f and g be two arrows with $\mathsf{dom.f} = \mathsf{cod.g}$ then

$$\mathsf{F.(f{\circ}g) = F.f \circ F.g}$$

and for each object x in $\mathcal{D}$

$$\mathsf{F.id_x = id_{F.x}} \quad .$$

□

The two coherence requirements on functors reflect the fact that there are two proofs of each of the properties:

$$\mathsf{F.x \sqsupseteq F.z \Leftarrow x \sqsupseteq y \wedge y \sqsupseteq z}$$

and

$$\mathsf{F.x \sqsupseteq F.x}$$

given that F is a monotonic function.

An anti-monotonic function in lattice theory corresponds to a contravariant functor.

Definition 2.8 (Contravariant Functor) Given are two categories $\mathcal{C}$ and $\mathcal{D}$. F is called a *contravariant functor* to $\mathcal{C}$ from $\mathcal{D}$ if F maps objects of $\mathcal{D}$ to objects of $\mathcal{C}$ and arrows of $\mathcal{D}$ to arrows of $\mathcal{C}$ such that: for all objects x and y in $\mathcal{D}$

$$\mathsf{F.f \in F.y \overset{\mathcal{C}}{\longleftarrow} F.x \;\Leftarrow\; f \in x \overset{\mathcal{D}}{\longleftarrow} y} \quad .$$

(Note the reversal of x and y.) Furthermore, F must also satisfy the following two coherence requirements. Let f and g be two arrows with $\mathsf{dom.f} = \mathsf{cod.g}$ then

$$\mathsf{F.(f{\circ}g) = F.g \circ F.f}$$

and for each object x in $\mathcal{D}$

$$\mathsf{F.id_x = id_{F.x}} \quad .$$

(Note the reversal of f and g.)

□

If $\mathcal{C}$ is a category, then $\mathsf{Id}_{\mathcal{C}}$ denotes the identity functor on $\mathcal{C}$. It maps objects and arrows of $\mathcal{C}$ to themselves. The subscript $\mathcal{C}$ is often omitted.

2.4 Natural Transformations

A natural transformation between functors in category theory corresponds to the pointwise ordering of monotonic functions. More specifically, in lattice theory we have the pointwise ordering $\dot{\sqsupseteq}$ on monotonic functions defined by: let f and g be monotonic functions

$$f \mathrel{\dot{\sqsupseteq}} g \equiv \forall(x::\ f.x \sqsupseteq g.x) \quad .$$

In category theory we have a similar definition, but — as usual in category theory — with an extra coherence requirement.

Definition 2.9 (Natural Transformation) Given are two categories $\mathcal{C}$ and $\mathcal{D}$ and two functors F and G to $\mathcal{C}$ from $\mathcal{D}$. A *natural transformation* η to F from G is a family of arrows, one for each object $x{\in}\mathcal{D}$, such that

$$\forall(x{:}\ x{\in}\mathcal{D}{:}\ \eta_x \in F.x \overset{\mathcal{C}}{\longleftarrow} G.x)$$

and, for each arrow $f \in x \overset{\mathcal{D}}{\longleftarrow} y$,

$$F.f \circ \eta_y = \eta_x \circ G.f \quad .$$

□

The coherence requirement arises from the fact that an arrow to $F.x$ from $G.y$ can be constructed in two different ways. This is shown in the following diagram which provides a useful way of remembering the requirement.

$$\begin{tikzcd}
F.x & G.x \arrow[l, "\eta_x"'] \\
F.y \arrow[u, "F.f"] & G.y \arrow[l, "\eta_y"'] \arrow[u, "G.f"']
\end{tikzcd}$$

Note that arrows in the diagram point from right to left or from bottom to top. This facilitates reading off the coherence requirement: just remember to always read from left to right and from top to bottom in the normal way.

2.5 Examples of Categories

2.5.1 Discrete Category

Trivial examples of categories are the *discrete* categories. A discrete category is a category in which the only arrows are the identity arrows. Formally, $\mathcal{C}$ is a discrete category whenever, for all objects x and y, and all arrows f,

$$f \in x \overset{\mathcal{C}}{\longleftarrow} y \equiv x = y \wedge f = \mathrm{id}_x \quad .$$

The discrete categories with a finite number of objects will be denoted by their number of objects in a bold font. For example, $\mathbf{0}$ denotes the (discrete) category with no objects, and $\mathbf{2}$ the discrete category with exactly two objects.

Trivial though they may be, we often make use of discrete categories, in particular as so-called "shape categories".

Exercise 2.10 Simplify the definition of a functor in the case that the domain category is $\mathbf{0}$, $\mathbf{1}$ or $\mathbf{2}$. Suppose F and G are both functors of type $\mathcal{C}\leftarrow\mathbf{2}$. What does it mean for α to be a natural transformation between F and G?

□

2.5.2 Opposite Category

Given a category $\mathcal{C}$ we also have a so-called *opposite* category $\mathcal{C}^{op}$. This category has the same class of objects and the same class of arrows but with the codomain and domain turned around. That is, $f\in x\leftarrow y$ is an arrow in $\mathcal{C}^{op}$ if and only if $f\in y\leftarrow x$ is an arrow in $\mathcal{C}$.

Because of the existence of an opposite category, we don't have to give special treatment to contravariant functors separately from (covariant) functors. More specifically, a contravariant functor F to $\mathcal{C}$ from $\mathcal{D}$ can be viewed as a (covariant) functor by replacing its codomain by $\mathcal{C}^{op}$ or its domain by $\mathcal{D}^{op}$.

Let a be an initial object in the category $\mathcal{C}$, i.e. there is a unique arrow *from* a *to* every other object in the category $\mathcal{C}$. So, in the opposite category $\mathcal{C}^{op}$ there is unique arrow *from* every other object *to* a. In the category $\mathcal{C}^{op}$ such an object a is called a *terminal object*.

2.5.3 Category of Categories

Having introduced the concept of a small category, we can talk about a category of small categories denoted by Cat. The objects of Cat are small categories and the arrows of Cat are functors. Using the notation previously introduced for arrows, we let $\mathsf{F}\in\mathcal{C}\leftarrow\mathcal{D}$ denote the functor to the category $\mathcal{C}$ from the category $\mathcal{D}$. (We use this notation also for non-small categories.) Composition of two functors $\mathsf{F}\in\mathcal{C}\leftarrow\mathcal{D}$ and $\mathsf{G}\in\mathcal{D}\leftarrow\mathcal{E}$ is denoted by $\mathsf{F}\bullet\mathsf{G}\in\mathcal{C}\leftarrow\mathcal{E}$. Note that Cat itself is not a small category in just the same way as there is no set containing all sets.

2.5.4 Functor Category

Given the pre-ordered sets $\mathcal{C}$ and $\mathcal{D}$, the set of monotonic functions to $\mathcal{C}$ from $\mathcal{D}$ forms a pre-ordered set under the pointwise ordering of functions. Correspondingly, let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $\mathsf{Fun}(\mathcal{C},\mathcal{D})$, constructed as follows, is a category. The objects are the functors to $\mathcal{C}$ from $\mathcal{D}$ and the arrows are natural transformations. Using the notation previously introduced for arrows, we let $\eta \in F \leftarrow G$ denote the natural transformation to F from G. Composition of arrows $\eta \in F \leftarrow G$ and $\tau \in G \leftarrow H$ is defined by

(2.11) $(\eta \circ \tau)_x = \eta_x \circ \tau_x$, where $x \in \mathcal{D}$.

The identity arrows are the identity transformations. I.e. $\mathsf{id} \bullet F \in F \leftarrow F$ is defined by

(2.12) $(\mathsf{id} \bullet F)_x = \mathsf{id}_{F.x}$, where $x \in \mathcal{D}$.

The category of endofunctors on $\mathcal{C}$, i.e. $\mathsf{Fun}(\mathcal{C},\mathcal{C})$, will be denoted by $\mathsf{End}.\mathcal{C}$.

Note that functors are arrows in the category Cat and objects in the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$. Sometimes we write $F \in \mathcal{C} \leftarrow \mathcal{D}$, meaning that F is an arrow in the category Cat with codomain $\mathcal{C}$ and domain $\mathcal{D}$, and sometimes we write $F \in \mathsf{Fun}(\mathcal{C},\mathcal{D})$, meaning that F is an object in the functor category. Both meanings boil down to the same thing, but in the first case the emphasis is on the fact that a functor is an arrow, and in the second case that it is an object. It may be confusing at first but after a while one gets used to the idea!

Given a monotonic function $f \in \mathcal{C} \leftarrow \mathcal{D}$ and a pre-ordered set $\mathcal{E}$ we can construct two monotonic functions $f\bullet \in (\mathcal{C} \leftarrow \mathcal{E}) \leftarrow (\mathcal{D} \leftarrow \mathcal{E})$ and $\bullet f \in (\mathcal{E} \leftarrow \mathcal{D}) \leftarrow (\mathcal{E} \leftarrow \mathcal{C})$ from monotonic functions to monotonic functions. They are defined by $(f\bullet).g = f \bullet g$ and $(\bullet f).g = g \bullet f$ where $\bullet$ denotes function composition. Similarly, given a functor $F \in \mathcal{C} \leftarrow \mathcal{D}$ and a category $\mathcal{E}$, we can construct $F\bullet \in \mathsf{Fun}(\mathcal{C},\mathcal{E}) \leftarrow \mathsf{Fun}(\mathcal{D},\mathcal{E})$ and $\bullet F \in \mathsf{Fun}(\mathcal{E},\mathcal{D}) \leftarrow \mathsf{Fun}(\mathcal{E},\mathcal{C})$. These are functors from a functor category to a functor category. The definition of $F\bullet$ is as follows. For every functor $G \in \mathsf{Fun}(\mathcal{D},\mathcal{E})$ we have:

$$(F\bullet).G = F \bullet G \quad .$$

Application of $F\bullet$ to the natural transformation $\eta \in G \leftarrow H$, where $G,H \in \mathsf{Fun}(\mathcal{D},\mathcal{E})$, is denoted by $F\bullet\eta$ and defined pointwise by

$$(F\bullet\eta)_x = F.\eta_x \quad \text{for each object } x \text{ in } \mathcal{E}.$$

Similarly, for every functor $G \in \mathsf{Fun}(\mathcal{E},\mathcal{C})$ we have:

$$(\bullet F).G = G \bullet F \quad .$$

Application of $\bullet F$ to the natural transformation $\eta \in G \leftarrow H$, where $G,H \in \mathsf{Fun}(\mathcal{E},\mathcal{C})$, is denoted by $\eta\bullet F$ and defined pointwise by

$$(\eta\bullet F)_x = \eta_{F.x} \quad \text{for each object } x \text{ in } \mathcal{D}.$$

We leave the verification of the coherence requirements needed to prove that we have indeed defined two functors as an exercise. Finally, having introduced the functors $F\bullet$ and $\bullet F$ there are all kinds of coherence properties relating natural transformations $F\bullet\eta$ and $\eta\bullet F$, commonly referred to as *Godement's rules*. Without proof we state them here. Suppose F and G are functors and η and τ are natural transformations all of appropiate type. Then

(2.13) $F \bullet \mathsf{id} = \mathsf{id} \bullet F \quad ,$

(2.14) $(\mathsf{F}\bullet\mathsf{G})\bullet\eta = \mathsf{F}\bullet(\mathsf{G}\bullet\eta)$,

(2.15) $\tau\bullet(\mathsf{F}\bullet\mathsf{G}) = (\tau\bullet\mathsf{F})\bullet\mathsf{G}$,

(2.16) $(\mathsf{G}\bullet\eta)\bullet\mathsf{F} = \mathsf{G}\bullet(\eta\bullet\mathsf{F})$,

(2.17) $\mathsf{F}\bullet(\eta\circ\tau) = (\mathsf{F}\bullet\eta)\circ(\mathsf{F}\bullet\tau)$,

(2.18) $(\eta\circ\tau)\bullet\mathsf{F} = (\eta\bullet\mathsf{F})\circ(\tau\bullet\mathsf{F})$.

Finally, suppose $\eta\in\mathsf{F}\leftarrow\mathsf{G}$ and $\tau\in\mathsf{H}\leftarrow\mathsf{K}$. Then η and τ can be composed to form a natural transformation, denoted $\eta*\tau$, of type $\mathsf{F}\bullet\mathsf{H}\leftarrow\mathsf{G}\bullet\mathsf{K}$ called their *vertical composition*. Specifically,

(2.19) $\eta*\tau = (\eta\bullet\mathsf{H})\circ(\mathsf{G}\bullet\tau) = (\mathsf{F}\bullet\tau)\circ(\eta\bullet\mathsf{K})$.

The two different but equal ways of defining vertical composition (expressed by the equality between the last two terms) is commonly known as the *interchange law* for functors and natural transformations.

Let $\eta\in\mathsf{F}\leftarrow\mathsf{G}$. Define $\eta\bullet$ by $(\eta\bullet)_{\mathsf{H}} = \eta\bullet\mathsf{H}$ for all functors H. Then we have for all functors H:

$$(\eta\bullet)_{\mathsf{H}} \in \mathsf{F}\bullet\mathsf{H}\leftarrow\mathsf{G}\bullet\mathsf{H}$$

and by (2.19) for all $\tau\in\mathsf{H}\leftarrow\mathsf{K}$

$$(\mathsf{F}\bullet).\tau\circ(\eta\bullet)_{\mathsf{K}} = (\eta\bullet)_{\mathsf{H}}\circ(\mathsf{G}\bullet).\tau \quad .$$

Thus, $\eta\bullet$ is a natural transformation. More specifically,

$$\eta\bullet \in \mathsf{F}\bullet\leftarrow\mathsf{G}\bullet \quad .$$

Furthermore, by (2.18), $(\eta\circ\tau)\bullet = \eta\bullet\circ\tau\bullet$ and, by (2.16), $(\mathsf{id}\bullet\mathsf{F})\bullet = \mathsf{id}\bullet(\mathsf{F}\bullet)$. In other words, *postcomposition* — the function mapping F to $\mathsf{F}\bullet$ and η to $\eta\bullet$ — is a functor to the category $\mathsf{Fun}(\mathsf{Fun}(\mathcal{C},\mathcal{E}),\mathsf{Fun}(\mathcal{D},\mathcal{E}))$ from the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$. Similarly, let $\tau\in\mathsf{H}\leftarrow\mathsf{K}$. Define $\bullet\tau$ by $(\bullet\tau)_{\mathsf{F}} = \mathsf{F}\bullet\tau$ for all functors F. Then, *precomposition* — the function mapping F to $\bullet\mathsf{F}$ and τ to $\bullet\tau$ — is a functor to the category $\mathsf{Fun}(\mathsf{Fun}(\mathcal{E},\mathcal{D}),\mathsf{Fun}(\mathcal{E},\mathcal{C}))$ from the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.

2.5.5 Sum Category

Suppose $\mathcal{C}$ is a category and $\mathsf{J}\in\mathsf{Cat}\leftarrow\mathcal{C}$. That is, $\mathsf{J}.x$ is a category for each object x in $\mathcal{C}$ and $\mathsf{J}.f$ is a functor for each arrow f in $\mathcal{C}$. Then we define the *sum category* $\Sigma_{\mathcal{C}}\mathsf{J}$ of J in the following way. The objects are pairs (x,y) where $x\in\mathcal{C}$ and $y\in\mathsf{J}.x$. (Thus the objects are elements, y, of the collections of objects $\mathsf{J}.x$ accompanied by a "tag", x, indicating from which collection they are drawn.) The arrows are pairs (f,g) satisfying

(2.20) $(f,g)\in(u,v)\xleftarrow{\Sigma_{\mathcal{C}}\mathsf{J}}(x,y) \;\equiv\; f\in u\xleftarrow{\mathcal{C}}x \wedge g\in v\xleftarrow{\mathsf{J}.u}(\mathsf{J}.f).y$.

The identity arrow on object (x,y) is $(\mathsf{id}^{\mathcal{C}}_x, \mathsf{id}^{J.x}_y)$. Composition of arrows is defined by

(2.21) $(f,g)\circ(h,k) \;=\; (f\circ h\;,\; g\circ(J.f).k)$.

In the case that J is a contravariant functor, i.e. a functor of type $\mathsf{Cat}\leftarrow\mathcal{C}^{op}$ we define $\Sigma_{\mathcal{C}}J$ by: the objects are pairs (x,y) where $x\in\mathcal{C}$ and $y\in J.x$, and the arrows are pairs (f,g) satisfying

(2.22) $(f,g)\in(u,v)\overset{\Sigma_{\mathcal{C}}J}{\longleftarrow}(x,y) \;\equiv\; f\in u\overset{\mathcal{C}}{\longleftarrow}x \;\wedge\; g\in(J.f).v\overset{J.x}{\longleftarrow}y$.

The identity arrow on object (x,y) is $(\mathsf{id}^{\mathcal{C}}_x, \mathsf{id}^{J.x}_y)$ and composition of arrows is defined by

(2.23) $(f,g)\circ(h,k) \;=\; (f\circ h\;,\;(J.h).g\circ k)$.

For a contravariant functor, J, the above construction is called the *Grothendieck construction.* Such a functor is called an *indexed category*. Instead of $\Sigma_{\mathcal{C}}J$ the notation $\int J$ (or $\int_{\mathcal{C}} J$ if it is desired to make the category $\mathcal{C}$ explicit) is also used.

Exercise 2.24 The definition of the composition of arrows in $\Sigma_{\mathcal{C}}J$ simplifies considerably when the category $\mathcal{C}$ is a discrete category. What is this simplification?

Suppose $\mathcal{C}$ is the discrete category with two objects 0 and 1. Suppose $J\in\mathsf{Cat}\leftarrow\mathcal{C}$ is such that $J.0=\mathcal{A}$ and $J.1=\mathcal{B}$. It is usual to write $\mathcal{A}+\mathcal{B}$ instead of $\Sigma_{\mathcal{C}}J$.

Apply the definition of $\Sigma_{\mathcal{C}}J$ in this case to determine what the objects and arrows of $\mathcal{A}+\mathcal{B}$ are.

□

Exercise 2.25 In the construction of the sum category, Σ maps functors to categories. The question that immediately arises is whether the definition of Σ can be extended to map natural transformations (arrows in the functor category $\mathsf{Cat}\leftarrow\mathcal{C}$) to functors (arrows in the category Cat) in such a way that it is a functor. This can indeed be done. The exercise is a tedious one since it involves careful unfolding of several definitions. Nevertheless, it is worthwhile working through some of the details in order to gain familiarity with the various definitions. In order to derive the educational benefit without too much of the tedium we present some of the details for you.

We have to define $\Sigma\alpha$ for each natural transformation α. Suppose $\alpha\in J\leftarrow K$. Then $\Sigma\alpha$ should be a functor (an arrow in Cat) of type $\Sigma_{\mathcal{C}}J\leftarrow\Sigma_{\mathcal{C}}K$. Thus, the first task is to define the application of $\Sigma\alpha$ to objects and to arrows in the category $\Sigma_{\mathcal{C}}K$.

Guided by the type requirements, we define $\Sigma\alpha$ on objects by $\Sigma\alpha.(u,v)=(u\;,\;\alpha_u.v)$ for each $u\in\mathcal{C}$ and $v\in K.u$. Verify that this meets the requirement that $\Sigma\alpha.(u,v)\in\Sigma_{\mathcal{C}}J$. (You will need to use the fact that α_u is a functor. How much of the definition of a functor do you actually use?)

Also guided by the type requirements, we define $\Sigma\alpha$ on arrows by $\Sigma\alpha.(f,g)=(f\;,\;\alpha_u.g)$ whenever $f\in u\overset{\mathcal{C}}{\longleftarrow}x$ and $g\in v\overset{K.u}{\longleftarrow}(K.f).y$. Verify that this meets the requirement that $\Sigma\alpha.(f,g)\in\Sigma\alpha.(u,v)\overset{\Sigma_{\mathcal{C}}J}{\longleftarrow}\Sigma\alpha.(x,y)$. (Here you will need to use the fact that α is a natural transformation.) Verify in addition that $\Sigma\alpha$ preserves identities and distributes through composition.

$\square$

Another important category, the category of algebras, is treated in chapter 5.

Chapter 3

Elementary Illustrations

In this chapter we present two elementary illustrations of the methodology of deriving categorical results by generalising from lattice theoretic results. The second of the two —"Yoneda's lemma"— is (in our view) transformed in this way from a relatively difficult proposition to a straightforward, but tedious, one.

3.1 The Initial Functor

In this section we present a relatively straightforward but nevertheless substantial illustration of the constructive nature of category theory. The lemma we prove combines the notion of a sum category with the notion of an initial object and is thus a good illustration of categorical concepts *per se*. It will also be useful to us later on.

We begin with the statement of the lemma in lattice theory. Suppose $(\mathsf{D}, \leq)$ is a poset and $\{x\colon x{\in}\mathsf{D}\colon \mathsf{S}.x\}$ is a set of sets and suppose that each set $\mathsf{S}.x$ has a least element $\mathsf{least}.x$. Then, if S is an antimonotonic function (i.e. $\mathsf{S}.x \supseteq \mathsf{S}.y \Leftarrow x \leq y$), the function least is monotonic. The proof is elementary.

To obtain the categorical version of this fact we replace "poset" by "category", "antimonotonic function S" by "contravariant functor S", the indexed set of sets "$\{x\colon x{\in}\mathsf{D}\colon \mathsf{S}.x\}$" by $\Sigma\mathsf{S}$, "least" by "initial" and "monotonic function" by "(covariant) functor". We get:

Lemma 3.1 Suppose $\mathcal{D}$ is a category, and $\mathsf{S} \in \mathsf{Cat}{\leftarrow}\mathcal{D}$ is a contravariant functor. Suppose further that, for each x in $\mathcal{D}$, the category $\mathsf{S}.x$ has an initial object given by $\mathsf{InitObj}.x$. Then there is a functor $\mathsf{Init} \in \Sigma\mathsf{S}{\leftarrow}\mathcal{D}$.

□

In the future, whenever we state a lemma such as this one we will simultaneously provide all details. In this case, however, we postpone giving the details of the functor Init because we want to illustrate how its definition is *constructed*. Were we to provide the details immediately it would suggest that we always somehow *guess* what the definition is and then *verify* that our guess is correct, but that is not the case!

Referring to the definition of a functor (definition 2.7), we have to define Init on objects and arrows of $\mathcal{D}$. In the case of object x we define

$$\mathsf{Init}.x = (x, \mathsf{InitObj}.x)$$

this being the obvious way to satisfy the requirement that $\mathsf{Init}.x \in \Sigma S$. On arrows $f \in x \overset{\mathcal{D}}{\longleftarrow} y$ the requirement for Init to be a functor is:

$$\mathsf{Init}.f \in \mathsf{Init}.x \overset{\Sigma S}{\longleftarrow} \mathsf{Init}.y \quad .$$

Since arrows in ΣS are, by definition, pairs we postulate that $\mathsf{Init}.f = (\alpha, \beta)$ and we calculate suitable definitions of α and β as follows:

$$\begin{array}{ll}
 & (\alpha, \beta) \in \mathsf{Init}.x \overset{\Sigma S}{\longleftarrow} \mathsf{Init}.y \\
\equiv & \qquad \{ \qquad \text{definition of an arrow in } \Sigma S\text{: (2.22),} \\
 & \qquad\qquad\quad \text{definition of } \mathsf{Init}.x \text{ and } \mathsf{Init}.y \quad \} \\
 & \alpha \in x \overset{\mathcal{D}}{\longleftarrow} y \ \wedge \ \beta \in (S.\alpha).(\mathsf{InitObj}.x) \overset{S.y}{\longleftarrow} \mathsf{InitObj}.y \\
\equiv & \qquad \{ \qquad \bullet \quad \alpha = f \quad \} \\
 & \beta \in (S.f).(\mathsf{InitObj}.x) \overset{S.y}{\longleftarrow} \mathsf{InitObj}.y \\
\equiv & \qquad \{ \qquad \mathsf{InitObj}.y \text{ is initial in } S.y \quad \} \\
 & \beta = (\!| S.y \,;\, (S.f).(\mathsf{InitObj}.x) =: \mathsf{InitObj}.y |\!)
\end{array}$$

The *unique* way of satisfying the type requirement on $\mathsf{Init}.f$ is thus by letting:

$$\text{(3.2)} \qquad \mathsf{Init}.f = (f\,,\ (\!| S.y \,;\, (S.f).(\mathsf{InitObj}.x) =: \mathsf{InitObj}.y |\!)) \quad .$$

We now have to verify that Init maps identity arrows to identity arrows as well as distributing through the composition of two arrows. For brevity we use $\mathsf{InitArr}.f$ to denote the second component of $\mathsf{Init}.f$. First the verification that identity arrows are mapped to identity arrows.

$$\begin{array}{ll}
 & \mathsf{Init}.id_x = id_{\mathsf{Init}.x} \\
\equiv & \qquad \{ \qquad \text{definition of } \mathsf{Init}\text{: (3.2),} \\
 & \qquad\qquad\quad \text{definition of identity arrows in } \Sigma S \quad \} \\
 & (id_x\,,\ \mathsf{InitArr}.id_x) = (id_x, id_{\mathsf{InitObj}.x}) \\
\equiv & \qquad \{ \qquad \text{pair forming} \quad \} \\
 & \mathsf{InitArr}.id_x = id_{\mathsf{InitObj}.x} \\
\equiv & \qquad \{ \qquad \text{initiality of } \mathsf{InitObj}.x \quad \} \\
 & \mathsf{InitArr}.id_x \in \mathsf{InitObj}.x \overset{S.x}{\longleftarrow} \mathsf{InitObj}.x \\
\Leftarrow & \qquad \{ \qquad S.id_x = id_{S.x} \quad \}
\end{array}$$

$$
\begin{array}{ll}
 & \mathsf{InitArr.id_x} \in (\mathsf{S.id_x}).(\mathsf{InitObj.x}) \overset{\mathsf{S.x}}{\longleftarrow} \mathsf{InitObj.x} \\
\equiv & \qquad \{ \quad \text{by construction} \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

Now we prove that Init distributes through composition. Suppose $\mathsf{f} \in \mathsf{x} \leftarrow \mathsf{y}$ and $\mathsf{g} \in \mathsf{y} \leftarrow \mathsf{z}$. Then,

$$
\begin{array}{ll}
 & \mathsf{Init.(f \circ g)} = \mathsf{Init.f} \circ \mathsf{Init.g} \\
\equiv & \qquad \{ \quad \text{definition} \quad \} \\
 & (\mathsf{f \circ g}\,,\ \mathsf{InitArr.(f \circ g)}) \ = \ (\mathsf{f}, \mathsf{InitArr.f}) \circ (\mathsf{g}, \mathsf{InitArr.g}) \\
\equiv & \qquad \{ \quad \text{definition of composition in } \Sigma\mathsf{S} \ : \ (2.23) \quad \} \\
 & (\mathsf{f \circ g}\,,\ \mathsf{InitArr.(f \circ g)}) \ = \ (\mathsf{f \circ g}\ ,\ (\mathsf{S.g}).(\mathsf{InitArr.f}) \circ \mathsf{InitArr.g}) \\
\Leftarrow & \qquad \{ \quad \text{pair forming} \quad \} \\
 & \mathsf{InitArr.(f \circ g)} \ = \ (\mathsf{S.g}).(\mathsf{InitArr.f}) \circ \mathsf{InitArr.g} \\
\equiv & \qquad \{ \quad \text{initiality: both terms have domain } \mathsf{InitObj.z} \quad \} \\
 & \mathsf{cod.(InitArr.(f \circ g))} = \mathsf{cod.((S.g).(InitArr.f) \circ InitArr.g)} \\
\Leftarrow & \qquad \{ \quad \text{construction of } \mathsf{InitArr}\,,\ \mathsf{cod.(f \circ g)} = \mathsf{cod.f} \quad \} \\
 & (\mathsf{S.(f \circ g)}).(\mathsf{InitObj.x}) = \mathsf{cod.((S.g).(InitArr.f))} \\
\equiv & \qquad \{ \quad \mathsf{S} \text{ is a contravariant functor,} \\
 & \qquad\qquad \mathsf{S.g} \text{ is a (covariant) functor} \quad \} \\
 & (\mathsf{S.g} \bullet \mathsf{S.f}).(\mathsf{InitObj.x}) = (\mathsf{S.g}).(\mathsf{cod.(InitArr.f)}) \\
\equiv & \qquad \{ \quad \mathsf{cod.(InitArr.f)} = (\mathsf{S.f}).(\mathsf{InitObj.x}) \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

This completes the proof.

3.2 Yoneda's Lemma

In this section we illustrate the idea of category theory as coherently constructive lattice theory by showing how to formulate and prove Yoneda's lemma.

3.2.1 In Lattice Theory

By 'a monotonic predicate' we mean a predicate p such that:

$$(\mathsf{p.x} \Leftarrow \mathsf{p.y}) \Leftarrow \mathsf{x} \sqsupseteq \mathsf{y} \quad .$$

In lattice theory we have the following lemma.

Lemma 3.3 Suppose p is a monotonic predicate and x is an arbitrary element of the domain of p. Then,

$$p.x \equiv \forall(y{::}\ p.y \Leftarrow y \sqsupseteq x) \quad . \tag{3.4}$$

□

The lemma is so simple that it doesn't have a name. Among its corollaries is the rule we call *indirect equality* — a rule that is equally simple but which deserves a name because of its ubiquity.

Corollary 3.5 (indirect equality) For arbitrary elements x and z,

$$x = z \equiv \forall(y{::}\ y \sqsupseteq z \equiv y \sqsupseteq x) \quad .$$

Proof Instantiate $p.x := (x \sqsupseteq z)$ in lemma 3.3 (noting that p is indeed monotonic). Then we have

$$x \sqsupseteq z \equiv \forall(y{::}\ y \sqsupseteq z \Leftarrow y \sqsupseteq x) \quad . \tag{3.6}$$

The result follows from (3.6), (3.6) with x and z interchanged, and anti-symmetry.
□

We now formulate and prove a lemma similar to (3.3) but within a categorical framework. We first make the following observation. Suppose we denote function f with domain A by $[x : x{\in}A : f.x]$, or by $[x :: f.x]$ [1] if the domain is clear from the context. Then, using extensionality three times, we can rewrite (3.4) as the equality:

$$[p, x : \mathit{monotonic}.p : p.x] = [p, x : \mathit{monotonic}.p : p \Leftarrow [y :: y \sqsupseteq x]] \quad . \tag{3.7}$$

Now note that the three predicates in this equality are all *monotonic* functions. The predicate $[p, x :: p.x]$ is the *evaluation function* (restricted to monotonic predicates). It is the function denoted by the infix dot. (We have refrained from writing just a dot on the lefthand side of the equation because it would most probably confuse a great many readers!) It is monotonic in the argument p —*by definition*— with respect to the pointwise ordering on predicates:

$$p \Leftarrow q \equiv \forall(x{::}\ p.x \Leftarrow q.x) \quad .$$

It is also monotonic in the argument x by virtue of the following simple calculation:

$$
\begin{array}{ll}
 & [p : \mathit{monotonic}.p : p.x] \Leftarrow [p : \mathit{monotonic}.p : p.y] \\
\equiv & \qquad \{ \qquad \text{definition of pointwise ordering} \quad \} \\
 & \forall(p{:}\ \mathit{monotonic}.p{:}\ p.x \Leftarrow p.y) \\
\Leftarrow & \qquad \{ \qquad \text{definition of monotonic predicate} \quad \} \\
 & x \sqsupseteq y \quad .
\end{array}
$$

[1] In the more familiar lambda notation this would be written $\lambda x.fx$. The freedom to include domain information is a useful device, particularly for restricting the domain of a function.

Similarly, the other two functions are also monotonic. We urge you to write down the (rather trivial) verifications. It will help you shortly to understand the categorical equivalents.

In order to "lift" (3.7) to category theory we replace equality by isomorphism, monotonic function by functor, and predicate p by a functor $F \in \mathsf{Set} \leftarrow \mathcal{C}$ for a locally small category $\mathcal{C}$. In addition, we have to define functorial counterparts to the three monotonic functions just discussed.

3.2.2 The Evaluation Functor

Let us begin with the evaluation function. We have to define an evaluation *functor* for a given locally small category $\mathcal{C}$. It must map $F \in \mathsf{Set} \leftarrow \mathcal{C}$ and $x \in \mathcal{C}$ to an object in Set : the object $F.x$ is the obvious candidate. Moreover, it must map an arrow $f \in x \overset{\mathcal{C}}{\longleftarrow} y$ and a natural transformation $\eta \in F \xleftarrow{\mathsf{Fun}(\mathsf{Set},\mathcal{C})} G$ to an arrow $\eta.f$ in Set. Being a binary functor means that it has to be *constructively* monotonic in both its arguments. That is, for given objects $x{,}y \in \mathcal{C}$, arrow $f \in x \overset{\mathcal{C}}{\longleftarrow} y$, functors $F{,}G \in \mathsf{Set} \leftarrow \mathcal{C}$ and natural transformation $\eta \in F \xleftarrow{\mathsf{Fun}(\mathsf{Set},\mathcal{C})} G$ we have to construct an arrow $\eta.f \in F.x \overset{\mathsf{Set}}{\longleftarrow} G.y$. In other words $\eta.f$ must satisfy

$$(3.8) \qquad \eta.f \in F.x \leftarrow G.y \;\Leftarrow\; \eta \in F \leftarrow G \,\wedge\, f \in x \leftarrow y \quad .$$

Compare this requirement with the monotonicity of the evaluation function:

$$(3.9) \qquad F.x \sqsupseteq G.y \Leftarrow F \mathrel{\dot{\sqsupseteq}} G \wedge x \sqsupseteq y \quad .$$

Compare also the proof of (3.9) (assuming that F and G are monotonic, proof left to the reader) with the following construction of $\eta.f$:

$$
\begin{array}{ll}
& \eta.f \in F.x \leftarrow G.y \\
\Leftarrow & \quad \{ \quad G \text{ is a functor. Thus, } G.f \in G.x \leftarrow G.y \\
& \qquad\quad \bullet \quad \eta.f = \alpha \circ G.f \quad \} \\
& \alpha \in F.x \leftarrow G.x \\
\Leftarrow & \quad \{ \quad \text{definition of natural transformation} \quad \} \\
& \alpha = \eta_x \quad .
\end{array}
$$

In conclusion, $\eta.f = \eta_x \circ G.f$ satisfies the requirement (3.8).

There is an alternative to the first step in the above calculation, namely, we could have used the fact that F rather than G is a functor. This leads to defining $\eta.f$ as $F.f \circ \eta_y$. The coherence property of natural transformations guarantees that the choice is irrelevant.

We still have to verify the coherence requirement on the evaluation functor. Since it is a binary functor we have to check that

$$(\eta \circ \tau).(f \circ g) = \eta.f \circ \tau.g$$

for all natural transformations η and τ, and all arrows f and g . (Note that all three compositions are in different categories.) We also have to check that

$$(\mathsf{id}\bullet\mathsf{F}).\mathsf{id}_x^{\mathcal{C}} = \mathsf{id}_{\mathsf{F}.x}^{\mathsf{Set}}$$

for all functors F and objects x. We leave these (simple) checks to the reader.

This completes the construction of the evaluation functor and the verification that it is indeed a functor. In order to maintain the suggestive link with lattice theory we shall denote it by $[\mathsf{F},x:\ \mathsf{F}{\in}\mathsf{Fun}(\mathsf{Set},\mathcal{C})\wedge x{\in}\mathcal{C}:\ \mathsf{F}.x]$ or, more commonly, $[\mathsf{F},x::\ \mathsf{F}.x]$. In general, when we introduce quantified expressions (as indicated by square brackets) the dummies range over both objects *and* arrows of the relevant categories. Thus, in this case, F ranges over objects and arrows of the functor category $\mathsf{Fun}(\mathsf{Set},\mathcal{C})$ and x ranges over objects and arrows of the category $\mathcal{C}$.

One final thing to note about the evaluation functor is that the structure of the category Set has never been used — it might just as well have been some arbitrary category.

3.2.3 Hom Functors (Generalised)

We still have to define the functors corresponding to the functions $[y::\ y\sqsupseteq x]$ and $[p,x::\ p\Leftarrow[y::\ y\sqsupseteq x]]$. It pays to first define a more general functor of which both are instances.

Suppose p and q are arbitrary monotonic predicates (not necessarily with the same domain). Then the predicate $p.x\Leftarrow q.y$ is monotonic in x and anti-monotonic in y. Let us "lift" this observation to a statement in category theory.

Suppose $\mathsf{F}\in\mathcal{C}{\leftarrow}\mathcal{D}$ and $\mathsf{G}\in\mathcal{C}{\leftarrow}\mathcal{E}$ are arbitrary functors. Consider the function mapping $x{\in}\mathcal{D}$ and $y{\in}\mathcal{E}$ to the set[2] $\mathsf{F}.x\overset{\mathcal{C}}{\longleftarrow}\mathsf{G}.y$ (the arrows in the category $\mathcal{C}$ to $\mathsf{F}.x$ from $\mathsf{G}.y$). This function can be extended to a binary functor that is covariant in the argument x and contravariant in the argument y. The codomain of the functor is the category Set. Its domain is the cartesian product of the category $\mathcal{D}$ and (the opposite of) category $\mathcal{E}$. On arrows $f\in x\overset{\mathcal{D}}{\longleftarrow}u$ and $g\in y\overset{\mathcal{E}}{\longleftarrow}v$ we define $\mathsf{F}.f\leftarrow\mathsf{G}.g$ by

$$(\mathsf{F}.f\leftarrow\mathsf{G}.g).h = \mathsf{F}.f\circ h\circ\mathsf{G}.g \quad ,$$

for all $h\in\mathsf{F}.u\overset{\mathcal{C}}{\longleftarrow}\mathsf{G}.y$. Straightforwardly,

$$\mathsf{F}.f\leftarrow\mathsf{G}.g\in(\mathsf{F}.x\overset{\mathcal{C}}{\longleftarrow}\mathsf{G}.v)\leftarrow(\mathsf{F}.u\overset{\mathcal{C}}{\longleftarrow}\mathsf{G}.y) \quad .$$

(Note the switch in the order of y and v. The unlabelled arrow on the right denotes a class of arrows in the category Set. So, $\mathsf{F}.f\leftarrow\mathsf{G}.g$ is a function.) Moreover,

$$(\mathsf{F}.\mathsf{id}_x\leftarrow\mathsf{G}.\mathsf{id}_v).h = h \quad ,$$

and

$$(\mathsf{F}.f\leftarrow\mathsf{G}.g)\bullet(\mathsf{F}.h\leftarrow\mathsf{G}.k) \;=\; \mathsf{F}.(f\circ h)\leftarrow\mathsf{G}.(k\circ g) \quad ,$$

[2]For convenience we assume that the category is locally small. That is, the arrows between a pair of objects form a set. We shall see later that this can be avoided.

for all f, g, h and k of suitable type. (Note the switch in the order of g and k.) Thus we have indeed defined a functor. Let us denote this functor by $[x,y :: F.x \leftarrow G.y]$, the square brackets indicating abstraction from the dummies x and y.

If F and G are both instantiated to the identity functor on some category $\mathcal{C}$ then the functor we obtain, $[x,y :: x{\leftarrow}y]$, maps a pair of objects x and y onto the set of arrows to x from y. This set is called a *hom set* ("hom" being short for "homomorphism") and the functor is called the hom functor. Category theoreticians will recognize the functor $[x,y :: F.x \leftarrow G.y]$ as the composition of the hom functor on the (common) codomains of F and G and the product of the functors F and the opposite of G.

3.2.4 The Lemma

Given a binary functor we can always construct a unary functor by fixing one of its arguments. The binary functor $[x,y :: F.x \leftarrow G.y]$ thus has instances the covariant functor $[x :: F.x \leftarrow y]$ defined on arrow f by

$$(F.f \leftarrow \mathsf{id}_y).h = F.f \circ h \quad ,$$

and the contravariant functor $[y :: x \leftarrow G.y]$ defined on arrow g by

$$(\mathsf{id}_x \leftarrow G.g).h = h \circ G.g \quad .$$

So we now have all the bits and pieces needed to formulate the categorical counterpart of lemma (3.3), a lemma commonly referred to as *Yoneda's lemma*.

Lemma 3.10 (Yoneda's lemma) Suppose $\mathcal{C}$ is locally small. Then,

$$[F,x : F{\in}\mathsf{Fun}(\mathsf{Set},\mathcal{C}) \wedge x{\in}\mathcal{C} : F.x] \cong [F,x : F{\in}\mathsf{Fun}(\mathsf{Set},\mathcal{C}) \wedge x{\in}\mathcal{C} : F{\leftarrow}[y :: y{\leftarrow}x]] \quad .$$

□

For the proof of Yoneda's lemma we are required to construct two witnesses φ and ψ such that

$$\varphi \in [F,x :: F.x] \cong [F,x :: F{\leftarrow}[y :: y{\leftarrow}x]] \ni \psi \quad .$$

We begin by constructing candidates for the natural transformations φ and ψ.

Candidate φ: Suppose x is an arbitrary object of $\mathcal{C}$. For φ we observe that

$$\varphi_{F,x} \in F.x \overset{\mathsf{Set}}{\longleftarrow} (F{\leftarrow}[y :: y{\leftarrow}x]) \quad .$$

So, $\varphi_{F,x}$ is a function and on $\eta \in F{\leftarrow}[y :: y{\leftarrow}x]$ we define it by:

$$\varphi_{F,x}.\eta = \eta_x.\mathsf{id}_x \quad . \tag{3.11}$$

Candidate ψ: For ψ we observe that

$$\psi_{F,x} \in (F{\leftarrow}[y :: y{\leftarrow}x]) \overset{\mathsf{Set}}{\longleftarrow} F.x \quad .$$

Take an object $a \in F.x$ then

$$\psi_{F,x}.a \in F \xleftarrow{\mathsf{Fun}(\mathsf{Set},\mathcal{C})} [y :: y \leftarrow x] \quad .$$

So, $\psi_{F,x}.a$ is a natural transformation. Thus for all objects y and arrows $f \in u \leftarrow v$ we must have

$$(\psi_{F,x}.a)_y \in F.y \xleftarrow{\mathsf{Set}} (y \leftarrow x) \;\wedge\; F.f \circ (\psi_{F,x}.a)_v = (\psi_{F,x}.a)_u \circ (f\circ) \quad .$$

So, $(\psi_{F,x}.a)_y$ is a function and on $g \in y \leftarrow x$ we define it by:

$$(3.12) \quad (\psi_{F,x}.a)_y.g = (F.g).a \quad .$$

It remains to verify the naturality property of $\psi_{F,x}.a$. Take $g \in v \leftarrow x$. Then,

$$
\begin{array}{ll}
 & (F.f \circ (\psi_{F,x}.a)_v).g \\
= & \{ \quad \text{definition } \psi_{F,x}.a \quad \} \\
 & (F.f).((F.g).a) \\
= & \{ \quad \text{composition, } F \text{ functor} \quad \} \\
 & (F.(f \circ g)).a \\
= & \{ \quad \text{definition } \psi_{F,x}.a \quad \} \\
 & (\psi_{F,x}.a)_u.(f \circ g) \quad .
\end{array}
$$

Thus we have derived candidates for both φ and ψ.
Inverses: We now prove that they are others' inverses. First,

$$
\begin{array}{ll}
 & \varphi_{F,x}.(\psi_{F,x}.a) \\
= & \{ \quad \text{definition } \varphi \quad \} \\
 & (\psi_{F,x}.a)_x.\mathsf{id}_x \\
= & \{ \quad \text{definition } \psi \quad \} \\
 & (F.\mathsf{id}_x).a \\
= & \{ \quad F \text{ functor} \quad \} \\
 & \mathsf{id}_{F.x}.a \quad .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & (\psi_{F,x}.(\varphi_{F,x}.\eta))_y.g \\
= & \{ \quad \text{definition } \varphi \text{ and } \psi \quad \} \\
 & (F.g).(\eta_x.\mathsf{id}_x) \\
= & \{ \quad \eta \in F \leftarrow [y :: y \leftarrow x] \text{, so } F.g \circ \eta_x = \eta_y \circ (g\circ) \quad \} \\
 & \eta_y.g \quad .
\end{array}
$$

Naturality φ: Finally we prove the naturality property for φ, i.e. for all $f \in u \leftarrow v$, and all $\tau \in F \leftarrow G$,

$$\tau.f \circ \varphi_{G,v} = \varphi_{F,u} \circ (\tau \leftarrow [y :: y \leftarrow f]) \quad .$$

Note,

$$\tau \leftarrow [y :: y \leftarrow f] \in (F \leftarrow [y :: y \leftarrow v]) \xleftarrow{\mathsf{Set}} (G \leftarrow [y :: y \leftarrow u]) \quad ,$$

i.e. it is a function and for $\eta \in G \leftarrow [y :: y \leftarrow u]$ it is defined by

$$((\tau \leftarrow [y :: y \leftarrow f]).\eta)_z = \tau_z \circ \eta_z \circ (\circ f) \quad . \tag{3.13}$$

Thus,

$$
\begin{array}{ll}
 & (\varphi_{F,u} \circ (\tau \leftarrow [y :: y \leftarrow f])).\eta \\
= & \quad \{ \quad \text{application} \quad \} \\
 & \varphi_{F,u}.((\tau \leftarrow [y :: y \leftarrow f]).\eta) \\
= & \quad \{ \quad \text{definition } \varphi \quad \} \\
 & ((\tau \leftarrow [y :: y \leftarrow f]).\eta)_u.\mathsf{id}_u \\
= & \quad \{ \quad (3.13) \quad \} \\
 & (\tau_u \circ \eta_u).f \\
= & \quad \{ \quad \eta \in G \leftarrow [y :: y \leftarrow x] \text{, so } G.f \circ \eta_v = \eta_u \circ (f \circ) \\
 & \quad \phantom{\{} \quad \text{In particular, } (G.f).(\eta_v.\mathsf{id}_v) = \eta_u.f \quad \} \\
 & (\tau_u \circ G.f).(\eta_v.\mathsf{id}_v) \\
= & \quad \{ \quad \text{definition of } \tau.f \text{ and of } \varphi \quad \} \\
 & (\tau.f).(\varphi_{G,v}.\eta) \\
= & \quad \{ \quad \text{composition} \quad \} \\
 & (\tau.f \circ \varphi_{G,v}).\eta \quad .
\end{array}
$$

The naturality of φ follows by extensionality.

We now have to verify the naturality property of ψ. However, this is unnecessary, and our proof of Yoneda's lemma is complete, because of the following lemma.

Lemma 3.14 (natural isomorphism) For all functors F and G,

$$
\begin{array}{ll}
 & \varphi \in F \cong G \ni \psi \\
\equiv & \quad \forall(x :: \ \varphi_x \in F.x \cong G.x \ni \psi_x) \\
 & \wedge \ \forall(f: \ f \in x \leftarrow y: \ F.f \circ \varphi_y = \varphi_x \circ G.f) \quad .
\end{array}
$$

Proof Lemma 3.14 follows almost immediately from the definition of isomorphism and natural transformation. The only remarkable thing is that in the $\Leftarrow$ part nothing is assumed about the naturality of ψ . That it holds follows by the following calculation. For all $f \in x \leftarrow y$ we have

$$\begin{array}{ll}
& G.f \circ \psi_y = \psi_x \circ F.f \\
\equiv & \quad \{ \quad \varphi_x \circ \psi_x = id_{F.x} \text{ and } \psi_x \circ \varphi_x = id_{G.x} , \\
& \quad\quad \text{Leibniz in both directions} \quad \} \\
& \varphi_x \circ (G.f \circ \psi_y) \circ \varphi_y = \varphi_x \circ (\psi_x \circ F.f) \circ \varphi_y \\
\equiv & \quad \{ \quad \varphi_x \circ \psi_x = id_{F.x} \text{ and } \psi_x \circ \varphi_x = id_{G.x} \quad \} \\
& \varphi_x \circ G.f = F.f \circ \varphi_y \quad .
\end{array}$$

□

3.2.5 Corollaries

Just as indirect equality is a direct consequence of lemma 3.4, the rule of indirect isomorphism is a direct consequence of Yoneda's lemma:

Lemma 3.15 (indirect isomorphism) Suppose x and y are objects of a locally small category $\mathcal{C}$. Then

$$x \cong z \equiv [y :: y \leftarrow x] \cong [y :: y \leftarrow z]$$

Proof By taking $F := [y :: y \leftarrow z]$ in lemma 3.10 we have (after some dummy renaming)

$$\varphi \in [x :: x \leftarrow z] \cong [x :: [y :: y \leftarrow z] \leftarrow [y :: y \leftarrow x]] \ni \psi \quad .$$

Since this is a natural isomorphism we also have, for an arbitrary object x ,

$$\text{(3.16)} \quad \varphi_x \in x \leftarrow z \cong [y :: y \leftarrow z] \leftarrow [y :: y \leftarrow x] \ni \psi_x \quad .$$

Similarly, we can derive the isomorphism, for an arbitrary object z ,

$$\text{(3.17)} \quad \rho_z \in z \leftarrow x \cong [y :: y \leftarrow x] \leftarrow [y :: y \leftarrow z] \ni \xi_z \quad .$$

In both (3.16) and (3.17) the witnesses are defined as in the proof of lemma 3.10.

Now, for the $\Leftarrow$ part, assume

$$\eta \in [y :: y \leftarrow z] \cong [y :: y \leftarrow x] \ni \tau \quad .$$

Then, $\varphi_x.\eta \in x \leftarrow z$ and $\rho_z.\tau \in z \leftarrow x$. We only prove $\varphi_x.\eta \circ \rho_z.\tau = id_x$. The composition in reverse order is completely symmetric:

$$
\begin{array}{ll}
 & \varphi_x.\eta \circ \rho_z.\tau \\
= & \{ \quad \text{definition: } \ \varphi_x.\eta = \eta_x.\mathsf{id}_x \quad \} \\
 & \eta_x.\mathsf{id}_x \circ \tau_z.\mathsf{id}_z \\
= & \{ \quad \text{let } f \in u \leftarrow v \text{ then } (f\circ) \circ \tau_v = \tau_u \circ (f\circ)\text{, i.e.} \\
 & \quad\ \ \text{let } g \in v \leftarrow z \text{ then } f \circ \tau_v.g = \tau_u.(f \circ g) \quad \} \\
 & \tau_x.(\eta_x.\mathsf{id}_x) \\
= & \{ \quad \tau \circ \eta = \mathsf{id}_{[y::\, y \leftarrow z]} \quad \} \\
 & \mathsf{id}_x \ \ .
\end{array}
$$

For the $\Rightarrow$ part, assume

$$f \in x \cong z \ni g \ \ .$$

Then, $\psi_x.f \in [y :: y \leftarrow z] \leftarrow [y :: y \leftarrow x]$ and $\xi_z.g \in [y :: y \leftarrow x] \leftarrow [y :: y \leftarrow z]$ Again, for reasons of symmetry, we only prove that $\psi_x.f \circ \xi_z.g = \mathsf{id}_{[y::\, y \leftarrow z]}$. Suppose $h \in y \leftarrow z$

$$
\begin{array}{ll}
 & (\psi_x.f \circ \xi_z.g)_y.h \\
= & \{ \quad \text{definition: } \ (\xi_z.g)_y.h = ([y :: y \leftarrow x].h).g = h \circ g \quad \} \\
 & (\psi_x.f)_y.(h \circ g) \\
= & \{ \quad \text{definition} \quad \} \\
 & h \circ g \circ f \\
= & \{ \quad g \circ f = \mathsf{id}_x \quad \} \\
 & h \ \ .
\end{array}
$$

This concludes our proof.
□

It is abundantly clear from this one example that the progression from lattice theory to category theory causes an explosion in the lengths of one's calculations. We hope to have demonstrated, nonetheless, that the process is systematic. The construction of witnesses does not add to the lengths of the proofs, although it does add extra detail. It is the verification of the coherence properties that is the additional burden, together with the fact that whenever a functor is introduced one must define it for both objects and arrows. (At the same time, however, it is the coherence properties that give the theorem in category theory its added value.)

Given the tremendous amount of detail that needs to be borne in mind, it is our view that much effort in organising and streamlining proofs in lattice theory is an essential prerequisite to tackling non-trivial problems in category theory. In subsequent chapters we present novel theorems that were discovered in this way but which we probably would never have been able to discover otherwise.

Chapter 4

Adjunctions

The concept of a *Galois connection* in lattice theory is captured in category theory by the concept of an *adjunction*. Before giving the definition of an adjunction, we first give the definition of a Galois connection and show some elementary examples and properties. In lattice theory there are several alternative, but equivalent, definitions of a Galois connection. We will show that we can also give several alternative, but equivalent, definitions of an adjunction. In sections 4.3 and 4.4 we will prove several properties concerning adjunctions.

4.1 Galois Connections

The combination of two partially ordered sets $(\mathsf{C},\sqsupseteq)$ and $(\mathsf{D},\succeq)$, and two functions, $\mathsf{F}\in\mathsf{C}\leftarrow\mathsf{D}$ and $\mathsf{G}\in\mathsf{D}\leftarrow\mathsf{C}$ forms a Galois connection if the following formula holds for all $x\in\mathsf{C}$ and $y\in\mathsf{D}$.

$$(4.1)\qquad x\sqsupseteq \mathsf{F}.y \equiv \mathsf{G}.x \succeq y \quad .$$

The function F is called the lower *adjoint* and the function G is called the upper *adjoint*.

The concept of a Galois connection is supposedly "well known", see e.g. [4, 5], but even if it is not well known to the reader it is such a simple and elegant concept that no difficulty should be experienced in verifying any properties that we state without proof. The concept was first introduced by Ore in 1944 [21] [1]; the definition given above is due to Schmidt [24].

The importance of Galois connections lies in their ubiquity and their simplicity. Mathematics and, in particular, computing science abounds with instances of Galois connections, although until very recently they have rarely been recognised as such. We present several elementary examples in this section. Those to whom the concept is indeed well known may skip the section.

[1] In spite of the name, "Galois" connections were not invented by the famous mathematician Évariste Galois, but an instance of a "Galois connection" is *the* "Galois correspondence" between groups and extension fields to which Galois owes his fame. We shall have nothing further to say about *the* Galois correspondence, our attention being devoted to connections between functions in a much broader setting. Those wishing to know more about the Galois correspondence are referred to [7].

4.1.1 Examples

Lots of examples of Galois connections can be given, in the first instance, by observing that two inverse functions are Galois connected. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two sets and $F \in \mathcal{A} \leftarrow \mathcal{B}$ and $g \in \mathcal{B} \leftarrow \mathcal{A}$ are inverse functions. Then their being inverse can be expressed by the equivalence, for all $x \in \mathcal{B}$ and $y \in \mathcal{A}$,

$$F.x = y \;\equiv\; x = G.y \quad .$$

This is a Galois connection in which we view $\mathcal{A}$ and $\mathcal{B}$ as ordered sets where the ordering relation is identical to the equality relation. (Two elements are ordered if and only if they are equal.) An example that will no doubt be very familiar is the connection between the exponential and logarithmic functions:

$$e^x = y \;\equiv\; x = \ln y \quad .$$

A yet simpler example is negation (in arithmetic):

$$-x = y \;\equiv\; x = -y \quad .$$

Negation is self-inverse and thus Galois connected to itself. Just slightly more complex is the connection between addition and subtraction:

$$x+y = z \;\equiv\; x = z-y \quad .$$

This formula states that, for all y, the function $(+y)$ (add y) is inverse to the function $(-y)$ (subtract y). It thus describes a *family* of Galois-connected functions, elements of the family being indexed by y.

That inverse functions are Galois connected is a useful observation — *not* because a study of Galois connections will tell us something we didn't already know about inverse functions, but because we can draw inspiration from our existing knowledge of properties of inverse functions to guide us in the study of Galois-connected functions. The "golden rule" of Galois connections discussed later was discovered in precisely this way.

Further examples of Galois connections are not hard to find although sometimes they are not immediately evident. One that is is the connection between conjunction and implication in the predicate calculus:

$$p \wedge q \Rightarrow r \equiv q \Rightarrow (p \Rightarrow r) \quad .$$

Here p, q and r denote predicates and the connection is between the functions $(p\wedge)$ and $(p\Rightarrow)$. To be more precise, both sets $\mathcal{A}$ and $\mathcal{B}$ in the definition of a Galois connection are taken to be the set of predicates, and the ordering relation is implication $(\Rightarrow)$. The above formula also describes a *family* of Galois connections, one for each instance of the variable p.

An interesting example is provided by negation (**not**) in the predicate calculus. We have:

$$\neg p \Rightarrow q \equiv p \Leftarrow \neg q \quad .$$

The example is interesting because it involves two different orderings on the same set. Specifically, we can order predicates by implication ($\Rightarrow$) or by the converse ordering follows-from ($\Leftarrow$). *Predicates ordered by implication and predicates ordered by follows-from are quite different partially ordered sets.* The point is that there are four elements to the definition of a Galois connection: two ordered sets and two functions between the ordered sets. All four elements form an integral part of the definition and mistakes in the exploitation of the properties of Galois connections may occur if one is not clear about all four. To reinforce the point let us return to addition and subtraction in real arithmetic. Not only do we have the identity:

$$x{+}y = z \equiv x = z{-}y \quad ,$$

we also have:

$$x{+}y \leq z \equiv x \leq z{-}y \quad .$$

These identities express two different (families of) Galois connections. Both involve the functions $(+y)$ and $(-y)$. In the first of the two the functions are endofunctions on the real numbers ordered by equality; in the second the ordering is the usual at most relation.

One elementary example of a Galois connection that is not immediately evident is afforded by the binary maximum operator on real numbers. Denoting it by the infix operator max, we have[2]:

$$x \,\mathsf{max}\, y \leq z \;\equiv\; x \leq z \wedge y \leq z \quad .$$

At first sight this doesn't look like a Galois connection principally on account of the conjunction on the righthand side of the equation. We can however identify it as such as follows. First note that max is a binary function, i.e. a function from the cartesian product $\mathbb{R}{\times}\mathbb{R}$ (the set of pairs of real numbers) to the set $\mathbb{R}$. Now $\mathbb{R}$ is ordered by the at-most relation ($\leq$), and this relation can be extended pointwise to an ordering relation on $\mathbb{R}{\times}\mathbb{R}$. Specifically, denoting the relation by $\leq{\times}\leq$, we define

$$(u,v) \leq{\times}\leq (x,y) \;\equiv\; u \leq x \wedge v \leq y \quad .$$

Finally, we define the *doubling function*, denoted by $\triangle$, by

$$\triangle z = (z,z) \quad .$$

Having done so we can rewrite the definition of max as follows:

$$\mathsf{max}.(x,y) \leq z \equiv (x,y) \leq{\times}\leq \triangle z \quad .$$

Thus max is a function mapping $\mathbb{R}{\times}\mathbb{R}$, ordered pointwise by the relation $\leq{\times}\leq$, to $\mathbb{R}$, ordered by the at-most relation ($\leq$), and is defined by a Galois-connection connecting it to the doubling function.

[2]Most texts define maximum via a case analysis and the above, beautifully compact, definition is little known. What a shame!

4.1.2 Introduction and Elimination Rules

There are several equivalent definitions of the notion of a Galois connection. So far we have presented all examples according to Schmidt's definition [24]. Gentzen's sequent calculus [12] defines the logic operators using so-called introduction and elimination rules. In this section we show, by example, that such rules define a Galois connection, albeit not in the form proposed by Schmidt.

Let us use disjunction as our first example. There are two introduction rules, namely:

$$p \Rightarrow p \vee q$$

and

$$q \Rightarrow p \vee q \quad .$$

(Gentzen would have used a turnstile rather than an implication operator. Such subtleties will be ignored in this discussion.)

There is one elimination rule for disjunction:

$$(p \Rightarrow r) \wedge (q \Rightarrow r) \;\Rightarrow\; (p \vee q \Rightarrow r) \quad .$$

It is not difficult to see that these three rules are equivalent to the one rule

$$(p \Rightarrow r) \wedge (q \Rightarrow r) \;\equiv\; p \vee q \Rightarrow r \quad .$$

In general, the combination of two partially ordered sets $(C, \sqsubseteq)$ and $(D, \preceq)$, and two functions, $F \in C \leftarrow D$ and $G \in D \leftarrow C$ forms a Galois connection if for all $x \in C$ and $y \in D$,

$$x \sqsupseteq F.y \Rightarrow G.x \succeq y \quad ,$$

if also, for all $x \in C$, we have the cancellation law

$$x \sqsupseteq F.(G.x) \quad ,$$

and, finally, F is monotonic.

To see that this is the form taken by Gentzen's rules we first rewrite the elimination rule in the same way as we did for maximum above. Doing so, we identify F as the doubling function and G as disjunction:

$$(p, q) \Rightarrow\!\times\!\Rightarrow (r, r) \;\;\Rightarrow\;\; (\vee.(p, q) \Rightarrow r)$$

We now check that the required cancellation law corresponds to the introduction rules. Formally, it is:

$$(p, q) \Rightarrow\!\times\!\Rightarrow (p \vee q\,,\, p \vee q)$$

which is indeed the same as the conjunction of $p \Rightarrow p \vee q$ and $q \Rightarrow p \vee q$. Finally, it is obvious that the doubling function is monotonic.

Gentzen's laws for existential quantification generalise the laws for (binary) disjunction. Let p be a predicate defined on some state space S. Then, $\exists.p$ (i.e. $\exists(x{:}\,x{\in}S{:}\,p.x)$), is defined by the elimination rule

$$\forall(x{::}\ p.x \Rightarrow r)\ \Rightarrow\ (\exists.p \Rightarrow r)$$

combined with the introduction rule

$$p.a \Rightarrow \exists.p\ \ .$$

Let us introduce the lifted ordering $\dot{\Rightarrow}$ on predicates on S. Specifically, $p\dot{\Rightarrow}q$ whenever $\forall(x{:}\ x{\in}S{:}\ p.x \Rightarrow q.x)$. Let us also introduce K to denote the constant function mapping proposition r to predicates on S. Specifically, the predicate $\mathsf{K}.r$ is such that, for all $x{\in}S$, $(\mathsf{K}.r).x = r$. With the aid of these definitions, the elimination rule for existential quantification is then

$$(p \dot{\Rightarrow} \mathsf{K}.r) \Rightarrow (\exists.p \Rightarrow r)$$

and the introduction rule is

$$p \dot{\Rightarrow} \mathsf{K}.(\exists.p)\ \ .$$

Noting that K is a monotonic function, we recognise yet another Galois connection.

4.1.3 Pointwise Ordering of Functions

Suppose $\mathcal{A} = (A,\sqsupseteq)$ is a partially ordered set. The set A is the *carrier* and $\sqsupseteq$ is the —reflexive, transitive and anti-symmetric— ordering relation on elements of the carrier. "Partially ordered set" will be abbreviated from now on to "poset". (In fact the assumption that $\mathcal{A}$ is a preorder suffices and fits in better with the discussion of the generalisation to category theory. However, this would introduce additional complications that we prefer to postpone.)

Suppose $\mathcal{B} = (B, \succeq)$ is also a poset. Then, a function $f \in A{\leftarrow}B$ is *monotonic* if $\forall(x,y{:}\ x \succeq y{:}\ f.x \sqsupseteq f.y)$. We write $f \in \mathcal{A}{\leftarrow}\mathcal{B}$ whenever this is the case. Indeed, we shall restrict our attention throughout to monotonic functions even though in specific cases this is unnecessary. Two trivial, but vital, observations are that the identity function on a set is monotonic, and the composition of two monotonic functions is also monotonic.

Functions f and g, both of type $\mathcal{A}{\leftarrow}\mathcal{B}$, can be ordered pointwise. Specifically, we define the relation $\sqsupseteq_{\mathcal{A}\leftarrow\mathcal{B}}$ on the carrier set $\mathcal{A}{\leftarrow}\mathcal{B}$ by

$$f \sqsupseteq_{\mathcal{A}\leftarrow\mathcal{B}} g\ \equiv\ \forall(x{::}\ f.x \sqsupseteq g.x)\ \ .$$

As is easily verified, $(\mathcal{A}{\leftarrow}\mathcal{B}\,, \sqsupseteq_{\mathcal{A}\leftarrow\mathcal{B}})$ is also a partially ordered set.

A very useful strategy —borrowed from category theory— in the study of posets is to "lift" statements about orderings between elements to statements about orderings between functions. We shall adopt this strategy frequently in the sequel. An example is the formulation of monotonicity of a function in terms purely of functions and function composition rather than function application. Specifically, denoting composition of functions by the infix operator $\bullet$, we have (for all $\mathcal{C}$):

$$\mathsf{f} \text{ is monotonic} \equiv \forall(\mathsf{g},\mathsf{h}\colon \mathsf{g} \sqsupseteq_{\mathcal{B}\leftarrow\mathcal{C}} \mathsf{h}\colon \mathsf{f}\bullet\mathsf{g} \sqsupseteq_{\mathcal{A}\leftarrow\mathcal{C}} \mathsf{f}\bullet\mathsf{h}) \quad .$$

Letting $\mathsf{f}\bullet$ denote the function $\mathsf{g} \mapsto \mathsf{f}\bullet\mathsf{g}$ we can express the above yet more strikingly:

$$\mathsf{f} \text{ is monotonic} \equiv \mathsf{f}\bullet \text{ is monotonic} \quad .$$

Letting $\bullet\mathsf{f}$ denote the function $\mathsf{g} \mapsto \mathsf{g}\bullet\mathsf{f}$ we also have:

$$\bullet\mathsf{f} \text{ is monotonic} \quad .$$

Rather than subscripting the symbol $\sqsupseteq$ with the type of the functions involved to indicate a pointwise ordering of functions we will sometimes use the symbol $\dot{\sqsupseteq}$ instead (the point serving to remind the reader that the ordering is pointwise). This has the disadvantage that the same symbol is used for different ordering relations, sometimes in one rule, but which ordering is intended will never be ambiguous.

4.1.4 Properties

Galois connections are interesting because as soon as we recognise one we can immediately deduce a number of useful properties of the adjoints. First of all, if we instantiate (4.1) in such a way that one side becomes true, we obtain two *cancellation properties*. Point-free these properties are expressed as follows

(4.2) $\quad \mathsf{G}\bullet\mathsf{F} \mathrel{\dot{\succeq}} \mathsf{Id} \quad ,$

(4.3) $\quad \mathsf{Id} \mathrel{\dot{\sqsupseteq}} \mathsf{F}\bullet\mathsf{G} \quad ,$

where Id denotes the identity function. Furthermore, with these cancellation properties it is straightforward to prove that both F and G are monotonic.

These properties of a Galois connection give rise to an alternative definition of a Galois connection that is equivalent to our previous formulation. (F,G) is a Galois connection iff the following two clauses hold:

$$\mathsf{G}\bullet\mathsf{F} \mathrel{\dot{\succeq}} I \quad \text{and} \quad I \mathrel{\dot{\sqsupseteq}} \mathsf{F}\bullet\mathsf{G} \quad ,$$

$$\mathsf{F} \text{ and } \mathsf{G} \text{ are monotonic.}$$

Instead of stating the definition of a Galois connection in terms of points it is useful to restate it in terms of functions. Specifically, the combination of two posets $(\mathsf{A},\sqsupseteq)$ and $(\mathsf{B},\succeq)$, and two functions, $\mathsf{F} \in \mathsf{A}\leftarrow\mathsf{B}$ and $\mathsf{G} \in \mathsf{B}\leftarrow\mathsf{A}$, forms a Galois connection if and only if for all functions h and k of appropriate type,

(4.4) $\quad \mathsf{h} \mathrel{\dot{\sqsupseteq}} \mathsf{F}\bullet\mathsf{k} \equiv \mathsf{G}\bullet\mathsf{h} \mathrel{\dot{\succeq}} \mathsf{k} \quad .$

Another way of expressing the equivalence of (4.1) and (4.4) is

(4.5) $\quad (F, G)$ forms a Galois connection $\equiv$ $(F\bullet\,,\ G\bullet)$ forms a Galois connection .

If two functions are inverses of each other then they are Galois connected. Suppose the inverse functions are F and G. Then we have, for all x in the domain of F, and y in the domain of G,

$$x{=}F.y \;\equiv\; G.x{=}y \quad .$$

The two poset orderings needed to establish the connection are the trivial orderings whereby the only ordered elements are equal elements.

This observation has no significance whatsoever for a study of inverse functions: nothing can be gained in such a study by instantiating general theorems about Galois connections that is not predicted by much simpler, direct calculations using the fact that a composition of the one function followed by the other is an identity function. The main benefit that is gained from the observation is that it can suggest properties that one might investigate of Galois-connected functions. An important example is that inverse functions have "inverse" algebraic properties. The exponential function, for instance, has as its inverse the logarithmic function, and

$$\exp(-x) \;=\; \frac{1}{\exp x} \qquad \text{whereas} \qquad -\ln x \;=\; \ln(\frac{1}{x})$$

and

$$\exp(x+y) \;=\; \exp x \cdot \exp y \qquad \text{whereas} \qquad \ln x + \ln y \;=\; \ln(x \cdot y) \quad .$$

In general, if F and G are inverse functions then, for any functions h and k of appropriate type,

$$\forall(x{::}\ h.(F.x) = F.(k.x)) \;\equiv\; \forall(y{::}\ G.(h.y) = k.(G.y)) \quad .$$

More generally, and expressed at function level, if (F_0,G_0) and (F_1,G_1) are pairs of inverse functions, then for all functions h and k of appropriate type,

$$h\bullet F_0 = F_1\bullet k \;\equiv\; G_1\bullet h = k\bullet G_0 \quad .$$

The generalisation to Galois connections takes the following form. Suppose, for $i = 0{,}1$, the combination of two posets $\mathcal{A}_i = (A_i, \sqsupseteq_i)$ and $\mathcal{B}_i = (B_i,\ \succeq_i)$ and two functions $F_i \in \mathcal{A}_i \leftarrow \mathcal{B}_i$ and $G_i \in \mathcal{B}_i \leftarrow \mathcal{A}_i$ forms a Galois connection. Let $h \in \mathcal{A}_0 \leftarrow \mathcal{A}_1$ and $k \in \mathcal{B}_1 \leftarrow \mathcal{B}_0$ be arbitrary monotonic functions. Then

(4.6) $\quad h\bullet F_0 \;\dot{\sqsupseteq}\; F_1\bullet k \;\equiv\; G_1\bullet h \;\dot{\succeq}\; k\bullet G_0 \quad .$

(As forewarned, subscripts have been omitted from the ordering relations since they can be inferred from the given type information.)

As a useful aide mémoire to property (4.6) we suggest the slogan "Galois-connected functions have pseudo-inverse algebraic properties".

Property (4.6) does not seem to be widely known but it is soon learnt and it is particularly useful since it captures in one rule several calculational properties of Galois-connected functions. As well as subsuming (4.4) it includes the following as special cases: First, by instantiating F_1 and G_1 to the identity function, and F_0 and G_0 to F and G, respectively, we obtain:

$$(F, G) \text{ forms a Galois connection} \Rightarrow (\bullet G\,,\ \bullet F) \text{ forms a Galois connection} \quad .$$

(In fact an equivalence can be proved.) Take care to note the switch in the order of F and G. The rule states that if F has *upper* adjoint G then $\bullet F$ has *lower* adjoint $\bullet G$. Second, by instantiating h and k to the identity function,

$$F_0 \mathrel{\dot{\sqsupseteq}} F_1 \;\equiv\; G_1 \mathrel{\dot{\succeq}} G_0 \quad .$$

Note this time the switch in the order of subscripts. Hence,

$$F_0 = F_1 \;\equiv\; G_1 = G_0 \quad .$$

Thus adjoints are uniquely defined.

4.1.5 Suprema

Defining Galois Connection

Suppose $\mathcal{A} = (A, \sqsupseteq)$ and $\mathcal{B} = (B, \succeq)$ are posets, and $f \in \mathcal{A} \leftarrow \mathcal{B}$ is a monotonic function. Then a *supremum* of f is a solution of the equation

$$(4.7) \qquad x :: \forall(a :: a \sqsupseteq x \equiv \forall(b :: a \sqsupseteq f.b)) \quad .$$

(Some readers may be more familiar with the definition of a supremum as a least upper bound. That is, x is a supremum of f if it is an upper bound:

$$\forall(b :: x \sqsupseteq f.b) \quad ,$$

and it is least among such upper bounds

$$\forall(a :: a \sqsupseteq x \Leftarrow \forall(b :: a \sqsupseteq f.b)) \quad .$$

This is entirely equivalent to (4.7), as is easily verified.)

It is easily seen that all solutions of (4.7) are equal. Specifically, for all $a \in A$,

$$\begin{array}{ll} & a \sqsupseteq x_0 \\ \equiv & \{ \quad x_0 \text{ solves (4.7)} \quad \} \\ & \forall(b :: a \sqsupseteq f.b) \\ \equiv & \{ \quad x_1 \text{ solves (4.7)} \quad \} \\ & a \sqsupseteq x_1 \quad . \end{array}$$

Hence, by indirect equality,

$$x_0 \text{ and } x_1 \text{ both solve (4.7)} \ \Rightarrow\ x_0 = x_1 \quad .$$

Equation (4.7) need not, of course, have a solution. If it does, for a given f, we denote its solution by $\mathsf{Sup}.f$. By definition, then,

$$(4.8) \qquad \forall(a{::}\ a \sqsupseteq \mathsf{Sup}.f \equiv \forall(b{::}\ a \sqsupseteq f.b)) \quad .$$

Suppose we fix the posets $\mathcal{A}$ and $\mathcal{B}$ and consider all functions of type $\mathcal{A}{\leftarrow}\mathcal{B}$. Suppose that there is a function Sup mapping all such functions to a supremum. Then we recognise (4.8) as a Galois connection. Specifically,

$$\begin{array}{ll}
 & \forall(b{::}\ a \sqsupseteq f.b) \\
\equiv & \qquad \{ \qquad \text{define the function } \ \mathsf{K} \in (\mathcal{A}{\leftarrow}\mathcal{B}){\leftarrow}\mathcal{A} \\
 & \qquad\qquad\quad \text{by } \ (\mathsf{K}.a).b = a \qquad \} \\
 & \forall(b{::}\ (\mathsf{K}.a).b \sqsupseteq f.b) \\
\equiv & \qquad \{ \qquad \text{definition of } \dot{\sqsupseteq} \text{ (pointwise ordering on functions)} \qquad \} \\
 & \mathsf{K}.a \mathrel{\dot{\sqsupseteq}} f \quad .
\end{array}$$

Thus, our supposition becomes that there is a function Sup that is the lower adjoint of the so-called "constant combinator" K of type $(\mathcal{A}{\leftarrow}\mathcal{B}){\leftarrow}\mathcal{A}$. That is, for all $a{\in}A$ and $f \in \mathcal{A}{\leftarrow}\mathcal{B}$,

$$(4.9) \qquad a \sqsupseteq \mathsf{Sup}.f \ \equiv\ \mathsf{K}.a \mathrel{\dot{\sqsupseteq}} f \quad .$$

The poset $\mathcal{B}$ is called the *shape* poset, and the existence of the Galois connection (4.9) (between the posets $\mathcal{A}$ and $(\mathcal{A}{\leftarrow}\mathcal{B}, \dot{\sqsupseteq})$) is put into words by saying that $\mathcal{A}$ is $\mathcal{B}$-*cocomplete*. If $\mathcal{A}$ is $\mathcal{B}$-cocomplete for all posets $\mathcal{B}$ then we say that $\mathcal{A}$ is *cocomplete*. (Dual to suprema, we may also define infima, leading to the dual notion of *completeness* of $\mathcal{A}$. Completeness and cocompleteness of posets are, however, equivalent and for this reason the "co" is redundant. We retain it, however, in order to emphasise the link with category theory .)

Examples

If $\mathcal{B}$ is the two-point set $\{0,1\}$ ordered by equality then the set of functions to $\mathcal{A}$ from $\mathcal{B}$ is in (1–1) correspondence with pairs of elements (a_0,a_1) (to be precise: $f{\mapsto}(f.0\,,f.1)$ and $(a_0,a_1){\mapsto}f$ where $f.0 = a_0$ and $f.1 = a_1$). The function $\mathsf{K}.a$ corresponds to the pair (a,a) , and the ordering relation on two functions both to $\mathcal{A}$ from $\mathcal{B}$ corresponds to the elementwise ordering on the pairs to which the functions correspond. Writing $f.0 \sqcup f.1$ instead of $\mathsf{Sup}.f$, the Galois connection (4.9) thus corresponds to

$$a \sqsupseteq x{\sqcup}y \ \equiv\ (a,a) \mathrel{\dot{\sqsupseteq}} (x,y) \quad .$$

That is,

$$a \sqsupseteq x \sqcup y \ \equiv\ a \sqsupseteq x \wedge a \sqsupseteq y \quad .$$

This is the well-known Galois connection defining the supremum of a bag of two elements.

If $\mathcal{B}$ is the empty poset then there is exactly one function of type $\mathcal{A} \leftarrow \mathcal{B}$, namely the identity function (or the empty function, which is the same thing). The right side of (4.9) is vacuously true and, thus, for all $a \in A$ and $f \in \mathcal{A} \leftarrow \emptyset$,

$$a \sqsupseteq \mathsf{Sup}.f \quad .$$

In words, the poset $\mathcal{A}$ is $\emptyset$-cocomplete equivales $\mathcal{A}$ has a least element.

A third example is the case that $\mathcal{B}$ is $(\mathbb{N}, \leq)$ (the natural numbers ordered in the usual way). Functions in $\mathcal{A} \leftarrow \mathcal{B}$ are then in (1–1) correspondence with chains $a_0 \leq a_1 \leq a_2 \leq \ldots$. To say that $\mathcal{A}$ is $(\mathbb{N}, \leq)$-cocomplete is equivalent to all such chains having a supremum in $\mathcal{A}$.

4.1.6 Parameterised Suprema

Suppose $\mathcal{A} = (A, \sqsupseteq_{\mathcal{A}})$, $\mathcal{B} = (B, \sqsupseteq_{\mathcal{B}})$ and $\mathcal{C} = (C, \sqsupseteq_{\mathcal{C}})$ are posets, and suppose $\mathcal{A}$ is $\mathcal{B}$-cocomplete. Motivated by the strategy of lifting statements about points to statements about functions we are naturally led to the question whether the poset $\mathcal{A} \leftarrow \mathcal{C}$ is $\mathcal{B}$-cocomplete.

We can answer this question by answering the following question. Consider function $\oplus \in (\mathcal{A} \leftarrow \mathcal{C}) \leftarrow \mathcal{B}$. For brevity we denote application of $\oplus$ to arguments b in $\mathcal{B}$ and c in $\mathcal{C}$ by $b \oplus c$. Assume that, for all $c \in C$, the function $\oplus c \in \mathcal{A} \leftarrow \mathcal{B}$ defined by $(\oplus c).b = b \oplus c$ has a supremum denoted by $\mathsf{Sup}.(\oplus c)$. Note that, since $\oplus c$ is a function of type $\mathcal{A} \leftarrow \mathcal{B}$, the existence of $\mathsf{Sup}.(\oplus c)$ is implied by $\mathcal{B}$-cocompleteness of $\mathcal{A}$. The question is whether this information is sufficient to guarantee that $\oplus$ has a supremum.

To express the question formally we need to refine our notation in order to avoid ambiguity. Where before we wrote K let us now write $\mathsf{K}_{\mathcal{A},\mathcal{B}}$. Thus $\mathsf{K}_{\mathcal{A},\mathcal{B}}$ is the function of type $(\mathcal{A} \leftarrow \mathcal{B}) \leftarrow \mathcal{A}$ defined by

$$(\mathsf{K}_{\mathcal{A},\mathcal{B}}.a).b = a \quad .$$

The assumption is thus that, for all $c \in C$, there is an element $\mathsf{Sup}.(\oplus c)$ in A satisfying

$$(4.10) \quad \forall (a{::}\ a \sqsupseteq_{\mathcal{A}} \mathsf{Sup}.(\oplus c) \ \equiv\ \mathsf{K}_{\mathcal{A},\mathcal{B}}.a \sqsupseteq_{\mathcal{A} \leftarrow \mathcal{B}} \oplus c) \quad .$$

The goal is to solve the equation

$$(4.11) \quad x{::}\ x \in \mathcal{A} \leftarrow \mathcal{C} \ \wedge\ \forall (f{::}\ f \sqsupseteq_{\mathcal{A} \leftarrow \mathcal{C}} x \equiv \mathsf{K}_{\mathcal{A} \leftarrow \mathcal{C},\mathcal{B}}.f \sqsupseteq_{(\mathcal{A} \leftarrow \mathcal{C}) \leftarrow \mathcal{B}} \oplus) \quad .$$

The obvious candidate for x is the function $c \mapsto \mathsf{Sup}.(\oplus c)$. This function is monotonic (and thus in $\mathcal{A} \leftarrow \mathcal{C}$) since it is the composition of two monotonic functions, the function Sup (which is monotonic because the adjoints in a Galois connection are inevitably monotonic) and the function $\hat{\oplus} \in (\mathcal{A} \leftarrow \mathcal{B}) \leftarrow \mathcal{C}$ defined by $(\hat{\oplus}.c).b = b \oplus c$. That it also satisfies (4.11) is a straightforward calculation. We present it nonetheless in order to show the benefit of exploiting (4.5).

$$
\begin{array}{ll}
 & \mathsf{f} \sqsupseteq_{\mathcal{A}\leftarrow\mathcal{C}} \mathsf{c}{\mapsto}\mathsf{Sup}.(\oplus\mathsf{c}) \\
\equiv & \{ \quad \text{see above for the definition of } \hat{\oplus} \text{ ; extensionality} \quad \} \\
 & \mathsf{f} \sqsupseteq_{\mathcal{A}\leftarrow\mathcal{C}} \mathsf{Sup}\bullet\hat{\oplus} \\
\equiv & \{ \quad \mathsf{Sup} \text{ has upper adjoint } \mathsf{K}_{\mathcal{A},\mathcal{B}} \text{ , by assumption,} \\
 & \quad \text{so, by (4.5), } (\mathsf{Sup}\bullet) \text{ has upper adjoint } (\mathsf{K}_{\mathcal{A},\mathcal{B}}\bullet) \quad \} \\
 & \mathsf{K}_{\mathcal{A},\mathcal{B}}\bullet\mathsf{f} \sqsupseteq_{(\mathcal{A}\leftarrow\mathcal{C})\leftarrow\mathcal{B}} \hat{\oplus} \\
\equiv & \{ \quad \text{We now try to express } \mathsf{K}_{\mathcal{A},\mathcal{B}}\bullet\mathsf{f} \text{ in terms of } \mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f} \text{ :} \\
 & \qquad ((\mathsf{K}_{\mathcal{A},\mathcal{B}}\bullet\mathsf{f}).\mathsf{c}).\mathsf{b} \\
 & \quad = \quad \{ \quad \text{definition} \quad \} \\
 & \qquad \mathsf{f}.\mathsf{c} \\
 & \quad = \quad \{ \quad \text{definition} \quad \} \\
 & \qquad ((\mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f}).\mathsf{b}).\mathsf{c} \\
 & \quad \text{Thus, defining the function } \mathsf{Flip} \text{ by} \\
 & \quad ((\mathsf{Flip}.\mathsf{g}).\mathsf{c}).\mathsf{b} = (\mathsf{g}.\mathsf{b}).\mathsf{c} \quad \text{, we have} \\
 & \quad \mathsf{K}_{\mathcal{A},\mathcal{B}}\bullet\mathsf{f} = \mathsf{Flip}.(\mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f}) \quad \} \\
 & \mathsf{Flip}.(\mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f}) \sqsupseteq_{(\mathcal{A}\leftarrow\mathcal{C})\leftarrow\mathcal{B}} \hat{\oplus} \\
\equiv & \{ \quad \mathsf{Flip} \text{ is clearly an order isomorphism,} \\
 & \quad \hat{\oplus} = \mathsf{Flip}.\oplus \quad \} \\
 & \mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f} \sqsupseteq_{(\mathcal{A}\leftarrow\mathcal{C})\leftarrow\mathcal{B}} \oplus \quad .
\end{array}
$$

Thus, we have indeed verified that $\mathsf{c}{\mapsto}\mathsf{Sup}.(\oplus\mathsf{c}) = \mathsf{Sup}.\oplus$.

The element $\mathsf{Sup}.(\oplus\mathsf{c})$ is called a *parameterised supremum* and the theorem just proved we call the *abstraction theorem* for suprema. In words it says that the result of abstracting from the parameter in a parameterised supremum is itself a supremum.

Since the existence of a supremum of $(\oplus\mathsf{c})$ for each c is implied by $\mathcal{B}$-cocompleteness of $\mathcal{A}$, a direct corollary is that if $\mathcal{A}$ is $\mathcal{B}$-cocomplete then $\mathcal{A}\leftarrow\mathcal{C}$ is also $\mathcal{B}$-cocomplete (independently of $\mathcal{C}$). In particular if $\mathcal{A}$ is (co)complete then $\mathcal{A}\leftarrow\mathcal{C}$ is (co)complete.

An example of a parameterised supremum is the binary supremum operator: let $\mathcal{B}$ be $(\{0,1\},=)$ as in the definition of $\sqcup$ and let $\mathcal{C}$ be $(\mathcal{A}\times\mathcal{A}, \sqsupseteq\times\sqsupseteq)$. Define $\oplus\in(\mathcal{A}\leftarrow\mathcal{C})\leftarrow\mathcal{B}$ by $0\oplus(a_0,a_1) = a_0$ and $1\oplus(a_0,a_1) = a_1$. Then, by definition,

$$a_0\sqcup a_1 = (\mathsf{c}{\mapsto}\mathsf{Sup}.(\oplus\mathsf{c})).(a_0,a_1) \quad .$$

Applying the abstraction theorem,

$$(\sqcup) = \mathsf{Sup}.\oplus \quad .$$

Category theoreticians will recognise in the above a special case of theorem 1 on page 111 of Mac Lane's classic text [15] concerning parameterised (co)limits. Neither the fact that $(\sqcup) = \mathsf{Sup}.\oplus$ nor the general abstraction theorem for suprema seem to be significant. The corresponding limit theorem in category theory is, however. The proof we have just given, in particular its use of (4.5), can be easily adapted to an attractive alternative to Mac Lane's proof, as we shall demonstrate in section 4.5.1.

4.2 Definition

The category-theoretic concept corresponding to the notion of a Galois connection is that of an *adjunction*. There is a large number of equivalent definitions of an adjunction. The one we give here is particularly well-suited to our goal.

Recall the defining equation for a Galois connection between functions F and G: for all x and y of the appropriate types,

$$x \sqsupseteq F.y \equiv G.x \succeq y \quad .$$

Using extensionality, we can rewrite this universally quantified proposition as an equation between functions. Specifically,

$$\begin{array}{ll} & \forall(x,y{::}\ x \sqsupseteq F.y \equiv G.x \succeq y) \\ \equiv & \{ \quad \text{extensionality: } F = G \equiv \forall(x{::}\ F.x = G.x) \quad \} \\ & [x,y :: \ x \sqsupseteq F.y] = [x,y :: \ G.x \succeq y] \quad . \end{array}$$

I.e. F and G form a Galois connection if the function mapping the pair x,y to $x \sqsupseteq F.y$ is equal to the function mapping the pair x,y to $G.x \succeq y$. By replacing "function" by "functor" we generalise this form of the definition to category theory.

Suppose that F and G are functors of type $\mathcal{C} \leftarrow \mathcal{D}$ and $\mathcal{D} \leftarrow \mathcal{C}$, respectively. Then $[x,y :: \ \mathsf{id}_\mathcal{C}.x \leftarrow F.y]$ and $[x,y :: \ G.x \leftarrow \mathsf{id}_\mathcal{D}.y]$ are functors of the same type. (Recall the definition of these functors given in section 3.2.) If they are isomorphic then F and G are said to be *adjoint functors*. Writing $[x,y :: \ x \leftarrow F.y]$ and $[x,y :: \ G.x \leftarrow y]$ instead of $[x,y :: \ \mathsf{id}_\mathcal{C}.x \leftarrow F.y]$ and $[x,y :: \ G.x \leftarrow \mathsf{id}_\mathcal{D}.y]$ we therefore have the following definition.

Definition 4.12 (Adjunction) Suppose $F \in \mathcal{C} \leftarrow \mathcal{D}$ and $G \in \mathcal{D} \leftarrow \mathcal{C}$ are two functors then

$$(F, G) \text{ is an adjunction} \equiv [x,y :: \ x \leftarrow F.y] \cong [x,y :: \ G.x \leftarrow y] \quad .$$

Here, functor F is called the *lower adjoint* and functor G is called the *upper adjoint*.

□

Let us spell out some of the details of definition 4.12. Expanding the definition of a natural isomorphism we get that there are two natural transformations $\lceil\ \rceil$ and $\lfloor\ \rfloor$ such that

(4.13) $\lceil\ \rceil \in [x,y :: \ x \leftarrow F.y] \leftarrow [x,y :: \ G.x \leftarrow y]$

(4.14) $\lfloor\ \rfloor \in [x,y ::\ G.x \leftarrow y] \leftarrow [x,y ::\ x \leftarrow F.y]$,

which are each others' inverses, i.e.

(4.15) $\lceil\lfloor\ \rfloor\rceil = \mathsf{id} \wedge \lfloor\lceil\ \rceil\rfloor = \mathsf{id}$.

The natural transformations $\lceil\ \rceil$ and $\lfloor\ \rfloor$ are called the *upper adjungate* and the *lower adjungate* respectively. Expanding the definition of a natural transformation we get from (4.13): for each $x \in \mathcal{C}$ and $y \in \mathcal{D}$

$$\lceil\ \rceil_{x,y} \in (x \leftarrow F.y) \leftarrow (G.x \leftarrow y)$$

and for each $(f,g) \in (x,v) \leftarrow (u,y)$

$$(f \leftarrow F.g) \circ \lceil\ \rceil_{u,v} = \lceil\ \rceil_{x,y} \circ (G.f \leftarrow g) \quad .$$

A similar result can be derived for the natural transformation (4.14). So, both $\lceil\ \rceil_{x,y}$ and $\lfloor\ \rfloor_{x,y}$ are functions for every pair (x,y). (Remark: had we defined $x \leftarrow F.y$ as a discrete category, then $\lceil\ \rceil_{x,y}$ and $\lfloor\ \rfloor_{x,y}$ would have been functors.) Now, we can use extensionality and the definition of the functor $[x,y ::\ F.x \leftarrow G.y]$ to give an alternative definition of an adjunction.

Theorem 4.16 F and G form an adjunction if we have two natural transformations $\lceil\ \rceil$ and $\lfloor\ \rfloor$ satisfying for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$

(a) $\lceil g \rceil_{x,y} \in x \overset{\mathcal{C}}{\longleftarrow} F.y \ \Leftarrow\ g \in G.x \overset{\mathcal{D}}{\longleftarrow} y$,

(b) $\lfloor f \rfloor_{x,y} \in G.x \overset{\mathcal{D}}{\longleftarrow} y \ \Leftarrow\ f \in x \overset{\mathcal{C}}{\longleftarrow} F.y$.

Furthermore, suppose $f \in x \overset{\mathcal{C}}{\longleftarrow} u$ and $g \in v \overset{\mathcal{D}}{\longleftarrow} y$, then the following equalities must hold. If $h \in G.u \overset{\mathcal{D}}{\longleftarrow} v$ then

(c) $f \circ \lceil h \rceil_{u,v} \circ F.g = \lceil G.f \circ h \circ g \rceil_{x,y}$

and dually if $h \in u \overset{\mathcal{D}}{\longleftarrow} F.v$ then

(d) $G.f \circ \lfloor h \rfloor_{u,v} \circ g = \lfloor f \circ h \circ F.g \rfloor_{x,y}$.

Finally, for $f \in x \overset{\mathcal{C}}{\longleftarrow} F.y$ and $g \in G.x \overset{\mathcal{D}}{\longleftarrow} y$ the equivalence

(e) $\lceil g \rceil_{x,y} = f \ \equiv\ g = \lfloor f \rfloor_{x,y}$

must hold.

□

We will usually omit the subscripts of the lower and upper adjungate.

Example 4.17 In section 4.1.2, we argued that the introduction and elimination rules for existential quantification amounted to a Galois connection. A categorical generalisation of that Galois connection is an adjunction between the sum functor $\Sigma_{\mathcal{C}}$ (discussed in section 2.5.5) and the constant functor K for a *discrete* category $\mathcal{C}$.

Suppose $\mathcal{C}$ is a discrete category. That is,

$$\mathsf{f}\in\mathsf{c}\overset{\mathcal{C}}{\longleftarrow}\mathsf{d}\equiv\mathsf{c}=\mathsf{d}\wedge\mathsf{f}=\mathsf{id_c}\quad .$$

Suppose $\mathsf{J}\in\mathsf{Cat}\leftarrow\mathcal{C}$ and $\mathsf{F}\in\mathcal{B}\overset{\mathsf{Cat}}{\longleftarrow}\Sigma_{\mathcal{C}}\mathsf{J}$. Note that, for a discrete category $\mathcal{C}$, the composition of arrows in $\Sigma_{\mathcal{C}}\mathsf{J}$ is componentwise.

We have to define $\lfloor\mathsf{F}\rfloor\in\mathsf{K}.\mathcal{B}\overset{\mathsf{Fun(Cat,}\mathcal{C})}{\longleftarrow}\mathsf{J}$. Thus we require that

$$(4.18)\quad \lfloor\mathsf{F}\rfloor_{\mathsf{c}}\in\mathcal{B}\overset{\mathsf{Cat}}{\longleftarrow}\mathsf{J.c}\ \Leftarrow\ \mathsf{c}\in\mathcal{C}\quad ,$$

$$(4.19)\quad \lfloor\mathsf{F}\rfloor_{\mathsf{c}}\bullet\mathsf{J.f}=\lfloor\mathsf{F}\rfloor_{\mathsf{d}}\ \Leftarrow\ \mathsf{c}\in\mathcal{C}\wedge\mathsf{d}\in\mathcal{C}\wedge\mathsf{f}\in\mathsf{c}\overset{\mathcal{C}}{\longleftarrow}\mathsf{d}\quad .$$

However, (4.19) is trivially satisfied since $\mathcal{C}$ is, by assumption, a discrete category. Property (4.18) expands to, for all $\mathsf{c}\in\mathcal{C}$,

$$(4.20)\quad \lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{x}\in\mathcal{B}\ \Leftarrow\ \mathsf{c}\in\mathcal{C}\wedge\mathsf{x}\in\mathsf{J.c}\quad ,$$

$$(4.21)\quad \lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{f}\in\lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{x}\overset{\mathcal{B}}{\longleftarrow}\lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{y}\ \Leftarrow\ \mathsf{x}\in\mathsf{J.c}\wedge\mathsf{y}\in\mathsf{J.c}\wedge\mathsf{f}\in\mathsf{x}\overset{\mathsf{J.c}}{\longleftarrow}\mathsf{y}\quad ,$$

$$(4.22)\quad \lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{id_x}=\mathsf{id}_{\lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{x}}\Leftarrow\mathsf{x}\in\mathsf{J.c}\quad ,$$

$$(4.23)\quad \lfloor\mathsf{F}\rfloor_{\mathsf{c}}.(\mathsf{f}\circ\mathsf{g})=\lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{f}\circ\lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{g}\ \Leftarrow\ \mathsf{f}\in\mathsf{x}\overset{\mathsf{J.c}}{\longleftarrow}\mathsf{y}\wedge\mathsf{g}\in\mathsf{y}\overset{\mathsf{J.c}}{\longleftarrow}\mathsf{z}\quad .$$

To satisfy (4.20) we define

$$\lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{x}=\mathsf{F.(c,x)}\quad .$$

To satisfy (4.21) we define

$$\lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{f}=\mathsf{F.(id_c,f)}\quad .$$

Property (4.22) is then

$$\begin{array}{ll} & \lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{id_x} \\ = & \quad\{\quad \text{definition}\quad\} \\ & \mathsf{F.(id_c,id_x)} \\ = & \quad\{\quad \mathsf{(id_c,id_x)}=\mathsf{id_{(c,x)}}\quad\} \\ & \mathsf{id_{F.(c,x)}} \\ = & \quad\{\quad \text{definition}\quad\} \\ & \mathsf{id}_{\lfloor\mathsf{F}\rfloor_{\mathsf{c}}.\mathsf{x}}\quad . \end{array}$$

Property (4.23) is also easily verified:

$$
\begin{array}{ll}
 & \lfloor F \rfloor_c.(f \circ g) \\
= & \{ \quad \text{definition} \quad \} \\
 & F.(id_c\,,\ f \circ g) \\
= & \{ \quad \text{composition in } \Sigma_{\mathcal{C}} J \text{ is componentwise when} \\
 & \quad\;\; \mathcal{C} \text{ is a discrete category} \quad \} \\
 & F.((id_c, f) \circ (id_c, g)) \\
= & \{ \quad F \text{ is a functor} \quad \} \\
 & F.(id_c, f) \circ F.(id_c, g) \\
= & \{ \quad \text{definition} \quad \} \\
 & \lfloor F \rfloor_c.f \circ \lfloor F \rfloor_c.g \quad .
\end{array}
$$

Now suppose $\alpha \in K.\mathcal{B} \xleftarrow{\mathsf{Fun(Cat},\mathcal{C})} J$. We have to construct $\lceil \alpha \rceil \in \mathcal{B} \xleftarrow{\mathsf{Cat}} \Sigma_{\mathcal{C}} J$. That is,

$$\lceil \alpha \rceil.(c, x) \in \mathcal{B} \Leftarrow c \in \mathcal{C} \wedge x \in J.c \quad ,$$

$$\lceil \alpha \rceil.(f, g) \in \lceil \alpha \rceil.(u, x) \xleftarrow{\mathcal{B}} \lceil \alpha \rceil.(v, y) \;\Leftarrow\; f \in u \xleftarrow{\mathcal{C}} v \wedge g \in x \xleftarrow{J.u} (J.f).y \quad ,$$

$$\lceil \alpha \rceil.((f, g) \circ (h, k)) = \lceil \alpha \rceil.(f, g) \circ \lceil \alpha \rceil.(h, k) \quad .$$

Recalling that $\mathcal{C}$ is a discrete category, these requirements are satisfied by defining

$$\lceil \alpha \rceil.(c, x) = \alpha_c.x$$

$$\lceil \alpha \rceil.(id_c, g) = \alpha_c.g \quad .$$

It remains to verify properties (c) through (e) of theorem 4.16. We leave the verification to the reader. Refer to exercise 2.25 for the complete definition of the functor $\Sigma_{\mathcal{C}}$.

□

Exercise 4.24 The two simplest examples of discrete categories are **1** and **2**, the discrete categories containing exactly 1 and 2 elements, respectively. Work out the details of example 4.17 in these two cases. In particular, give an interpretation to the lower and upper adjungates.

□

Elementary consequences of a Galois connection are the cancellation properties. For an adjunction a similar result can be derived. By construction of α we calculate as follows:

$$
\begin{array}{ll}
& \alpha \in \mathsf{G}\bullet\mathsf{F} \leftarrow \mathsf{Id}_{\mathcal{D}} \\
\equiv & \{ \quad \text{definition natural transformation} \quad \} \\
& \quad \forall(x:: \alpha_x \in \mathsf{G.F}.x \leftarrow x) \\
& \wedge\ \forall(f: f \in x \leftarrow y: \mathsf{G.F}.f \circ \alpha_y = \alpha_x \circ f) \\
\Leftarrow & \{ \quad (4.16b),\ \bullet \quad \alpha_x = \lfloor \mathsf{id}_{\mathsf{F}.x} \rfloor \quad \} \\
& \forall(f: f \in x \leftarrow y: \mathsf{G.F}.f \circ \lfloor \mathsf{id}_{\mathsf{F}.y} \rfloor = \lfloor \mathsf{id}_{\mathsf{F}.x} \rfloor \circ f) \\
\equiv & \{ \quad (4.16d) \quad \} \\
& \mathit{true} \quad .
\end{array}
$$

In category theory this natural transformation α is commonly known as the *unit* of the adjunction. So, an immediate consequence of an adjunction is that we have a natural transformation unit, such that

$$\mathsf{unit} \in \mathsf{G}\bullet\mathsf{F} \leftarrow \mathsf{Id}_{\mathcal{D}} \quad \text{, where } \mathsf{unit}_x = \lfloor \mathsf{id}_{\mathsf{F}.x} \rfloor \text{ for all } x.$$

Dually, we also have a natural transformation counit, known as the *co-unit* of the adjunction, such that

$$\mathsf{counit} \in \mathsf{Id}_{\mathcal{C}} \leftarrow \mathsf{F}\bullet\mathsf{G} \quad \text{, where } \mathsf{counit}_x = \lceil \mathsf{id}_{\mathsf{G}.x} \rceil \text{ for all } x.$$

Corresponding to lattice theory, we can give an alternative definition of an adjunction using the unit and co-unit.

Theorem 4.25 F and G form an adjunction if we have two natural transformations unit and counit such that

$$\mathsf{unit} \in \mathsf{G}\bullet\mathsf{F} \leftarrow \mathsf{Id}_{\mathcal{D}} \quad ,$$

$$\mathsf{counit} \in \mathsf{Id}_{\mathcal{C}} \leftarrow \mathsf{F}\bullet\mathsf{G} \quad ,$$

and the following two coherence requirements are satisfied

$$\mathsf{counit}\bullet\mathsf{F} \circ \mathsf{F}\bullet\mathsf{unit} = \mathsf{id}\bullet\mathsf{F} \quad ,$$

$$\mathsf{G}\bullet\mathsf{counit} \circ \mathsf{unit}\bullet\mathsf{G} = \mathsf{id}\bullet\mathsf{G} \quad .$$

Proof We have to construct a lower and upper adjungate. Under the assumption that $f \in x \leftarrow \mathsf{F}.y$ we construct the lower adjungate as follows:

$$
\begin{array}{ll}
& \alpha \in \mathsf{G}.x \leftarrow y \\
\Leftarrow & \{ \quad \bullet \quad \alpha = \beta \circ \mathsf{unit}_y \quad \} \\
& \beta \in \mathsf{G}.x \leftarrow \mathsf{G.F}.y \\
\Leftarrow & \{ \quad \mathsf{G} \text{ is a functor} \quad \} \\
& \beta = \mathsf{G}.f \quad .
\end{array}
$$

Thus,

$$\mathsf{G.f} \circ \mathsf{unit}_y \in \mathsf{G}.x \leftarrow y \;\Leftarrow\; f \in x \leftarrow \mathsf{F}.y \quad ,$$

and dually,

$$\mathsf{counit}_x \circ \mathsf{F.g} \in x \leftarrow \mathsf{F}.y \;\Leftarrow\; g \in \mathsf{G}.x \leftarrow y \quad .$$

This gives the candidates for (the pointwise definition of) the lower and upper adjungate respectively. The verification of the coherence requirements goes as follows. We only give two verifications the others' being similar.

$$\begin{array}{ll}
& \mathsf{G.g} \circ (\mathsf{G.f} \circ \mathsf{unit}_y) \circ h \\
= & \{ \quad \mathsf{unit} \in \mathsf{G}\bullet\mathsf{F} \leftarrow \mathsf{Id} \quad \} \\
& \mathsf{G.g} \circ \mathsf{G.f} \circ \mathsf{G.F}.h \circ \mathsf{unit}_{\mathsf{dom}.h} \\
= & \{ \quad \mathsf{G} \text{ is a functor} \quad \} \\
& \mathsf{G}.(g \circ f \circ \mathsf{F}.h) \circ \mathsf{unit}_{\mathsf{dom}.h}
\end{array}$$

And finally,

$$\begin{array}{ll}
& \mathsf{G.f} \circ \mathsf{unit}_y = g \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
& \mathsf{counit}_x \circ \mathsf{F}.(\mathsf{G.f} \circ \mathsf{unit}_y) = \mathsf{counit}_x \circ \mathsf{F.g} \\
\equiv & \{ \quad \mathsf{F} \text{ is a functor} \quad \} \\
& \mathsf{counit}_x \circ \mathsf{F.G.f} \circ \mathsf{F.unit}_y = \mathsf{counit}_x \circ \mathsf{F.g} \\
\equiv & \{ \quad \mathsf{counit} \in \mathsf{Id} \leftarrow \mathsf{F}\bullet\mathsf{G} \quad \} \\
& f \circ \mathsf{counit}_{\mathsf{F}.y} \circ \mathsf{F.unit}_y = \mathsf{counit}_x \circ \mathsf{F.g} \\
\equiv & \{ \quad \mathsf{counit}\bullet\mathsf{F} \circ \mathsf{F}\bullet\mathsf{unit} = \mathsf{id}\bullet\mathsf{F} \quad \} \\
& f = \mathsf{counit}_x \circ \mathsf{F.g} \quad .
\end{array}$$

□

We have shown that, given the lower and upper adjungate we can construct a unit and co-unit, and vice versa. An adjunction does not determine the lower and upper adjungate uniquely, nor the unit and co-unit. But, the above constructions show that we can always take the lower and upper adjungate and the unit and co-unit in such a way that the following properties hold

(4.26) $\lfloor f \rfloor = \mathsf{G.f} \circ \mathsf{unit}_y$, where $f \in x \leftarrow \mathsf{F}.y$,

(4.27) $\lceil g \rceil = \mathsf{counit}_x \circ \mathsf{F.g}$, where $g \in \mathsf{G}.x \leftarrow y$.

We'll do this in the rest of this chapter so that we can always use this property.

In the above proof a valuable property is proven. Specifically, for all $f \in x \leftarrow F.y$ and $g \in G.x \leftarrow y$,

(4.28) $\quad G.f \circ \mathsf{unit}_y = g \;\equiv\; f = \mathsf{counit}_x \circ F.g \quad .$

This property is equivalent to the two coherence requirements stipulated in theorem 4.25. More specifically, the two coherence requirements give rise to this property as shown in the last proof. For the other way around, by instantiating $f,y,g := \mathsf{counit}_x, G.x, \mathsf{id}_{G.x}$ in (4.28), we obtain

$$G.\mathsf{counit}_x \circ \mathsf{unit}_{G.x} = \mathsf{id}_{G.x} \quad .$$

By using extensionality we thus obtain one of the two coherence requirements. The other one can be derived in a similar way. So, we have as a third alternative definition of an adjunction

Theorem 4.29 F and G form an adjunction if we have two natural transformations unit and counit such that

$$\mathsf{unit} \in G{\bullet}F \leftarrow \mathsf{Id}_{\mathcal{D}} \quad ,$$

$$\mathsf{counit} \in \mathsf{Id}_{\mathcal{C}} \leftarrow F{\bullet}G \quad ,$$

and the following coherence requirement is satisfied: for two arrows $f \in x \leftarrow F.y$ and $g \in G.x \leftarrow y$

$$G.f \circ \mathsf{unit}_y = g \;\equiv\; f = \mathsf{counit}_x \circ F.g \quad .$$

□

A monotonic function corresponds to a (covariant) functor. An anti-monotonic function corresponds to a contravariant functor. Two (covariant) functors can give rise to a (covariant) adjunction. So, two contravariant functors can give rise to a *contravariant* adjunction.

Definition 4.30 (Contravariant adjunction) Two contravariant functors F and G form a *contravariant adjunction* if we have two mappings $\lceil\ \rceil$ and $\lfloor\ \rfloor$ such that

(a) $\quad \lceil g \rceil \in F.y \leftarrow x \;\Leftarrow\; g \in G.x \leftarrow y \quad ,$

(b) $\quad \lfloor f \rfloor \in G.x \leftarrow y \;\Leftarrow\; f \in F.y \leftarrow x \quad .$

(c) $\quad F.g \circ \lceil h \rceil \circ f = \lceil G.f \circ h \circ g \rceil \quad ,$

(d) $\quad G.f \circ \lfloor h \rfloor \circ g = \lfloor F.g \circ h \circ f \rfloor \quad ,$

(e) $\quad \lceil g \rceil = f \equiv g = \lfloor f \rfloor \quad .$

□

For the remainder of this chapter we will use the following notation. If (F,G) is an adjunction then we denote its unit and co-unit by unit_F and $\mathsf{counit}_\mathsf{G}$ and upper and lower adjungate by $\lceil \mathsf{F};\ \rceil$ and $\lfloor \mathsf{G};\ \rfloor$ respectively. Although we have introduced this notation, this does not mean that the other adjoint is unimportant for the definition of units and adjungates. These parameters are only introduced to remind the reader to which adjunction they belong. Using this notation we have the following computation rules:

$$(4.31)\quad \mathsf{G}.\lceil \mathsf{F};\mathsf{g}\rceil \circ (\mathsf{unit}_\mathsf{F})_\mathsf{x} = \mathsf{g} \quad \text{and} \quad (\mathsf{counit}_\mathsf{G})_\mathsf{y} \circ \mathsf{F}.\lfloor \mathsf{G};\mathsf{f}\rfloor = \mathsf{f} \quad ,$$

which follow immediately from (4.26), (4.27) and (4.28). If no confusion is possible the parameters F and G will be omitted.

Exercise 4.32 Let $(\mathsf{C},\sqsupseteq)$ and $(\mathsf{D},\succeq)$ be partially ordered sets. If $(\mathsf{F}\in\mathsf{C}\leftarrow\mathsf{D}\ ,\ \mathsf{G}\in\mathsf{D}\leftarrow\mathsf{C})$ is a Galois connection, then F maps the least element of D to the least element of C. More specifically, let $\perp\!\!\perp_\mathsf{C}$ and $\perp\!\!\perp_\mathsf{D}$ be the least elements in C and D respectively. Then, $\mathsf{F}.\perp\!\!\perp_\mathsf{D} \sqsupseteq \perp\!\!\perp_\mathsf{C}$ by definition of a least element. For the other inclusion we argue as follows

$$\begin{array}{ll}
 & \perp\!\!\perp_\mathsf{C} \sqsupseteq \mathsf{F}.\perp\!\!\perp_\mathsf{D} \\
\equiv & \qquad \{ \qquad \text{Galois connection} \qquad \} \\
 & \mathsf{G}.\perp\!\!\perp_\mathsf{C} \succeq \perp\!\!\perp_\mathsf{D}
\end{array}$$

and this last line also holds by definition of a least element. So, using mutual inclusion, we have proven $\mathsf{F}.\perp\!\!\perp_\mathsf{D} = \perp\!\!\perp_\mathsf{C}$, i.e. F preserves least elements.

Generalising to category theory, we propose the following exercise. Suppose the functor $\mathsf{F}\in\mathcal{C}\leftarrow\mathcal{D}$ is the lower adjoint in an adjunction. Show that F maps an initial object of $\mathcal{D}$ to an initial object of $\mathcal{C}$.

(Later we provide the proof of a more general theorem.)

□

4.3 Properties

Let $\mathcal{C}$,$\mathcal{D}$ and $\mathcal{E}$ be categories. Given a functor $\mathsf{F}\in\mathcal{C}\leftarrow\mathcal{D}$ we can construct a functor $\mathsf{F}\bullet\in\mathsf{Fun}(\mathcal{C},\mathcal{E})\leftarrow\mathsf{Fun}(\mathcal{D},\mathcal{E})$. The lifting of functors from object to functor level gives rise to the following theorem.

Theorem 4.33 (F,G) forms an adjunction iff $(\mathsf{F}\bullet\,,\ \mathsf{G}\bullet)$ forms an adjunction.

Proof First, we assume the adjunction (F,G) with unit and counit as unit and co-unit respectively and construct a unit and co-unit for the adjunction $(\mathsf{F}\bullet\,,\ \mathsf{G}\bullet)$. For the unit this construction goes as follows:

$$
\begin{array}{ll}
 & \alpha \in (\mathsf{G}\bullet)\bullet(\mathsf{F}\bullet) \leftarrow \mathsf{Id}_{\mathsf{Fun}(\mathcal{D},\mathcal{E})} \\
\equiv & \qquad \{ \qquad \text{definition natural transformation} \qquad \} \\
 & \quad \forall(\mathsf{H}::\ \alpha_{\mathsf{H}} \in \mathsf{G}\bullet\mathsf{F}\bullet\mathsf{H} \leftarrow \mathsf{H}) \\
 & \wedge\ \forall(\eta\colon\ \eta \in \mathsf{H} \leftarrow \mathsf{K}\colon\ \mathsf{G}\bullet\mathsf{F}\bullet\eta \circ \alpha_{\mathsf{K}} = \alpha_{\mathsf{H}} \circ \eta) \\
\Leftarrow & \qquad \{ \qquad \bullet \quad \alpha_{\mathsf{H}} = \mathsf{unit}\bullet\mathsf{H} \qquad \} \\
 & \forall(\eta\colon\ \eta \in \mathsf{H} \leftarrow \mathsf{K}\colon\ \mathsf{G}\bullet\mathsf{F}\bullet\eta \circ \mathsf{unit}\bullet\mathsf{K} = \mathsf{unit}\bullet\mathsf{H} \circ \eta) \\
\equiv & \qquad \{ \qquad \text{vertical composition (2.19):}\ \ \eta \in \mathsf{H} \leftarrow \mathsf{K}\ \text{ and }\ \mathsf{unit} \in \mathsf{G}\bullet\mathsf{F} \leftarrow \mathsf{Id} \qquad \} \\
 & \mathit{true} \quad .
\end{array}
$$

So, we define $\mathsf{unit}\bullet$, and dually $\mathsf{counit}\bullet$, by

(4.34) $(\mathsf{unit}\bullet)_{\mathsf{H}} = \mathsf{unit}\bullet\mathsf{H}$,

(4.35) $(\mathsf{counit}\bullet)_{\mathsf{H}} = \mathsf{counit}\bullet\mathsf{H}$.

What remains are the two coherence requirements (as stated in theorem 4.25), which are proven using extensionality.

$$
\begin{array}{ll}
 & ((\mathsf{counit}\bullet)\bullet(\mathsf{F}\bullet) \circ (\mathsf{F}\bullet)\bullet(\mathsf{unit}\bullet))_{\mathsf{H}} \\
= & \qquad \{ \qquad (2.11) \qquad \} \\
 & (\mathsf{counit}\bullet)_{(\mathsf{F}\bullet).\mathsf{H}} \circ (\mathsf{F}\bullet).((\mathsf{unit}\bullet)_{\mathsf{H}}) \\
= & \qquad \{ \qquad \text{definition } \mathsf{F}\bullet,\ \mathsf{unit}\bullet \text{ and } \mathsf{counit}\bullet \qquad \} \\
 & (\mathsf{counit}\bullet\mathsf{F}\bullet\mathsf{H}) \circ (\mathsf{F}\bullet\mathsf{unit}\bullet\mathsf{H}) \\
= & \qquad \{ \qquad \text{Godement's rules} \qquad \} \\
 & (\mathsf{counit}\bullet\mathsf{F} \circ \mathsf{F}\bullet\mathsf{unit})\bullet\mathsf{H} \\
= & \qquad \{ \qquad \text{coherence property of the adjunction } (\mathsf{F},\mathsf{G}) \qquad \} \\
 & (\mathsf{id}\bullet\mathsf{F})\bullet\mathsf{H} \\
= & \qquad \{ \qquad \text{Godement's rules} \qquad \} \\
 & (\mathsf{id}\bullet(\mathsf{F}\bullet))_{\mathsf{H}} \quad .
\end{array}
$$

The verification of the second coherence requirement is similar.

For the proof in the other direction use the constant functor K which maps an object x to the constant functor $\mathsf{K}.x$ on x, defined by: $(\mathsf{K}.x).y = x$ for all objects y and $(\mathsf{K}.x).f = \mathsf{id}_x$ for all arrows f, and maps an arrow f to the natural transformation $\mathsf{K}.f$ on objects y defined by: $(\mathsf{K}.f)_y = f$. This is left as an exercise for the reader.

□

The adjungates $\llbracket \mathsf{G}\bullet;\ \rrbracket$ and $\lceil\!\lceil \mathsf{F}\bullet;\ \rceil\!\rceil$ of the adjunction $(\mathsf{F}\bullet,\ \mathsf{G}\bullet)$ are of the following type:

$$\llbracket \mathsf{G}\bullet;\eta \rrbracket_{\mathsf{K},\mathsf{H}} \in \mathsf{G}\bullet\mathsf{K} \leftarrow \mathsf{H}\ \Leftarrow\ \eta \in \mathsf{K} \leftarrow \mathsf{F}\bullet\mathsf{H} \quad ,$$

$$[\![\mathsf{F}\bullet;\tau]\!]_{\mathsf{K},\mathsf{H}} \in \mathsf{K} \leftarrow \mathsf{F}\bullet\mathsf{H} \;\Leftarrow\; \tau \in \mathsf{G}\bullet\mathsf{K} \leftarrow \mathsf{H} \quad .$$

The relationship between the adjungates of the adjunction (F,G) and the adjungates of the adjunction $(\mathsf{F}\bullet\,,\,\mathsf{G}\bullet)$ can easily be constructed. Let η and τ be natural transformations as defined above, then

(4.36) $([\![\mathsf{G}\bullet;\eta]\!]_{\mathsf{K},\mathsf{H}})_x = \lfloor \mathsf{G};\eta_x \rfloor_{\mathsf{K}.x,\mathsf{H}.x}$ and $([\![\mathsf{F}\bullet;\tau]\!]_{\mathsf{K},\mathsf{H}})_y = \lceil \mathsf{F};\tau_y \rceil_{\mathsf{K}.y,\mathsf{H}.y}$.

Note, $[\![\mathsf{G}\bullet;\eta]\!] = \mathsf{G}\bullet\eta \circ \mathsf{unit}\bullet\mathsf{H}$ and $[\![\mathsf{F}\bullet;\tau]\!] = \mathsf{counit}\bullet\mathsf{K} \circ \mathsf{F}\bullet\tau$ so using the Godement's rules we can derive, for every functor L:

(4.37) $[\![\mathsf{G}\bullet;\eta]\!]\bullet\mathsf{L} = [\![\mathsf{G}\bullet;\eta\bullet\mathsf{L}]\!]$ and $[\![\mathsf{F}\bullet;\tau]\!]\bullet\mathsf{L} = [\![\mathsf{F}\bullet;\tau\bullet\mathsf{L}]\!]$

Inverse functions have inverse algebraic properties. Functions that form a Galois connection have pseudo-inverse algebraic properties. The latter is expressed by the following theorem. Suppose, for $i=0,1$, $(C_i, \sqsupseteq_{C_i})$ and $(D_i, \sqsupseteq_{D_i})$ are partially ordered sets and the pairs of functions $(\mathsf{F}_i \in C_i \leftarrow D_i\;,\; \mathsf{G}_i \in D_i \leftarrow C_i)$ form a Galois connection. Then for all monotonic functions $\mathsf{H} \in D_0 \leftarrow D_1$ and $\mathsf{K} \in C_0 \leftarrow C_1$,

(4.38) $\mathsf{K}\bullet\mathsf{F}_1 \mathrel{\dot{\sqsupseteq}} \mathsf{F}_0\bullet\mathsf{H} \equiv \mathsf{G}_0\bullet\mathsf{K} \mathrel{\dot{\sqsupseteq}} \mathsf{H}\bullet\mathsf{G}_1$.

This one rule captures in a nutshell all calculation properties of Galois connections. The corresponding theorem in category theory is that adjoint functors have constructively pseudo-inverse algebraic properties.

Theorem 4.39 Suppose, for $i=0,1$, $\mathcal{C}_i$ and $\mathcal{D}_i$ are categories and the pairs of functors $(\mathsf{F}_i \in \mathcal{C}_i \leftarrow \mathcal{D}_i\;,\; \mathsf{G}_i \in \mathcal{D}_i \leftarrow \mathcal{C}_i)$ form an adjunction. Then for all functors $\mathsf{H} \in \mathcal{D}_0 \leftarrow \mathcal{D}_1$ and $\mathsf{K} \in \mathcal{C}_0 \leftarrow \mathcal{C}_1$,

(4.40) $[\![\mathsf{G}_0\bullet\,;\,\mathsf{K}\bullet\mathsf{counit}_{\mathsf{G}_1} \circ \eta\bullet\mathsf{G}_1]\!] \in \mathsf{G}_0\bullet\mathsf{K} \leftarrow \mathsf{H}\bullet\mathsf{G}_1 \equiv \eta \in \mathsf{K}\bullet\mathsf{F}_1 \leftarrow \mathsf{F}_0\bullet\mathsf{H}$,

(4.41) $[\![\mathsf{F}_0\bullet\,;\,\tau\bullet\mathsf{F}_1 \circ \mathsf{H}\bullet\mathsf{unit}_{\mathsf{F}_1}]\!] \in \mathsf{K}\bullet\mathsf{F}_1 \leftarrow \mathsf{F}_0\bullet\mathsf{H} \equiv \tau \in \mathsf{G}_0\bullet\mathsf{K} \leftarrow \mathsf{H}\bullet\mathsf{G}_1$.

(Note that $[\![\mathsf{G}_0\bullet;\;]\!]$ and $[\![\mathsf{F}_0\bullet;\;]\!]$ denote the adjungates of the adjunction $(\mathsf{F}_0\bullet\,,\,\mathsf{G}_0\bullet)$ whilst $\mathsf{counit}_{\mathsf{G}_1}$ and $\mathsf{unit}_{\mathsf{F}_1}$ denote the counit and unit, respectively, of the adjunction $(\mathsf{F}_1,\mathsf{G}_1)$.)

Proof We first prove the follows-from in each of the equivalences. That is, we assume the right-hand side and construct the witness on the left-hand side. The implications are then proven by showing that the constructed witnesses are inverses of each other. By construction of α we calculate as follows:

$$
\begin{array}{ll}
 & \alpha \in \mathsf{G}_0\bullet\mathsf{K} \leftarrow \mathsf{H}\bullet\mathsf{G}_1 \\
\Leftarrow & \{ \quad (\mathsf{F}_0\bullet\,,\,\mathsf{G}_0\bullet) \text{ is an adjunction,} \\
 & \quad\; \bullet \quad \alpha = [\![\mathsf{G}_0\bullet;\beta]\!] \quad \}
\end{array}
$$

$$\beta \in K \leftarrow F_0 \bullet H \bullet G_1$$

$$\Leftarrow \quad \{ \quad \bullet \quad \beta = \gamma \circ \delta \quad \}$$

$$\gamma \in K \leftarrow K \bullet F_1 \bullet G_1 \;\wedge\; \delta \in K \bullet F_1 \bullet G_1 \leftarrow F_0 \bullet H \bullet G_1$$

$$\Leftarrow \quad \{ \quad \text{Godement's rules} \quad \}$$

$$\gamma = K \bullet \mathsf{counit}_{G_1} \;\wedge\; \delta = \eta \bullet G_1 \quad .$$

The construction of the witness in the left-hand side of equivalence (4.41) is similar. Finally, we prove that the constructed witnesses are each others' inverses.

$$\lceil\!\lceil F_0 \bullet \;;\; \lfloor\!\lfloor G_0 \bullet \;;\; K \bullet \mathsf{counit}_{G_1} \circ \eta \bullet G_1 \rfloor\!\rfloor \bullet F_1 \circ H \bullet \mathsf{unit}_{F_1} \rceil\!\rceil$$

$$= \quad \{ \quad (4.16c) \quad \}$$

$$\lceil\!\lceil F_0 \bullet ; \lfloor\!\lfloor G_0 \bullet \;;\; K \bullet \mathsf{counit}_{G_1} \circ \eta \bullet G_1 \rfloor\!\rfloor \bullet F_1 \rceil\!\rceil \circ F_0 \bullet (H \bullet \mathsf{unit}_{F_1})$$

$$= \quad \{ \quad (4.37),\ \lfloor\!\lfloor G_0 \bullet ;\ \rfloor\!\rfloor \text{ and } \lceil\!\lceil F_0 \bullet ;\ \rceil\!\rceil \text{ are each others' inverses} \quad \}$$

$$(K \bullet \mathsf{counit}_{G_1} \circ \eta \bullet G_1) \bullet F_1 \circ F_0 \bullet (H \bullet \mathsf{unit}_{F_1})$$

$$= \quad \{ \quad \text{Godement's rules} \quad \}$$

$$(K \bullet \mathsf{counit}_{G_1}) \bullet F_1 \circ \eta \bullet (G_1 \bullet F_1) \circ (F_0 \bullet H) \bullet \mathsf{unit}_{F_1}$$

$$= \quad \{ \quad \text{vertical composition (2.19): } \eta \in K \bullet F_1 \leftarrow F_0 \bullet H \text{ and } \mathsf{unit}_{F_1} \in G_1 \bullet F_1 \leftarrow \mathsf{Id} \quad \}$$

$$(K \bullet \mathsf{counit}_{G_1}) \bullet F_1 \circ (K \bullet F_1) \bullet \mathsf{unit}_{F_1} \circ \eta$$

$$= \quad \{ \quad \text{Godement's rules} \quad \}$$

$$K \bullet (\mathsf{counit}_{G_1} \bullet F_1 \circ F_1 \bullet \mathsf{unit}_{F_1}) \circ \eta$$

$$= \quad \{ \quad \text{coherence property of the adjunction } (F_1, G_1) \quad \}$$

$$K \bullet (\mathsf{id} \bullet F_1) \circ \eta$$

$$= \quad \{ \quad \text{identity} \quad \}$$

$$\eta \quad .$$

□

4.4 Sharp and Flat

Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories. In this section we suppose that two mappings, $^\sharp$ and $^\flat$, are given. The former maps a functor $F \in \mathcal{C} \leftarrow \mathcal{D}$ that is known to have an upper adjoint to a canonical upper adjoint $F^\sharp \in \mathcal{D} \leftarrow \mathcal{C}$. The latter does the opposite. It maps a functor $G \in \mathcal{D} \leftarrow \mathcal{C}$ that is known to have a lower adjoint to a canonical lower adjoint $G^\flat \in \mathcal{C} \leftarrow \mathcal{D}$. We assume, furthermore, that the relevant counits and units of the adjunctions are known. A final assumption is that $^\sharp$ and $^\flat$ are inverse functions (so that $F^{\sharp\flat} = F$ and $G^{\flat\sharp} = G$).

Theorem 4.42 $\sharp$ and $\flat$ are contravariant functors. Moreover, the functors are inverses of each other.

Proof On objects $\sharp$ maps a functor, that has an upper adjoint, to a given fixed upper adjoint. For arrows we have the following. Suppose F and G have upper adjoints. Then by theorem (4.39), where we take for both H and K the identity functor,

$$\llbracket \mathsf{F}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{G}^\sharp} \circ \eta\bullet\mathsf{G}^\sharp \rrbracket \in \mathsf{F}^\sharp \leftarrow \mathsf{G}^\sharp \quad \text{, where } \eta \in \mathsf{G} \leftarrow \mathsf{F} \quad .$$

Note the reversal in the typing of the arrows. So, for arrows we define $\sharp$ by

$$(4.43) \quad \eta^\sharp = \llbracket \mathsf{F}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{G}^\sharp} \circ \eta\bullet\mathsf{G}^\sharp \rrbracket \quad .$$

It remains to verify the coherence properties. Let F, G and H be functors that have an upper adjoint and let $\eta \in \mathsf{G} \leftarrow \mathsf{F}$ and $\tau \in \mathsf{H} \leftarrow \mathsf{G}$ be two natural transformations. Then,

$$
\begin{array}{ll}
 & \eta^\sharp \circ \tau^\sharp \\
= & \{ \quad \text{definition of } \sharp\text{: (4.43)} \quad \} \\
 & \llbracket \mathsf{F}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{G}^\sharp} \circ \eta\bullet\mathsf{G}^\sharp \rrbracket \circ \llbracket \mathsf{G}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{H}^\sharp} \circ \tau\bullet\mathsf{H}^\sharp \rrbracket \\
= & \{ \quad \text{(4.16d)} \quad \} \\
 & \llbracket \mathsf{F}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{G}^\sharp} \circ \eta\bullet\mathsf{G}^\sharp \circ \mathsf{F}\bullet \llbracket \mathsf{G}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{H}^\sharp} \circ \tau\bullet\mathsf{H}^\sharp \rrbracket \rrbracket \\
= & \{ \quad \text{vertical composition (2.19): } \eta \in \mathsf{G} \leftarrow \mathsf{F} \text{ and} \\
 & \qquad \llbracket \mathsf{G}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{H}^\sharp} \circ \tau\bullet\mathsf{H}^\sharp \rrbracket \in \mathsf{G}^\sharp \leftarrow \mathsf{H}^\sharp \quad \} \\
 & \llbracket \mathsf{F}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{G}^\sharp} \circ \mathsf{G}\bullet \llbracket \mathsf{G}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{H}^\sharp} \circ \tau\bullet\mathsf{H}^\sharp \rrbracket \circ (\eta\bullet\mathsf{H}^\sharp) \rrbracket \\
= & \{ \quad \text{(4.31)} \quad \} \\
 & \llbracket \mathsf{F}^\sharp\bullet \,;\, \mathsf{counit}_{\mathsf{H}^\sharp} \circ \tau\bullet\mathsf{H}^\sharp \circ \eta\bullet\mathsf{H}^\sharp \rrbracket \\
= & \{ \quad \text{factorise, definition of } \sharp\text{: (4.43)} \quad \} \\
 & (\tau\circ\eta)^\sharp \quad .
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & (\mathsf{id}\bullet\mathsf{F})^\sharp \\
= & \{ \quad \text{definition of } \sharp\text{: (4.43)} \quad \} \\
 & \llbracket \mathsf{F}^\sharp\bullet \;;\; \mathsf{counit}_{\mathsf{F}^\sharp} \circ \mathsf{id}\bullet\mathsf{F}\bullet\mathsf{F}^\sharp \rrbracket \\
= & \{ \quad \text{identity, } \mathsf{counit}_{\mathsf{F}^\sharp} = \llbracket \mathsf{F}\bullet ; \mathsf{id}\bullet\mathsf{F}^\sharp \rrbracket \quad \} \\
 & \mathsf{id}\bullet\mathsf{F}^\sharp \quad .
\end{array}
$$

Symmetrically, let F and G be functors that have a lower adjoint, then the arrow part of $\flat$ is defined by

$$(4.44) \quad \tau^\flat = \llbracket \mathsf{G}^\flat\bullet \,;\, \tau\bullet\mathsf{F}^\flat \circ \mathsf{unit}_{\mathsf{F}^\flat} \rrbracket (\in \mathsf{F}^\flat \leftarrow \mathsf{G}^\flat) \quad \text{, where } \tau \in \mathsf{G} \leftarrow \mathsf{F} \quad .$$

The final part of the theorem —that $\sharp$ and $\flat$ are inverses of each other— comprises two statements, namely that they are inverses of each other as mappings on functors, and they are inverses of each other as mappings on natural transformations. The first statement is true by assumption, the second statement is immediate from the (more general) fact that the constructions given by (4.40) and (4.41) are inverses of each other.
□

Theorem 4.45 $\sharp$ and $\flat$ form a contravariant adjunction.

Proof Let F be functor that has an upper adjoint, so we can apply $\sharp$ to F. Furthermore, let G be a functor that has a lower adjoint, so we can apply $\flat$ to G. Then we have

$$\eta^\sharp \in \mathsf{F}^\sharp \leftarrow \mathsf{G} \Leftarrow \eta \in \mathsf{G}^\flat \leftarrow \mathsf{F} \quad ,$$

$$\tau^\flat \in \mathsf{G}^\flat \leftarrow \mathsf{F} \Leftarrow \tau \in \mathsf{F}^\sharp \leftarrow \mathsf{G} \quad .$$

Thus the lower and upper adjungates of the adjunction we want to construct are the functors $\sharp$ and $\flat$. It remains to verify the coherence requirements specified in definition 4.30.

The property (4.30e) has already been verified. (It is the statement that the functors $\sharp$ and $\flat$ are inverses of each other.) To verify (4.30c), let H be a functor that has an upper adjoint and let K be a functor that has a lower adjoint. Furthermore suppose $\tau \in \mathsf{F} \leftarrow \mathsf{H}$, $\eta \in \mathsf{G}^\flat \leftarrow \mathsf{F}$ and $\sigma \in \mathsf{G} \leftarrow \mathsf{K}$ are natural transformations. Then the appropriate instantiation of (4.30c) is

$$\tau^\sharp \circ \eta^\sharp \circ \sigma = (\sigma^\flat \circ \eta \circ \tau)^\sharp \quad .$$

The verification is elementary:

$$\begin{array}{ll} & \tau^\sharp \circ \eta^\sharp \circ \sigma \\ = & \qquad \{ \quad \sharp \text{ and } \flat \text{ are inverses} \quad \} \\ & \tau^\sharp \circ \eta^\sharp \circ \sigma^{\flat\sharp} \\ = & \qquad \{ \quad \sharp \text{ is a contravariant functor} \quad \} \\ & (\sigma^\flat \circ \eta \circ \tau)^\sharp \quad . \end{array}$$

□

An upper adjoint of a functor may not be unique but, in the jargon of category theory, is “unique up to isomorphism”.

Theorem 4.46 With the same assumptions as in theorem (4.39) we have

$$\mathsf{F}_0 \cong \mathsf{F}_1 \equiv \mathsf{G}_0 \cong \mathsf{G}_1 \quad .$$

In words, adjoint functors are unique up to isomorphism.

Proof Suppose $\eta \in F_0 \cong F_1$. Then we have two natural transformations $\eta \in F_0 \leftarrow F_1$ and $\tau \in F_1 \leftarrow F_0$, whose composition in either order gives an identity transformation. Defining $(F_0)^\sharp = G_0$, $(F_1)^\sharp = G_1$, $(G_0)^\flat = F_0$ and $(G_1)^\flat = F_1$, we have $\eta^\sharp \in G_1 \leftarrow G_0$ and $\tau^\sharp \in G_0 \leftarrow G_1$. Hence,

$$
\begin{array}{ll}
 & \eta^\sharp \circ \tau^\sharp \\
= & \quad \{ \quad \sharp \text{ is a contravariant functor } \} \\
 & (\tau \circ \eta)^\sharp \\
= & \quad \{ \quad \tau \text{ is the inverse of } \eta \quad \} \\
 & \mathsf{id}^\sharp \quad .
\end{array}
$$

In the same way, $\tau^\sharp \circ \eta^\sharp$ is also an identity transformation.
□

4.5 Limits and Colimits

The category-theoretic notion corresponding to the notion of supremum is *colimit*. In order to define colimits we need to first define the diagonal[3] *functor* K in place of the diagonal *function* K . This is straightforward enough. Given categories $\mathcal{A}$ and $\mathcal{B}$ we define $\mathsf{K}.x$, for each x in $\mathcal{A}$, to be the functor that maps objects y in $\mathcal{B}$ to x, and arrows $g \in y \overset{\mathcal{B}}{\longleftarrow} z$ to the arrow $\mathsf{id}_x \in (\mathsf{K}.x).y \overset{\mathcal{A}}{\longleftarrow} (\mathsf{K}.x).z$. It is left to the reader to check that this does indeed define $\mathsf{K}.x$ to be a functor of type $\mathcal{A} \leftarrow \mathcal{B}$ (in particular that the coherence properties are satisfied). Now we extend the definition of K to arrows in the category $\mathcal{A}$: if $f \in w \overset{\mathcal{A}}{\longleftarrow} x$ we define $\mathsf{K}.f$ to be a natural transformation to the functor $\mathsf{K}.w$ from the functor $\mathsf{K}.x$ by letting $(\mathsf{K}.f)_y$ equal f for all objects y in the category $\mathcal{B}$. It is left to the reader to check that this does indeed define a natural transformation. The conclusion is that $\mathsf{K} \in \mathsf{Fun}(\mathcal{A},\mathcal{B}) \leftarrow \mathcal{A}$, i.e. K is a functor to the functor category $\mathsf{Fun}(\mathcal{A},\mathcal{B})$ from the category $\mathcal{A}$.

Suppose $F \in \mathcal{A} \leftarrow \mathcal{B}$. A *colimit* of F is a solution of the equation:

(4.47) $\quad x :: [a :: a \leftarrow x] \cong [a :: \mathsf{K}.a \leftarrow F] \quad .$

Here both dummies a range over objects and arrrows of $\mathcal{A}$. The functors $[a :: a \leftarrow x]$ and $[a :: \mathsf{K}.a \leftarrow F]$ are defined according to a minor variation on the definition of the functor $[x, y :: F.x \leftarrow G.y]$ introduced in section 4.2. Specifically, for all $a \in \mathcal{A}$, $[a :: a \leftarrow x].a$ is the set of arrows to a from x in the category $\mathcal{A}$, and $[a :: \mathsf{K}.a \leftarrow F].a$ is the set of all arrows to $\mathsf{K}.a$ from F in the category $\mathsf{Fun}(\mathcal{A},\mathcal{B})$; also, for all $f \in u \overset{\mathcal{A}}{\longleftarrow} v$, $[a :: a \leftarrow x].f$ is the

[3] The name "diagonal functor" is that used by Mac Lane [15, p67]. Mac Lane uses the symbol $\triangle$, however, instead of K . Confusingly, on p. 62 Mac Lane also uses the name "diagonal functor" and the symbol $\triangle$ for the functor that doubles its argument (the justification presumably being that the former generalises the latter). The functor K defined here is the more general of the two; the "doubling" functor $\triangle$ is introduced later.

function $f\circ$ (mapping arrows to ν from x into arrows to u from x) and $[a :: \mathsf{K}.a \leftarrow \mathsf{F}].f$ is the function $(\mathsf{K}.f)\circ$. (Unfolding the definition of $\mathsf{K}.f$, the function $(\mathsf{K}.f)\circ$ maps natural transformation $\eta \in \mathsf{K}.\nu \leftarrow \mathsf{F}$ to the natural transformation $\mathsf{K}.f \circ \eta \in \mathsf{K}.u \leftarrow \mathsf{F}$ where, for each object y, $(\mathsf{K}.f \circ \eta)_y = f \circ \eta_y$.)

It is easy to see that all colimits of a functor are isomorphic. Suppose x and y are both colimits of functor F. Then,

$$\begin{array}{ll} & [a :: a \leftarrow x] \\ \cong & \quad \{ \quad x \text{ is a colimit of } \mathsf{F} \quad \} \\ & [a :: \mathsf{K}.a \leftarrow \mathsf{F}] \\ \cong & \quad \{ \quad y \text{ is a colimit of } \mathsf{F} \quad \} \\ & [a :: a \leftarrow y] \quad . \end{array}$$

It follows by the law of indirect isomorphism, lemma 3.15, that $x \cong y$.

The category $\mathcal{A}$ is *$\mathcal{B}$-cocomplete* iff the diagonal functor $\mathsf{K} \in \mathsf{Fun}(\mathcal{A},\mathcal{B}) \leftarrow \mathcal{A}$ has a lower adjoint. Thus, $\mathcal{A}$ is *$\mathcal{B}$-cocomplete* if there is a functor $\mathsf{Col} \in \mathcal{A} \leftarrow \mathsf{Fun}(\mathcal{A},\mathcal{B})$ such that

$$(4.48) \quad [a, \mathsf{F} :: a \leftarrow \mathsf{Col}.\mathsf{F}] \cong [a, \mathsf{F} :: \mathsf{K}.a \leftarrow \mathsf{F}] \quad .$$

The dummies a and F in (4.48) range over objects of the category $\mathcal{A}$ and objects of the category $\mathsf{Fun}(\mathcal{A},\mathcal{B})$ (i.e. functors to $\mathcal{A}$ from $\mathcal{B}$), respectively.

Dually, a *limit* of F is a solution of the equation:

$$(4.49) \quad x :: [a :: x \leftarrow a] \cong [a :: \mathsf{F} \leftarrow \mathsf{K}.a] \quad .$$

Example I
Let $\emptyset$ denote the empty category. Then the functor category $\mathcal{A} \leftarrow \emptyset$ is the category consisting of only one object, say $\mathbb{1}$, and one arrow (the identity arrow on $\mathbb{1}$). So, the diagonal functor $\mathsf{K}_{\mathcal{A},\emptyset} \in (\mathcal{A} \leftarrow \emptyset) \leftarrow \mathcal{A}$ is the functor which maps an object of $\mathcal{A}$ to $\mathbb{1}$.

Suppose x is a colimit of the functor $\mathbb{1}$, i.e. x satisfies

$$(4.50) \quad [a :: a \overset{\mathcal{A}}{\longleftarrow} x] \cong [a :: \mathsf{K}_{\mathcal{A},\emptyset}.a \overset{\mathcal{A}\leftarrow\emptyset}{\longleftarrow} \mathbb{1}] \quad .$$

But, for every object a in $\mathcal{A}$, $\mathsf{K}_{\mathcal{A},\emptyset}.a = \mathbb{1}$, so $\mathsf{K}_{\mathcal{A},\emptyset}.a \overset{\mathcal{A}\leftarrow\emptyset}{\longleftarrow} \mathbb{1}$ consists of precisely one arrow, viz. $\mathsf{id}_{\mathbb{1}}$. Therefore, by (4.50), there is also precisely one arrow in $\mathcal{A}$ to a from x, i.e. x is an initial object in $\mathcal{A}$. Thus, a colimit of $\mathbb{1}$ is an initial object. Moreover, by the arguments given here it's obvious that the converse is also true.
End of Example

4.5.1 Parameterised Colimits

Just as in lattice theory if (poset) $\mathcal{A}$ is $\mathcal{B}$-cocomplete then $\mathcal{A}\leftarrow\mathcal{C}$ is $\mathcal{B}$-cocomplete for all (posets) $\mathcal{C}$ it is the case that if category $\mathcal{A}$ is $\mathcal{B}$-cocomplete then the functor category $\mathsf{Fun}(\mathcal{A},\mathcal{C})$ is $\mathcal{B}$-cocomplete for all categories $\mathcal{C}$. As mentioned earlier, this is theorem 1 on p.111 of [15]. Let us present an alternative proof to illustrate the process of transforming proofs in lattice theory into proofs in category theory. encouraged to whilst reading

Suppose $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are categories. Suppose $\oplus\in(\mathcal{A}\leftarrow\mathcal{C})\leftarrow\mathcal{B}$. For brevity we denote application of $\oplus$ to arguments b in $\mathcal{B}$ and c in $\mathcal{C}$ by $\mathsf{b}\oplus\mathsf{c}$. Assume that, for all $\mathsf{c}\in\mathcal{C}$, the functor $\oplus\mathsf{c}\in\mathcal{A}\leftarrow\mathcal{B}$ defined by $(\oplus\mathsf{c}).\mathsf{b}=\mathsf{b}\oplus\mathsf{c}$ has a colimit denoted by $\mathsf{Col}.(\oplus\mathsf{c})$. Note that, since $\oplus\mathsf{c}$ is a functor of type $\mathcal{A}\leftarrow\mathcal{B}$, the existence of $\mathsf{Col}.(\oplus\mathsf{c})$ is implied by $\mathcal{B}$-cocompleteness of $\mathcal{A}$. The question is whether this information is sufficient to guarantee that $\oplus$ has a colimit.

We need to differentiate between diagonal functors: where above we wrote K let us now write $\mathsf{K}_{\mathcal{A},\mathcal{B}}$. The assumption is thus that, for all $\mathsf{c}\in\mathcal{C}$, there is an object $\mathsf{Col}.(\oplus\mathsf{c})$ in $\mathcal{A}$ satisfying

$$(4.51)\quad [\mathsf{a} :: \mathsf{a}\leftarrow\mathsf{Col}.(\oplus\mathsf{c})] \cong [\mathsf{a} :: \mathsf{K}_{\mathcal{A},\mathcal{B}}.\mathsf{a}\leftarrow\oplus\mathsf{c}] \quad ,$$

The goal is to solve the equation:

$$(4.52)\quad \mathsf{x}{::}\ \mathsf{x}\in\mathcal{A}\leftarrow\mathcal{C}\ \wedge\ [\mathsf{f} :: \mathsf{f}\leftarrow\mathsf{x}]\cong[\mathsf{f} :: \mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f}\leftarrow\oplus] \quad .$$

A candidate for x is obtained by extending the mapping $\mathsf{c}\mapsto\mathsf{Col}.(\oplus\mathsf{c})$ on objects of the category $\mathcal{C}$ to a functor. This is easily done if we observe that $\mathsf{c}\mapsto\mathsf{Col}.(\oplus\mathsf{c})$ coincides with the object part of the functor $\mathsf{Col}\bullet\mathsf{Flip}.\oplus$ where Flip is the functor that “flips” the order of the arguments of a functor of type $(\mathcal{A}\leftarrow\mathcal{C})\leftarrow\mathcal{B}$ to form a functor of type $(\mathcal{A}\leftarrow\mathcal{B})\leftarrow\mathcal{C}$. (That is, $\mathsf{Flip}.\oplus$ is the functor $\hat{\oplus}$ of type $(\mathcal{A}\leftarrow\mathcal{B})\leftarrow\mathcal{C}$ such that $\mathsf{b}\oplus\mathsf{c}=\mathsf{c}\hat{\oplus}\mathsf{b}$.) Accordingly we verify that $\mathsf{Col}\bullet\mathsf{Flip}.\oplus$ satisfies (4.52). It clearly satisfies the first conjunct (since it is a composition of two functors of types $\mathcal{A}\leftarrow(\mathcal{A}\leftarrow\mathcal{B})$ and $(\mathcal{A}\leftarrow\mathcal{B})\leftarrow\mathcal{C}$). For the second conjunct we have:

$$
\begin{array}{ll}
 & [\mathsf{f} :: \mathsf{f}\leftarrow\mathsf{Col}\bullet\mathsf{Flip}.\oplus] \\
\cong & \{\quad \mathsf{Col}\text{ has upper adjoint }\mathsf{K}_{\mathcal{A},\mathcal{B}}\text{, by assumption.} \\
 & \quad\ \text{So, by (4.33), }(\mathsf{Col}\bullet)\text{ has upper adjoint }\mathsf{K}_{\mathcal{A},\mathcal{B}}\bullet \quad \} \\
 & [\mathsf{f} :: \mathsf{K}_{\mathcal{A},\mathcal{B}}\bullet\mathsf{f}\leftarrow\mathsf{Flip}.\oplus] \\
= & \{\quad \mathsf{K}_{\mathcal{A},\mathcal{B}}\bullet\mathsf{f}=\mathsf{Flip}.(\mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f})\ \text{ (just as in lattice theory)} \quad \} \\
 & [\mathsf{f} :: \mathsf{Flip}.(\mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f})\leftarrow\mathsf{Flip}.\oplus] \\
\cong & \{\quad \mathsf{Flip}\text{ is clearly an isomorphism of categories} \quad \} \\
 & [\mathsf{f} :: \mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f}\leftarrow\oplus] \quad .
\end{array}
$$

What emerges very clearly from this proof is that the parameterised colimit theorem is an instance of (4.33).

Mac Lane's [15, p.111] proof of this theorem involves the explicit construction of the witnesses to the isomorphism followed by a verification of their naturality properties. The above, equational, proof includes a construction of the witnesses as a (mechanical) by-product. Specifically, letting $\llbracket\ \rrbracket$ and $\lceil\!\lceil\ \rceil\!\rceil$ denote the lower and upper adjungate of the adjunction $(\mathsf{Col}_{\mathcal{A}}\bullet\,,\ \mathsf{K}_{\mathcal{A},\mathcal{C}}\bullet)$ they are:

$$\lceil\!\lceil\ \rceil\!\rceil_{\_\,,\,\mathsf{Flip}.\oplus}\bullet\mathsf{K}.\mathsf{Flip}\in[\mathsf{f}::\ \ \mathsf{f}\leftarrow\mathsf{Col}\bullet\mathsf{Flip}.\oplus]\cong[\mathsf{f}::\ (\mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f})\leftarrow\oplus]$$

and

$$\mathsf{K}.\mathsf{Flip}\bullet\llbracket\ \rrbracket_{\_\,,\,\mathsf{Flip}.\oplus}\in[\mathsf{f}::\ (\mathsf{K}_{\mathcal{A}\leftarrow\mathcal{C},\mathcal{B}}.\mathsf{f})\leftarrow\oplus]\cong[\mathsf{f}::\ \ \mathsf{f}\leftarrow\mathsf{Col}\bullet\mathsf{Flip}.\oplus]\quad .$$

(The compositions arise from the transitivity of isomorphism, and the four subterms are the witnesses to the two isomorphism steps in the above proof.)

The standard example of a parameterised colimit is the coproduct functor. Let $\mathcal{B}$ be the category consisting of exactly two objects, 0 and 1, and two arrows, namely the identity arrows on 0 and 1. Let $\mathcal{C}$ be $(\mathcal{A}\times\mathcal{A}\,,\sqsupseteq\times\sqsupseteq)$. Define $\oplus\in(\mathcal{A}\leftarrow\mathcal{C})\leftarrow\mathcal{B}$ by: on objects $0\oplus(a_0,a_1)=a_0$ and $1\oplus(a_0,a_1)=a_1$; on arrows $\mathsf{id}_0\oplus(f_0,f_1)=f_0$ and $\mathsf{id}_1\oplus(f_0,f_1)=f_1$. Assuming that $\mathcal{A}$ is $\mathcal{B}$-cocomplete, this defines the *coproduct* functor $+$ to be the functor $\mathsf{Col}\bullet\mathsf{Flip}.\oplus$. That is, by definition,

$$a_0+a_1\ =\ \mathsf{Col}.(\oplus(a_0,a_1))\quad .$$

Applying the abstraction theorem, coproduct is the colimit of the functor $\oplus$ we have just defined.

Expanding the definitions of colimit, adjunction, etc., we find that this gives several "theorems for free", as they have been called by Wadler [27]. Briefly, the lower adjungate in the adjunction defining Col boils down to a mapping from arrows $f\in a\leftarrow b+c$ to the pair $f\circ\mathsf{inl}_{b,c}\,,f\circ\mathsf{inr}_{b,c}$ where inl and inr are both natural transformations; the upper adjungate boils down to a mapping from pairs of arrows f and g with the same codomain to an arrow $f\triangledown g$ with codomain the common codomain of f and g and domain the coproduct of the domains of f and g. Moreover, the property (4.16e) yields

$$f\circ\mathsf{inl}=g\wedge f\circ\mathsf{inr}=h\ \equiv\ f=g\triangledown h\quad ,$$

the property (4.16d) yields the conjunction of

$$(h+k)\circ\mathsf{inl}=\mathsf{inl}\circ h$$

and

$$(h+k)\circ\mathsf{inr}=\mathsf{inr}\circ k\quad .$$

(Several simplifications are needed to reduce what is obtained from (4.16d) to this compact form. The conjunctions arise from equalities between pairs.) Finally, (4.16c) boils down to

$$(f\circ g\circ k)\ \triangledown\ (f\circ h\circ l)\ =\ f\circ(g\triangledown h)\circ(k+l)\quad .$$

4.5.2 Colimit Preservation

The lower adjoint in a Galois connection distributes universally over all suprema. Correspondingly:

Theorem 4.53 Lower adjoints preserve colimits.

Proof Let $(\mathsf{F} \in \mathcal{A} \leftarrow \mathcal{B}\,, \mathsf{F}^\sharp \in \mathcal{B} \leftarrow \mathcal{A}\,, \lfloor\ \rfloor\,, \lceil\ \rceil)$ be an adjunction. Suppose x is a colimit of the functor $\mathsf{G} \in \mathcal{B} \leftarrow \mathcal{C}$, i.e.

(4.54) $[\mathsf{b} :: \mathsf{b} \overset{\mathcal{B}}{\longleftarrow} x] \cong [\mathsf{b} :: \mathsf{K}_{\mathcal{B},\mathcal{C}}.\mathsf{b} \xleftarrow{\mathsf{Fun}(\mathcal{B},\mathcal{C})} \mathsf{G}]$.

We prove that

$$[a :: a \overset{\mathcal{A}}{\longleftarrow} \mathsf{F}.x] \cong [a :: \mathsf{K}_{\mathcal{A},\mathcal{C}}.a \xleftarrow{\mathsf{Fun}(\mathcal{A},\mathcal{C})} \mathsf{F}{\bullet}\mathsf{G}] \ ,$$

i.e. $\mathsf{F}.x$ is a colimit of $\mathsf{F}{\bullet}\mathsf{G}$.

$$
\begin{array}{ll}
 & [a :: a \overset{\mathcal{A}}{\longleftarrow} \mathsf{F}.x] \\
\cong & \quad \{ \quad \mathsf{F} \text{ has } \mathsf{F}^\sharp \text{ as upper adjoint,} \quad \} \\
 & [a :: \mathsf{F}^\sharp.a \overset{\mathcal{B}}{\longleftarrow} x] \\
\cong & \quad \{ \quad (4.54)\text{, note } \mathsf{H} \cong \mathsf{K} \Rightarrow \mathsf{H}{\bullet}\mathsf{F}^\sharp \cong \mathsf{K}{\bullet}\mathsf{F}^\sharp\,, \quad \} \\
 & [a :: \mathsf{K}_{\mathcal{B},\mathcal{C}}.(\mathsf{F}^\sharp.a) \xleftarrow{\mathsf{Fun}(\mathcal{B},\mathcal{C})} \mathsf{G}] \\
= & \quad \{ \quad (\mathsf{K}_{\mathcal{B},\mathcal{C}}.(\mathsf{F}^\sharp.a)).c = \mathsf{F}^\sharp.a = \mathsf{F}^\sharp.((\mathsf{K}_{\mathcal{A},\mathcal{C}}.a).c)\,, \\
 & \quad\quad\ \ (\mathsf{K}_{\mathcal{B},\mathcal{C}}.(\mathsf{F}^\sharp.a)).f = \mathsf{id}_{\mathsf{F}^\sharp.a} = \mathsf{F}^\sharp.\mathsf{id}_a = \mathsf{F}^\sharp.((\mathsf{K}_{\mathcal{A},\mathcal{C}}.a).f)\,, \\
 & \quad\quad\ \ \text{i.e. by extensionality, } \mathsf{K}_{\mathcal{B},\mathcal{C}}.(\mathsf{F}^\sharp.a) = \mathsf{F}^\sharp{\bullet}(\mathsf{K}_{\mathcal{A},\mathcal{C}}.a) \quad \} \\
 & [a :: \mathsf{F}^\sharp{\bullet}(\mathsf{K}_{\mathcal{A},\mathcal{C}}.a) \xleftarrow{\mathsf{Fun}(\mathcal{B},\mathcal{C})} \mathsf{G}] \\
\cong & \quad \{ \quad \mathsf{F}^\sharp{\bullet} \text{ has } \mathsf{F}{\bullet} \text{ as lower adjoint,} \quad \} \\
 & [a :: \mathsf{K}_{\mathcal{A},\mathcal{C}}.a \xleftarrow{\mathsf{Fun}(\mathcal{A},\mathcal{C})} \mathsf{F}{\bullet}\mathsf{G}] \ .
\end{array}
$$

□

Theorem 4.53 is a beautiful theorem with many applications. We are particularly proud of the above calculational proof of the theorem. For a traditional-style proof see [15, p. 114].

We showed that an initial object is a special instance of a colimit. So the above theorem immediately gives as a corollary:

Corollary 4.55 Lower adjoints preserve initial objects.

□

In lattice theory, if a set has a minimal element then that minimal element is also the infimum of the set. In category theory we have a theorem which has some resemblance to this property.

Theorem 4.56 If x is an initial object in the category $\mathcal{B}$ then the functor $\mathsf{F}\in\mathcal{A}\leftarrow\mathcal{B}$ has a limit viz. $\mathsf{F}.x$.

Proof See Mac Lane.
□

4.6 Existence of adjoints

Definition 4.57 (under-cone category) Let $\mathsf{G}\in\mathcal{D}\leftarrow\mathcal{C}$ be a functor and $z\in\mathcal{D}$. We will denote the *under-cone* category of z under G by $z{\downarrow}\mathsf{G}$. Its objects are pairs (f,x) such that $f\in\mathsf{G}.x\leftarrow z$. An arrow in $z{\downarrow}\mathsf{G}$ between two objects (f,x) and (g,y) is defined as follows:

$$\varphi\in(f,x)\overset{z\downarrow\mathsf{G}}{\longleftarrow}(g,y)\;\equiv\;\varphi\in x\overset{\mathcal{C}}{\longleftarrow}y\;\wedge\;f=\mathsf{G}.\varphi\circ g\quad.$$

We will refer to the x component of the pair (f,x) as the *carrier* of (f,x).

□

Note: in lattice theory this category corresponds to the set $\{x\colon \mathsf{G}.x\sqsupseteq z\colon x\}$.

Dually we can define the *over-cone* category $\mathsf{G}{\downarrow}z$. Its objects are pairs (f,x) such that $f\in z\leftarrow\mathsf{G}.x$.

Lemma 4.58 x is the carrier of an initial object in $z{\downarrow}\mathsf{G}$ iff,

$$[a::\ a\leftarrow x]\cong[a::\ \mathsf{G}.a\leftarrow z]\quad.$$

Proof We first prove the $\Leftarrow$ part. Suppose

$$\lceil\ \rceil\in[a::\ a\leftarrow x]\cong[a::\ \mathsf{G}.a\leftarrow z]\ni\lfloor\ \rfloor\quad.$$

Then, $\lfloor\mathsf{id}_x\rfloor_x\in\mathsf{G}.x\leftarrow z$, i.e. $(\lfloor\mathsf{id}_x\rfloor_x,x)\in z{\downarrow}\mathsf{G}$. We conjecture that $(\lfloor\mathsf{id}_x\rfloor_x,x)$ is initial in $z{\downarrow}\mathsf{G}$. We construct a *unique* arrow α in $z{\downarrow}\mathsf{G}$ to an object (g,y) from $(\lfloor\mathsf{id}_x\rfloor_x,x)$ as follows:

$$\begin{array}{ll}
 & \alpha\in(g,y)\overset{z\downarrow\mathsf{G}}{\longleftarrow}(\lfloor\mathsf{id}_x\rfloor_x,x)\\
\equiv & \{\quad\text{definition arrow in } z{\downarrow}\mathsf{G}\quad\}\\
 & \alpha\in y\leftarrow x\;\wedge\;g=\mathsf{G}.\alpha\circ\lfloor\mathsf{id}_x\rfloor_x\\
\equiv & \{\quad\text{computation rule: } (\mathsf{G}.h\circ)\circ\lfloor\ \rfloor=\lfloor\ \rfloor\circ(h\circ)\quad\}\\
 & \alpha\in y\leftarrow x\wedge g=\lfloor\alpha\rfloor_y\\
\equiv & \{\quad\lfloor\ \rfloor\text{ and }\lceil\ \rceil\text{ inverses}\quad\}\\
 & \alpha=\lceil g\rceil_y\quad.
\end{array}$$

For the $\Rightarrow$ part assume (f,x) to be initial in $z{\downarrow}G$. We have to define two mappings $\lceil\ \rceil$ and $\lfloor\ \rfloor$ such that

$$\lceil\ \rceil \in [a :: a \leftarrow x] \cong [a :: G.a \leftarrow z] \ni \lfloor\ \rfloor \quad .$$

We do this pointwise and then prove that they are natural in a and each others' inverses.

$$\begin{array}{ll}
 & g \in G.a \leftarrow z \\
\Rightarrow & \{ \quad (g,a) \in (z{\downarrow}G)\ ,\ (f,x) \text{ initial in } z{\downarrow}G \quad \} \\
 & (\!|(g,a) =:\ (f,x)|\!) \in (g,a) \overset{z\downarrow G}{\longleftarrow} (f,x) \\
\equiv & \{ \quad \text{definition arrow in } z{\downarrow}G \quad \} \\
 & (\!|(g,a) =:\ (f,x)|\!) \in a \leftarrow x \ \wedge\ g = G.(\!|(g,a) =:\ (f,x)|\!) \circ f \quad .
\end{array}$$

Thus, we define $\lceil g \rceil_a = (\!|(g,a) =:\ (f,x)|\!)$ and by definition $g = G.\lceil g \rceil_a \circ f$.

$$\begin{array}{ll}
 & \varphi \in a \leftarrow x \\
\Rightarrow & \{ \quad G \text{ functor, } f \in G.x \leftarrow z \quad \} \\
 & G.\varphi \circ f \in G.a \leftarrow z \quad .
\end{array}$$

Thus, we define $\lfloor \varphi \rfloor_a = G.\varphi \circ f$. Now we prove that $\lfloor\ \rfloor_a$ is natural in a. The naturality of $\lceil\ \rceil_a$ in a then follows automatically. Suppose $h \in a \leftarrow b$ and $\varphi \in b \leftarrow x$.

$$\begin{array}{ll}
 & ((G.h \circ) \circ \lfloor\ \rfloor_b).\varphi = (\lfloor\ \rfloor_a \circ (h \circ)).\varphi \\
\equiv & \{ \quad \text{application} \quad \} \\
 & G.h \circ \lfloor \varphi \rfloor_b = \lfloor h \circ \varphi \rfloor_a \\
\equiv & \{ \quad \lfloor \varphi \rfloor_a = G.\varphi \circ f \quad \} \\
 & G.h \circ G.\varphi \circ f = G.(h \circ \varphi) \circ f \\
\equiv & \{ \quad G \text{ is a functor} \quad \} \\
 & \mathit{true} \quad .
\end{array}$$

Finally the proof that $\lceil\ \rceil_a$ and $\lfloor\ \rfloor_a$ are inverse functions. Suppose $g \in G.a \leftarrow z$ and $\varphi \in a \leftarrow x$

$$\begin{array}{ll}
 & \lceil g \rceil_a = \varphi \\
\equiv & \{ \quad \text{definition } \lceil g \rceil_a \text{, unique arrow} \quad \} \\
 & \varphi \in (g,a) \overset{(z\downarrow G)}{\longleftarrow} (f,x) \\
\equiv & \{ \quad \varphi \in a \leftarrow x \quad \} \\
 & g = G.\varphi \circ f \\
\equiv & \{ \quad \text{defintion } \lfloor\ \rfloor_a \quad \} \\
 & g = \lfloor \varphi \rfloor_a \quad .
\end{array}$$

This completes our proof.
□

As an immediate consequence of this lemma we have the following examples.

Example II (yellowbook 5.32 (implication))
Suppose that (F,G) forms an adjunction. Thus by the above lemma $F.z$ is the carrier of an initial object in $z{\downarrow}G$. Moreover, if we take a look at the above proof we see that in the case that (F,G) forms an adjunction that $\lfloor \mathsf{id}_x \rfloor_x$ corresponds to the unit of the adjunction applied to x. Thus, if (F,G) forms an adjunction, then $(\mathsf{unit}_z\,,\,F.z)$ is initial in $z{\downarrow}G$.
□

Example III
A colimit $x{\in}\mathcal{C}$ of the functor $F \in \mathcal{C}{\leftarrow}\mathcal{D}$ satisfies

$$[a :: a{\leftarrow}x] \cong [a :: K.a \leftarrow F] \ .$$

Thus by the above lemma a colimit of F is the carrier of an initial object in the category $F{\downarrow}K$ where $K \in (\mathcal{C}{\leftarrow}\mathcal{D}){\leftarrow}\mathcal{C}$.
□

Theorem 4.59 The functor $G \in \mathcal{D}{\leftarrow}\mathcal{C}$ has a lower adjoint iff for each object z of $\mathcal{D}$ the category $z{\downarrow}G$ has an initial object.

Proof Using the above lemma, that "for each object z of $\mathcal{D}$ the category $z{\downarrow}G$ has an initial object" equivales:

$$\forall(z{::}\,[a :: a{\leftarrow}x] \cong [a :: G.a \leftarrow z] \text{ has a solution in } x) \ .$$

We first prove the $\Rightarrow$ part; suppose $G^\flat$ is a lower adjoint of G, then we have

$$
\begin{array}{ll}
 & [a,b :: a \leftarrow G^\flat.b] \cong [a,b :: G.a \leftarrow b] \\
\Rightarrow & \{ \quad \text{fix second argument} \quad \} \\
 & \forall(z{::}\,[a :: a \leftarrow G^\flat.z] \cong [a :: G.a \leftarrow z]) \ .
\end{array}
$$

So, $G^\flat.z$ is the carrier of an initial object in $z{\downarrow}G$. For the $\Leftarrow$ part; suppose

(4.60) $\forall(z{::}\ \lceil\ \rceil_z \in [a :: a{\leftarrow}\delta_z] \cong [a :: G.a \leftarrow z] \ni \lfloor\ \rfloor_z)$

We now construct a lower adjoint $G^\flat$ for G. An obvious candidate is the functor which maps an object z to δ_z. To see that this indeed defines a lower adjoint for G we need to verify that the natural transformation $\lceil\ \rceil_z$ (or $\lfloor\ \rfloor_z$) in (4.60) is also natural in z. We first define $[a :: a{\leftarrow}\delta_f]$ for $f \in u{\leftarrow}v$:

$$
\begin{array}{ll}
 & [a :: a{\leftarrow}\delta_f] \in [a :: a{\leftarrow}\delta_v]{\leftarrow}[a :: a{\leftarrow}\delta_u] \\
\Leftarrow & \{ \quad \textit{take}\text{: } [a :: a{\leftarrow}\delta_f] = \lceil\ \rceil_v \circ \alpha \circ \lfloor\ \rfloor_u \quad \} \\
 & \alpha \in [a :: G.a \leftarrow v]{\leftarrow}[a :: G.a \leftarrow u] \\
\Leftarrow & \{ \quad f \in u{\leftarrow}v \quad \} \\
 & \alpha = K.(\circ f) \ .
\end{array}
$$

I.e. $[a :: a \leftarrow \delta_f] = \lceil\ \rceil_v \circ \mathsf{K}.(\circ f) \circ \lfloor\ \rfloor_u$. To verify the naturality of $\lceil\ \rceil_z$ in z we calculate as follows:

$$
\begin{array}{ll}
 & [a :: a \leftarrow \delta_f] \circ \lceil\ \rceil_u \\
= & \{ \quad \text{definition } [a :: a \leftarrow \delta_f] \quad \} \\
 & \lceil\ \rceil_v \circ \mathsf{K}.(\circ f) \circ \lfloor\ \rfloor_u \circ \lceil\ \rceil_u \\
= & \{ \quad \lfloor\ \rfloor_u \text{ and } \lceil\ \rceil_u \text{ are each others' inverses} \quad \} \\
 & \lceil\ \rceil_v \circ [a :: \mathsf{G}.a \leftarrow f] \quad .
\end{array}
$$

We leave it to the reader to define $\mathsf{G}^\flat$ on arrows. (Hint: $\mathsf{G}^\flat$ is contravariant).
□

Suppose $\mathsf{G} \in \mathcal{D} \leftarrow \mathcal{C}$ is a functor. Define the forgetful functor $\mathsf{U}_z \in \mathcal{C} \leftarrow (z {\downarrow} \mathsf{G})$ by: let $(g, x) \in (z {\downarrow} \mathsf{G})$, i.e. $g \in \mathsf{G}.x \leftarrow z$, then $\mathsf{U}_z.(g, x) = x$. Note: a limit of U_z corresponds in lattice theory to $\sqcap(x\colon \mathsf{G}.x \sqsupseteq z\colon x)$.

Theorem 4.61 If the functor $\mathsf{G} \in \mathcal{D} \leftarrow \mathcal{C}$ has a lower adjoint then G preserves all limits and $\mathsf{U}_z \in \mathcal{C} \leftarrow (z {\downarrow} \mathsf{G})$ has a limit for all $z \in \mathcal{C}$.

Proof The dual of theorem 4.53 gives us that upper adjoints preserves limits. Furthermore, from example II we know that if G is the upper adjoint of an adjunction then for every object z the category $z {\downarrow} \mathsf{G}$ has an initial object, so by theorem 4.56 the functor $\mathsf{U}_z \in \mathcal{C} \leftarrow (z {\downarrow} \mathsf{G})$ has a limit.
□

Chapter 5

Algebras

The notion in category theory that corresponds to the notion of a prefix point in lattice theory is known as an *algebra*.

5.1 Definition and Properties

For a monotonic endofunction F an element x such that $x \sqsupseteq \mathsf{F}.x$ is called a *prefix point* of F. In category theory this corresponds to an object x and a (not necessarily unique) arrow f such that $f \in x \leftarrow \mathsf{F}.x$, where F is a functor. However, given the arrow f we can always deduce its corresponding object, namely $\mathsf{cod}.f$. This leads to the following definition.

Definition 5.1 (Algebra) Let $\mathcal{C}$ be a category and let F be an endofunctor on $\mathcal{C}$. An F-*algebra* is an arrow f of $\mathcal{C}$ such that $\mathsf{dom}.f = \mathsf{F}.(\mathsf{cod}.f)$. I.e. $f \in x \leftarrow \mathsf{F}.x$, where $x = \mathsf{cod}.f$.

□

If f is an F-algebra it is often the case that its codomain is referred to as the *carrier*.

If in lattice theory a prefix point x of F contains a prefix point y of F, then we have the following four containments. First, x and y are prefix points of F. Thus we have $x \sqsupseteq \mathsf{F}.x$ and $y \sqsupseteq \mathsf{F}.y$. Second, x contains y. Thus we have $x \sqsupseteq y$. Finally, from the monotonicity of F it follows that we also have the containment $\mathsf{F}.x \sqsupseteq \mathsf{F}.y$. As a result, the containment $x \sqsupseteq \mathsf{F}.y$ can be constructed in two different ways.

In category theory containment of prefix points is captured by the construction of arrows between algebras. Thus, F-algebras are arrows between certain objects in the underlying category, the so-called *base category*. But, we can also view F-algebras as objects and construct arrows between F-algebras. In other words, we can construct a category of F-algebras.

Definition 5.2 (Category $\mathsf{Alg}_{\mathcal{C}}.\mathsf{F}$) Let $\mathcal{C}$ be a category and let F be an endofunctor on $\mathcal{C}$. The category $\mathsf{Alg}_{\mathcal{C}}.\mathsf{F}$ is defined as follows. The objects of $\mathsf{Alg}_{\mathcal{C}}.\mathsf{F}$ are the F-algebras.

Its arrows are constructed from the arrows of the base category $\mathcal{C}$. Specifically, suppose f and g are F-algebras. Then the arrows φ to f from g are characterised by the equation:

$$\varphi \in \mathsf{f} \xleftarrow{\mathsf{Alg}_{\mathcal{C}}.\mathsf{F}} \mathsf{g} \;\equiv\; \varphi \in \mathsf{cod.f} \xleftarrow{\mathcal{C}} \mathsf{cod.g} \;\wedge\; \mathsf{f} \circ \mathsf{F}.\varphi = \varphi \circ \mathsf{g} \quad .$$

□

Note, that the coherence condition $\mathsf{f} \circ \mathsf{F}.\varphi = \varphi \circ \mathsf{g}$ follows from the fact that the arrow to $\mathsf{cod.f}$ from $\mathsf{F}.(\mathsf{cod.g})$ can be constructed in two different ways, which corresponds in lattice theory to two different proofs of $x \sqsupseteq \mathsf{F}.y$ given that $x \sqsupseteq y$ as discussed above.

The decoration '$\mathcal{C}$' in $\mathsf{Alg}_{\mathcal{C}}.\mathsf{F}$ is omitted when it can be deduced from the context. The claim that $\mathsf{Alg.F}$ is a category should be proven, of course. It is a straightforward exercise, however, and the fact is well-known. One "trivial" element of the proof that we make abundant use of is the fact that the identity arrows in $\mathsf{Alg.F}$ are the identity arrows in the base category. More precisely, if f is an F-algebra then

$$\mathsf{id}_{\mathsf{f}}^{\mathsf{Alg.F}} \;=\; \mathsf{id}_{\mathsf{cod.f}}^{\mathcal{C}} \quad ,$$

i.e. $\mathsf{id}_{\mathsf{cod.f}}^{\mathcal{C}} \in \mathsf{f} \xleftarrow{\mathsf{Alg.F}} \mathsf{f}$. The superscripts '$\mathsf{Alg.F}$' and '$\mathcal{C}$' are usually omitted. As a result we have

Lemma 5.3 If two algebras are isomorphic in the algebra category, then their carriers are isomorphic in the underlying base category. Moreover, the witnesses to the isomorphisms are the same.

Proof Let $\mathcal{C}$ be category and let F be an endofunctor on $\mathcal{C}$. Suppose f and g are two isomorphic F-algebras. So, we have two arrows

$$\varphi \in \mathsf{f} \xleftarrow{\mathsf{Alg.F}} \mathsf{g} \quad \text{and} \quad \phi \in \mathsf{g} \xleftarrow{\mathsf{Alg.F}} \mathsf{f} \quad ,$$

which are each others' inverses, i.e. $\varphi \circ \phi = \mathsf{id}_{\mathsf{f}}^{\mathsf{Alg.F}}$ and $\phi \circ \varphi = \mathsf{id}_{\mathsf{g}}^{\mathsf{Alg.F}}$. Arrows in an algebra category are also arrows in the base category. More specifically,

$$\varphi \in \mathsf{cod.f} \xleftarrow{\mathcal{C}} \mathsf{cod.g} \quad \text{and} \quad \phi \in \mathsf{cod.g} \xleftarrow{\mathcal{C}} \mathsf{cod.f} \quad .$$

Furthermore, the identity arrows in the algebra category are the identity arrows in the base category, so we also have $\varphi \circ \phi = \mathsf{id}_{\mathsf{cod.f}}^{\mathcal{C}}$ and $\phi \circ \varphi = \mathsf{id}_{\mathsf{cod.g}}^{\mathcal{C}}$. Thus, we conclude that φ and ϕ witness the isomorphism between $\mathsf{cod.f}$ and $\mathsf{cod.g}$ in the category $\mathcal{C}$.
□

We sometimes wish to emphasise that we have an isomorphism between two algebras, and what the relevant functor is. To this end, we introduce the infix operator $\cong_{\mathsf{F}}$ to denote isomorphism in the category $\mathsf{Alg.F}$. That is, we write $\mathsf{f} \cong_{\mathsf{F}} \mathsf{g}$ if f and g are isomorphic F-algebras.

We leave the proof of the following lemma as a (simple) exercise in the use of the notation.

Lemma 5.4

$$\sigma \in \alpha \cong_{\mathsf{F}} \beta \wedge \tau \in \mathsf{F} \cong \mathsf{G} \;\Rightarrow\; \sigma \in \alpha \circ \tau_{\mathsf{cod}.\alpha} \cong_{\mathsf{G}} \beta \circ \tau_{\mathsf{cod}.\beta} \quad .$$

□

Given the partially ordered sets $\mathcal{C} = (\mathsf{C}, \sqsupseteq)$ and $\mathcal{D} = (\mathsf{D}, \succeq)$ and the monotonic functions $\mathsf{F} \in \mathcal{C} \leftarrow \mathcal{D}$, $\mathsf{G} \in \mathcal{D} \leftarrow \mathcal{D}$ and $\mathsf{H} \in \mathcal{C} \leftarrow \mathcal{C}$ in the lattice theory, it is easily shown that F maps prefix points of G to prefix points of H provided that $\mathsf{F} \bullet \mathsf{G} \mathrel{\dot{\sqsupseteq}} \mathsf{H} \bullet \mathsf{F}$.

$$
\begin{array}{ll}
 & x \succeq \mathsf{G}.x \\
\Rightarrow & \{ \quad \mathsf{F} \text{ monotonic} \quad \} \\
 & \mathsf{F}.x \sqsupseteq \mathsf{F}.(\mathsf{G}.x) \\
\Rightarrow & \{ \quad \mathsf{F} \bullet \mathsf{G} \mathrel{\dot{\sqsupseteq}} \mathsf{H} \bullet \mathsf{F} \text{, transitivity} \quad \} \\
 & \mathsf{F}.x \sqsupseteq \mathsf{H}.(\mathsf{F}.x) \quad .
\end{array}
$$

Denote the set of prefix points of G by $\mathsf{Pre.G}$. Then formally, we have:

(5.5) $\quad \forall(x\colon x \in \mathsf{Pre.G}\colon \mathsf{F}.x \in \mathsf{Pre.H}) \Leftarrow \mathsf{F} \bullet \mathsf{G} \mathrel{\dot{\sqsupseteq}} \mathsf{H} \bullet \mathsf{F} \quad .$

In category theory this corresponds to the existence of a functor to $\mathsf{Alg.H}$ from $\mathsf{Alg.G}$ under similar conditions.

Lemma 5.6 Given are three functors $\mathsf{F} \in \mathcal{C} \leftarrow \mathcal{D}$, $\mathsf{G} \in \mathcal{D} \leftarrow \mathcal{D}$ and $\mathsf{H} \in \mathcal{C} \leftarrow \mathcal{C}$. Suppose that there is a natural transformation $\eta \in \mathsf{F} \bullet \mathsf{G} \leftarrow \mathsf{H} \bullet \mathsf{F}$. Then we can define a functor K to $\mathsf{Alg.H}$ from $\mathsf{Alg.G}$ in the following way:

(5.7) $\quad \mathsf{K}.g = \mathsf{F}.g \circ \eta_{\mathsf{cod}.g}$, for every object $g \in \mathsf{Alg.G}$.

(5.8) $\quad \mathsf{K}.\varphi = \mathsf{F}.\varphi$, for every arrow φ in $\mathsf{Alg.G}$.

Proof The first step is to prove that K maps G-algebras to H-algebras.

$$
\begin{array}{ll}
 & g \in x \leftarrow \mathsf{G}.x \\
\Rightarrow & \{ \quad \mathsf{F} \text{ is a functor} \quad \} \\
 & \mathsf{F}.g \in \mathsf{F}.x \leftarrow \mathsf{F}.(\mathsf{G}.x) \\
\Rightarrow & \{ \quad \eta \in \mathsf{F} \bullet \mathsf{G} \leftarrow \mathsf{H} \bullet \mathsf{F} \quad \} \\
 & \mathsf{F}.g \circ \eta_x \in \mathsf{F}.x \leftarrow \mathsf{H}.(\mathsf{F}.x) \quad .
\end{array}
$$

The next step is to prove that K maps arrows of $\mathsf{Alg.G}$ to arrows of $\mathsf{Alg.H}$. More precisely, given two G-algebras f and g and an arrow $\varphi \in f \xleftarrow{\mathsf{Alg.G}} g$, we have to construct a definition of $\mathsf{K}.\varphi$ such that $\mathsf{K}.\varphi \in \mathsf{K}.f \xleftarrow{\mathsf{Alg.H}} \mathsf{K}.g$. From the definition of K on objects and the fact that arrows in an algebra category are also arrows in the base category, we conjecture that $\mathsf{K}.\varphi = \mathsf{F}.\varphi$. Let $\mathsf{cod}.f = x$ and $\mathsf{cod}.g = y$. Then we verify this conjecture as follows:

$$\begin{array}{ll}
& \mathsf{F}.\varphi \in \mathsf{F}.\mathsf{f}\circ\eta_{\mathsf{x}} \xleftarrow{\mathsf{Alg.H}} \mathsf{F}.\mathsf{g}\circ\eta_{\mathsf{y}}. \\
\equiv & \{ \quad \text{definition of an arrow in } \mathsf{Alg.H}:\ (5.2) \quad \} \\
& \quad \mathsf{F}.\varphi \in \mathsf{cod}.(\mathsf{F}.\mathsf{f}\circ\eta_{\mathsf{x}}) \xleftarrow{\mathcal{C}} \mathsf{cod}.(\mathsf{F}.\mathsf{g}\circ\eta_{\mathsf{y}.}) \\
& \wedge\ \mathsf{F}.\mathsf{f}\circ\eta_{\mathsf{x}}\circ\mathsf{H}.\mathsf{F}.\varphi = \mathsf{F}.\varphi\circ\mathsf{F}.\mathsf{g}\circ\eta_{\mathsf{y}.} \\
\equiv & \{ \quad \text{domains ; } \eta \in \mathsf{F}\bullet\mathsf{G} \leftarrow \mathsf{H}\bullet\mathsf{F},\ \mathsf{F} \text{ is a functor} \quad \} \\
& \mathsf{F}.\varphi \in \mathsf{F}.\mathsf{x} \xleftarrow{\mathcal{C}} \mathsf{F}.\mathsf{y}\ \wedge\ \mathsf{F}.(\mathsf{f}\circ\mathsf{G}.\varphi)\circ\eta_{\mathsf{y}} = \mathsf{F}.(\varphi\circ\mathsf{g})\circ\eta_{\mathsf{y}.} \\
\Leftarrow & \{ \quad \mathsf{F} \text{ is a functor ; Leibniz} \quad \} \\
& \varphi \in \mathsf{x} \xleftarrow{\mathcal{D}} \mathsf{y}\ \wedge\ \mathsf{f}\circ\mathsf{G}.\varphi = \varphi\circ\mathsf{g} \\
\equiv & \{ \quad \text{definition of an arrow in } \mathsf{Alg.G}:\ (5.2) \quad \} \\
& \varphi \in \mathsf{f} \xleftarrow{\mathsf{Alg.G}} \mathsf{g}\ .
\end{array}$$

So,

$$\mathsf{F}.\varphi \in \mathsf{F}.\mathsf{f}\circ\eta_{\mathsf{x}} \xleftarrow{\mathsf{Alg.H}} \mathsf{F}.\mathsf{g}\circ\eta_{\mathsf{y}.} \Leftarrow \varphi \in \mathsf{f} \xleftarrow{\mathsf{Alg.G}} \mathsf{g}\ .$$

We still have to verify the two coherence requirements on functors. For K these are

(5.9) $\mathsf{K}.\mathsf{id}_{\mathsf{g}} = \mathsf{id}_{\mathsf{K}.\mathsf{g}}$, for every object $\mathsf{g}\in\mathsf{Alg.G}$ and

(5.10) $\mathsf{K}.(\varphi\circ\psi) = \mathsf{K}.\varphi\circ\mathsf{K}.\psi$, for every arrow φ and ψ in $\mathsf{Alg.G}$.

It is easy to verify (5.9) using the fact that the identity arrows in $\mathsf{Alg.G}$ are the identity arrows in the base category. (5.10) follows immediately from the definition of K on arrows (5.8) and the assumption that F is a functor.

□

Lemma 5.6 has two corollaries that prove useful later. Their proofs are easily supplied.

Corollary 5.11 With the same assumptions as in lemma 5.6 we have:

$$\mathsf{F}.\varphi\circ\eta_{\mathsf{cod.f}} \in \mathsf{F}.\mathsf{f} \cong_{\mathsf{H}} \mathsf{F}.\mathsf{g} \quad \Leftarrow \quad \varphi \in \mathsf{f} \cong_{\mathsf{G}} \mathsf{g}\ .$$

□

Corollary 5.12 Let $\mathsf{F}\in\mathcal{C}\leftarrow\mathcal{D}$ and $\mathsf{G}\in\mathcal{D}\leftarrow\mathcal{C}$ be functors. Then F is a functor to $\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G})$ from $\mathsf{Alg}.(\mathsf{G}\bullet\mathsf{F})$, i.e.

$$\mathsf{F}.\mathsf{f}\in\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G}) \Leftarrow \mathsf{f}\in\mathsf{Alg}.(\mathsf{G}\bullet\mathsf{F})\ ,$$

and

$$\mathsf{F}.\varphi \in \mathsf{F}.\mathsf{f} \xleftarrow{\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G})} \mathsf{F}.\mathsf{g} \quad \Leftarrow \quad \varphi \in \mathsf{f} \xleftarrow{\mathsf{Alg}.(\mathsf{G}\bullet\mathsf{F})} \mathsf{g}\ .$$

Specialising to isomorphisms, we have:

$$\mathsf{F}.\varphi \in \mathsf{F}.\mathsf{f} \cong_{\mathsf{F}\bullet\mathsf{G}} \mathsf{F}.\mathsf{g} \;\Leftarrow\; \varphi \in \mathsf{f} \cong_{\mathsf{G}\bullet\mathsf{F}} \mathsf{g} \quad .$$

□

A particular instance of corollary 5.12 is obtained by taking F to be an endofunctor and G to be the identity functor. We deduce that, if f is an F-algebra, then $\mathsf{F}.\mathsf{f}$ is also an F-algebra.

If we instantiate the function F in (5.5) to the identity function we get

(5.13) $\mathsf{Pre}.\mathsf{G} \subseteq \mathsf{Pre}.\mathsf{H} \Leftarrow \mathsf{G} \mathrel{\dot{\sqsupseteq}} \mathsf{H}$.

In words, the function that maps a function G to the set of prefix points of G is anti-monotonic. Note the reversal of the orderings. The set of prefix points of G, $\mathsf{Pre}.\mathsf{G}$, corresponds to the category $\mathsf{Alg}.\mathsf{G}$. So, in category theory (5.13) gives rise to the theorem that Alg is a contravariant functor.

Theorem 5.14 (Functor Alg) For each category $\mathcal{C}$ there is a contravariant functor $\mathsf{Alg}_{\mathcal{C}}$ from the category of endofunctors on $\mathcal{C}$ to the category of categories based on $\mathcal{C}$, i.e. $\mathsf{Alg}_{\mathcal{C}} \in \mathsf{Cat}_{\mathcal{C}} \leftarrow \mathsf{End}.\mathcal{C}$.

Proof The definition of $\mathsf{Alg}.\mathsf{F}$ (5.2) defines Alg (omitting the subscript) for each object $\mathsf{F} \in \mathsf{End}.\mathcal{C}$. On arrows of $\mathsf{End}.\mathcal{C}$, i.e. natural transformations, Alg has to satisfy

$$\mathsf{Alg}.\eta \in \mathsf{Alg}.\mathsf{H} \leftarrow \mathsf{Alg}.\mathsf{G} \quad \text{, whenever } \eta \in \mathsf{G} \leftarrow \mathsf{H}.$$

Note the reversal in the typing of the arrows and that $\mathsf{Alg}.\eta$ has to be an arrow in $\mathsf{Cat}_{\mathcal{C}}$, i.e. a functor. The definition of the functor $\mathsf{Alg}.\eta$ follows from lemma 5.6. Specifically,

(5.15) $(\mathsf{Alg}.\eta).\mathsf{g} = \mathsf{g} \circ \eta_{\mathsf{cod}.\mathsf{g}}$,

for each object $\mathsf{g} \in \mathsf{Alg}.\mathsf{G}$, and

(5.16) $(\mathsf{Alg}.\eta).\varphi = \varphi$,

for each arrow φ in $\mathsf{Alg}.\mathsf{G}$. It remains to check the coherence requirements to prove that Alg is indeed functorial. But this is trivial.

□

A trivial lemma that nevertheless proves its worth is the following:

Lemma 5.17 If f is an F-algebra then $\mathsf{f} \in \mathsf{f} \xleftarrow{\mathsf{Alg}.\mathsf{F}} \mathsf{F}.\mathsf{f}$.

Proof

$$\begin{array}{ll}
 & \mathsf{f} \in \mathsf{f} \xleftarrow{\mathsf{Alg}.\mathsf{F}} \mathsf{F}.\mathsf{f} \\
\equiv & \qquad \{ \quad \text{definition arrow in } \mathsf{Alg}.\mathsf{F}\text{: (5.2)} \quad \} \\
 & \mathsf{f} \in \mathsf{cod}.\mathsf{f} \xleftarrow{\mathcal{C}} \mathsf{F}.(\mathsf{cod}.\mathsf{f}) \;\wedge\; \mathsf{f} \circ \mathsf{F}.\mathsf{f} = \mathsf{f} \circ \mathsf{F}.\mathsf{f} \\
\equiv & \qquad \{ \quad \text{definition 5.1} \quad \} \\
 & \mathsf{f} \text{ is an } \mathsf{F}\text{-algebra.}
\end{array}$$

$\square$

(Although trivial, lemma 5.17 can be confusing. It says that every object of $\mathsf{Alg.F}$ is an arrow of $\mathsf{Alg.F}$. The confusion may occur when we consider a functor on $\mathsf{Alg.F}$; when applying such a functor one must always be very clear about whether it is being applied to an object or an arrow.)

5.2 Initial Algebras

In lattice theory we have the notion of a least prefix point of a monotonic endofunction f. 'Translating' all the concepts to the corresponding concepts in category theory, we end up with the notion of an *initial algebra of an endofunctor* F. In lattice theory we denote the least prefix point of a function f by $\mu\mathsf{f}$. In category theory we will usually denote an initial algebra of an endofunctor F by muF. Its codomain, i.e. its carrier, will be denoted by $\mu\mathsf{F}$. So, $\mathsf{muF} \in \mu\mathsf{F} \leftarrow \mathsf{F}.\mu\mathsf{F}$.

For initial algebras we have an equivalence obtained by instantiating (2.3) appropriately. Specifically, let F be an endofunctor on $\mathcal{C}$. Suppose f is an F-algebra and muF is an initial F-algebra. Then

(5.18) $\varphi \in \mathsf{f} \xleftarrow{\mathsf{Alg.F}} \mathsf{muF} \;\equiv\; \varphi = (\!|\mathsf{Alg.F}\,;\mathsf{f} =\colon \mathsf{muF}|\!)$.

As announced at the time the notation was introduced, we will most often omit the parameter $\mathsf{Alg.F}$ in expressions of the form $(\!|\mathsf{Alg.F};\mathsf{f} =\colon \mathsf{g}|\!)$. (Very often g is muF and so it is easy to see that $\mathsf{Alg.F}$ is the intended category.) Where it is necessary to avoid confusion we will write $(\!|\mathsf{F};\mathsf{f} =\colon \mathsf{muF}|\!)$.

We state some of the consequences of this equivalence in the following lemma.

Lemma 5.19 Let F be an endofunctor on $\mathcal{C}$. Suppose f is an F-algebra and muF is an initial F-algebra. Then:

(a) $(\!|\mathsf{f} =\colon \mathsf{muF}|\!) \in \mathsf{f} \xleftarrow{\mathsf{Alg.F}} \mathsf{muF}$.

(b) $\varphi \in \mathsf{cod.f} \xleftarrow{\mathcal{C}} \mu\mathsf{F} \wedge \mathsf{f}\circ\mathsf{F}.\varphi = \varphi\circ\mathsf{muF} \;\equiv\; \varphi = (\!|\mathsf{f} =\colon \mathsf{muF}|\!)$.

(c) $\mathsf{f}\circ\mathsf{F}.(\!|\mathsf{f} =\colon \mathsf{muF}|\!) = (\!|\mathsf{f} =\colon \mathsf{muF}|\!)\circ\mathsf{muF}$.

(d) $\mathsf{id}_{\mathsf{muF}} = \mathsf{id}_{\mu\mathsf{F}} = (\!|\mathsf{muF} =\colon \mathsf{muF}|\!)$.

(e) $\mathsf{f}\circ(\!|\mathsf{g} =\colon \mathsf{muF}|\!) = (\!|\mathsf{h} =\colon \mathsf{muF}|\!) \Leftarrow \mathsf{f}\in\mathsf{h} \xleftarrow{\mathsf{Alg.F}} \mathsf{g}$.

If the typing of f in the base category is already known when using property 5.19e then the right-hand side of the 'follows-from' can be replaced by $\mathsf{f}\circ\mathsf{g} = \mathsf{h}\circ\mathsf{F.f}$.

We will refer to these properties by using the terms "typing", "uep", "computation rule", "identity" and "(catamorphism) fusion" respectively. Here, "uep" is short for "unique extension property".

Proof By instantiating $\varphi := (\!|\mathsf{f} =: \mathsf{muF}|\!)$ in (5.18) we obtain property 5.19a.

Property 5.19b follows from 5.18 and writing out the definition of an arrow in an algebra category.

Property 5.19c follows immediately from 5.19b by instantiating φ such that the right-hand side becomes true.

For property 5.19d we argue as follows. By instantiating $\varphi,\mathsf{f} := \mathsf{id}_{\mathsf{muF}},\mathsf{muF}$ in (5.18) we obtain $\mathsf{id}_{\mathsf{muF}} = (\!|\mathsf{muF} =: \mathsf{muF}|\!)$. The equality $\mathsf{id}_{\mathsf{muF}} = \mathsf{id}_{\mu\mathsf{F}}$ follows from the remark made earlier that the identity arrows in an algebra category are the identity arrows in the base category.

Property 5.19e is proven as follows

$$\begin{array}{ll}
 & \mathsf{f} \circ (\!|\mathsf{g} =: \mathsf{muF}|\!) = (\!|\mathsf{h} =: \mathsf{muF}|\!) \\
\equiv & \qquad \{ \qquad (5.18) \quad \} \\
 & \mathsf{f} \circ (\!|\mathsf{g} =: \mathsf{muF}|\!) \in \mathsf{h} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{muF} \\
\Leftarrow & \qquad \{ \qquad (5.19a) \quad \} \\
 & \mathsf{f} \in \mathsf{h} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{g} \quad .
\end{array}$$

□

Theorem 2.5 is about initial objects in general. For initial algebras we can derive the following.

Lemma 5.20 Let F and G be endofunctors on $\mathcal{C}$. Suppose muF is an initial F-algebra, f is an arbitrary F-algebra and muG is an initial G-algebra. Finally, suppose we have a natural transformation $\eta \in \mathsf{F} \leftarrow \mathsf{G}$, then:

$$(\!|\mathsf{F}\,;\, \mathsf{f} =: \mathsf{muF}|\!) \circ (\!|\mathsf{G}\,;\, \mathsf{muF} \circ \eta_{\mu\mathsf{F}} =: \mathsf{muG}|\!) = (\!|\mathsf{G}\,;\, \mathsf{f} \circ \eta_{\mathsf{cod.f}} =: \mathsf{muG}|\!) \quad .$$

Proof

$$\begin{array}{ll}
 & (\!|\mathsf{f} =: \mathsf{muF}|\!) \circ (\!|\mathsf{muF} \circ \eta_{\mu\mathsf{F}} =: \mathsf{muG}|\!) = (\!|\mathsf{f} \circ \eta_{\mathsf{cod.f}} =: \mathsf{muG}|\!) \\
\equiv & \qquad \{ \qquad (5.19e) \quad \} \\
 & (\!|\mathsf{f} =: \mathsf{muF}|\!) \in \mathsf{f} \circ \eta_{\mathsf{cod.f}} \overset{\mathsf{Alg.G}}{\longleftarrow} \mathsf{muF} \circ \eta_{\mu\mathsf{F}} \\
\equiv & \qquad \{ \qquad \eta \in \mathsf{F} \leftarrow \mathsf{G}\,,\ (5.6) \quad \} \\
 & (\!|\mathsf{f} =: \mathsf{muF}|\!) \in \mathsf{f} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{muF} \\
\equiv & \qquad \{ \qquad (5.19a) \quad \} \\
 & \mathit{true} \quad .
\end{array}$$

□

In lattice theory, a least prefix point is a fixed point. In category theory, an initial F-algebra is also a fixed point, in the following sense:

Theorem 5.21 If muF is an initial F-algebra then $\mathsf{F.muF}$ is also an F-algebra and isomorphic to muF. The witnesses are

$$\mathsf{muF} \in \mathsf{muF} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{F.muF}$$

and

$$(\!|\mathsf{F.muF} =: \mathsf{muF}|\!) \in \mathsf{F.muF} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{muF} \quad .$$

Proof Assume that muF is an initial F-algebra. By corollary 5.12, $\mathsf{F.muF}$ is an F-algebra. By lemma 5.17,

$$\mathsf{muF} \in \mathsf{muF} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{F.muF} \quad .$$

Moreover, by the initiality of muF,

$$(\!|\mathsf{F.muF} =: \mathsf{muF}|\!) \in \mathsf{F.muF} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{muF} \quad .$$

It thus suffices to prove that

(5.22) $\mathsf{muF} \circ (\!|\mathsf{F.muF} =: \mathsf{muF}|\!) = \mathsf{id}_{\mathsf{muF}}$ and

(5.23) $(\!|\mathsf{F.muF} =: \mathsf{muF}|\!) \circ \mathsf{muF} = \mathsf{id}_{\mathsf{F.muF}}$.

We have,

$$
\begin{array}{ll}
 & \mathsf{muF} \circ (\!|\mathsf{F.muF} =: \mathsf{muF}|\!) = \mathsf{id}_{\mathsf{muF}} \\
\equiv & \qquad \{ \quad \text{identity: (5.19d)} \quad \} \\
 & \mathsf{muF} \circ (\!|\mathsf{F.muF} =: \mathsf{muF}|\!) = (\!|\mathsf{muF} =: \mathsf{muF}|\!) \\
\Leftarrow & \qquad \{ \quad \text{fusion: (5.19e)} \quad \} \\
 & \mathsf{muF} \in \mathsf{muF} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{F.muF} \\
\equiv & \qquad \{ \quad (5.17) \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & (\!|\mathsf{F.muF} =: \mathsf{muF}|\!) \circ \mathsf{muF} \\
= & \qquad \{ \quad \text{computation rule: (5.19c)} \quad \} \\
 & \mathsf{F.muF} \circ \mathsf{F.}(\!|\mathsf{F.muF} =: \mathsf{muF}|\!) \\
= & \qquad \{ \quad \mathsf{F} \text{ is a functor} \quad \} \\
 & \mathsf{F.}(\mathsf{muF} \circ (\!|\mathsf{F.muF} =: \mathsf{muF}|\!)) \\
= & \qquad \{ \quad (5.22) \quad \} \\
 & \mathsf{F.id}_{\mathsf{muF}} \\
= & \qquad \{ \quad \mathsf{F} \text{ is a functor} \quad \} \\
 & \mathsf{id}_{\mathsf{F.muF}} \quad .
\end{array}
$$

This completes the proof.
□

This theorem is due to Lambek [13] and is often referred to as *Lambek's lemma*.

5.3 The Initial Algebra Functor

In lattice theory the function μ is monotonic in the sense that $\mu f \sqsupseteq \mu g \Leftarrow f \mathrel{\dot{\sqsupseteq}} g$. In category theory μ is constructively monotonic, i.e. functorial. In this section we formulate and prove this theorem.

A complication in our formulation is that we do not wish to assume that all endofunctors have initial algebras. (It is very well-known, for example, that the power set functor cannot have an initial algebra.) Instead, the assumption we make is that all endofunctors of a certain "shape" have initial algebras. The assumed "shape" will be specified in more detail later; for the moment it has no relevance to our discussion.

Following the standard technique in category theory, we postulate a shape category, $\mathcal{D}$ say, and a (covariant) functor S to the category $\mathsf{End}.\mathcal{C}$ from $\mathcal{D}$. We further postulate that, for each $x \in \mathcal{D}$, the functor $\mathsf{S}.x$ has a canonical initial algebra $\mathsf{mu}(\mathsf{S}.x)$ with carrier $\mu(\mathsf{S}.x)$.

Noting that $\mathsf{mu}(\mathsf{S}.x)$ is an initial object in the category $\mathsf{Alg}.(\mathsf{S}.x)$, which equals $(\mathsf{Alg}\bullet\mathsf{S}).x$, and the functor $\mathsf{Alg}\bullet\mathsf{S}$ is contravariant we see that we can immediately apply lemma 3.1. We obtain: there is a functor $\mathsf{Mu} \in \Sigma(\mathsf{Alg}\bullet\mathsf{S}) \leftarrow \mathcal{D}$ defined by

$$\mathsf{Mu}.x \;=\; (x\,,\,\mathsf{mu}(\mathsf{S}.x)) \quad ,$$

for all $x \in \mathcal{D}$, and

$$\mathsf{Mu}.f \;=\; (f, (\!|\mathsf{Alg}.(\mathsf{S}.y)\,;\; (\mathsf{Alg}.(\mathsf{S}.f)).\mathsf{mu}(\mathsf{S}.x) =\!:\; \mathsf{mu}(\mathsf{S}.y)|\!)) \quad ,$$

for all $f \in x \leftarrow y$. Expanding the definition of the functor Alg on arrows (see theorem 5.14), we obtain:

Lemma 5.24 Suppose $\mathsf{S} \in \mathsf{End}.\mathcal{C} \leftarrow \mathcal{D}$ is a (covariant) functor such that, for each $x \in \mathcal{D}$, the functor $\mathsf{S}.x$ has a canonical initial algebra $\mathsf{mu}(\mathsf{S}.x)$ with carrier $\mu(\mathsf{S}.x)$. Then there is a functor $\mathsf{Mu} \in \Sigma(\mathsf{Alg}\bullet\mathsf{S}) \leftarrow \mathcal{D}$ defined by

$$\mathsf{Mu}.x \;=\; (x\,,\,\mathsf{mu}(\mathsf{S}.x)) \quad ,$$

for all $x \in \mathcal{D}$, and

$$\mathsf{Mu}.f \;=\; (f, (\!|\mathsf{Alg}.(\mathsf{S}.y)\,;\; \mathsf{mu}(\mathsf{S}.x) \circ (\mathsf{S}.f)_{\mu(\mathsf{S}.x)} =\!:\; \mathsf{mu}(\mathsf{S}.y)|\!)) \quad ,$$

for all $f \in x \overset{\mathcal{D}}{\longleftarrow} y$.

□

Remark: The name Mu is purely local to the current construction and will only once be used outside this section. This explains why we do not make explicit the fact that it is parameterised by S.

Now what we need is some sort of *carrier* functor that maps the pair $(\mathsf{F}, \mathsf{muF})$ to $\mu\mathsf{F}(= \mathsf{cod.muF})$.

Lemma 5.25 $\mathsf{car} \in \mathcal{C} \leftarrow \Sigma(\mathsf{Alg} \bullet \mathsf{S})$ is the functor defined in the following way. On objects car is defined by:

(5.26) $\mathsf{car}.(x, f) = \mathsf{cod}.f$,

and on arrows car is defined by:

(5.27) $\mathsf{car}.(h, \varphi) = \varphi$.

Proof

$$
\begin{array}{ll}
 & (h, \varphi) \in (x, f) \xleftarrow{\Sigma(\mathsf{Alg}\bullet\mathsf{S})} (y, g) \\
\Rightarrow & \{ \quad (2.22) \quad \} \\
 & \varphi \in ((\mathsf{Alg}\bullet\mathsf{S}).h).f \xleftarrow{(\mathsf{Alg}\bullet\mathsf{S}).y} g \\
\equiv & \{ \quad \text{definition } \mathsf{Alg}\text{: } (5.15) \quad \} \\
 & \varphi \in f \circ (\mathsf{S}.h)_{\mathsf{cod}.f} \xleftarrow{(\mathsf{Alg}\bullet\mathsf{S}).y} g \\
\Rightarrow & \{ \quad \text{definition arrow in } \mathsf{Alg.G}\text{, codomain calculus} \quad \} \\
 & \varphi \in \mathsf{cod}.f \xleftarrow{\mathcal{C}} \mathsf{cod}.g \\
\equiv & \{ \quad \text{definition } \mathsf{car} \text{ on objects} \quad \} \\
 & \varphi \in \mathsf{car}.(x, f) \xleftarrow{\mathcal{C}} \mathsf{car}.(y, g) \quad .
\end{array}
$$

So, for arrows we define $\mathsf{car}.(h, \varphi) = \varphi$. The coherence properties are straightforward.

□

Combining the two functors car and Mu we have a functor μ_{S}:

Theorem 5.28 (The Functor μ_{S}) Suppose $\mathsf{S} \in \mathsf{End}.\mathcal{C} \leftarrow \mathcal{D}$ is a (covariant) functor such that, for each $x \in \mathcal{D}$, the functor $\mathsf{S}.x$ has a canonical initial algebra $\mathsf{mu}(\mathsf{S}.x)$ with carrier $\mu(\mathsf{S}.x)$. Then there is a functor $\mu_{\mathsf{S}} \in \mathcal{C} \leftarrow \mathcal{D}$ defined by

$$\mu_{\mathsf{S}}.x \;=\; \mu(\mathsf{S}.x) \quad ,$$

for all $x \in \mathcal{D}$, and

$$\mu_{\mathsf{S}}.f \;=\; (\!| \mathsf{Alg}.(\mathsf{S}.y)\,;\; \mathsf{mu}(\mathsf{S}.x) \circ (\mathsf{S}.f)_{\mu(\mathsf{S}.x)} =: \mathsf{mu}(\mathsf{S}.y) |\!) \quad ,$$

for all $f \in x \xleftarrow{\mathcal{D}} y$.

□

5.4 Map Operator

We now come to one of the most important constructions in computing science, namely the construction of inductive types.

The functor List is an example of an inductive type. Informally, $\mathsf{List}.x$ is defined by considering the binary functor $\oplus$ where $x{\oplus}y = \mathbb{1}{+}(x{\times}y)$, fixing the argument x, constructing an initial $(y \mapsto \mathbb{1}{+}(x{\times}y))$-algebra, and then abstracting from x. In this section we make this process precise.

Assume that $\oplus \in \mathcal{D} \leftarrow \mathcal{C}{\times}\mathcal{D}$ is a binary functor. Suppose that for all objects x the endofunctor $x\oplus$ has a canonical initial algebra, $\mathsf{mu}(x\oplus)$, with carrier $\mu(x\oplus)$. We are going to prove that the so-called *map operator* $\varpi_\oplus$, which maps objects x to the object $\mu(x\oplus)$ is a functor. The theorem is clearly an instance of theorem 5.28 but in order to instantiate the latter formally we need two elementary lemmas. The proof of the first is left to the reader.

Lemma 5.29 $\oplus$ is functorial in both arguments. In other words, by fixing one of the arguments of a binary functor we obtain a unary functor. If x is an object of $\mathcal{C}$ then $x\oplus$ is the endofunctor on $\mathcal{D}$ obtained by fixing the first argument of $\oplus$ to x. Suppose y is an object of $\mathcal{D}$, then $x\oplus$ is defined by:

$$(x\oplus).y = x{\oplus}y \quad ,$$

and on arrows f it is defined by:

$$(x\oplus).f = \mathsf{id}_x{\oplus}f \quad .$$

Symmetrically, $\oplus y \in \mathcal{D}\leftarrow\mathcal{C}$ is the unary functor obtained by fixing the second argument of $\oplus$ to y.

$\Box$

Lemma 5.30 $\mathsf{app} \in \mathsf{End}.\mathcal{D} \leftarrow \mathcal{C}$ is a functor, where, by definition, on objects $x{\in}\mathcal{C}$,

(5.31) $\mathsf{app}.x = x\oplus \quad ,$

and, on arrows f in $\mathcal{C}$,

(5.32) $\mathsf{app}.f = f\oplus \quad ,$

where $f\oplus$ is the natural transformation in $\mathsf{End}.\mathcal{D}$ defined by $(f\oplus)_y = f{\oplus}\mathsf{id}_y$, for all $y{\in}\mathcal{D}$.

Proof In lemma 5.29 we stated that $x\oplus$ is an endofunctor on $\mathcal{D}$. Let $f \in x{\leftarrow}y$ be an arrow in $\mathcal{C}$, then by construction of α:

$$
\begin{array}{ll}
 & \alpha \in \mathsf{app}.x \leftarrow \mathsf{app}.y \\
\equiv & \{ \quad (5.31) \quad \} \\
 & \alpha \in x\oplus \leftarrow y\oplus \\
\equiv & \{ \quad x\oplus \text{ and } y\oplus \text{ are functors,} \\
 & \quad\ \text{definition of a natural transformation} \quad \} \\
 & \quad \forall(z\colon z\in\mathcal{D}\colon \alpha_z \in x\oplus z \leftarrow y\oplus z) \\
 & \wedge\ \forall(g\colon g\in u\leftarrow v\colon (x\oplus).g \circ \alpha_v = \alpha_u \circ (y\oplus).g) \\
\Leftarrow & \{ \quad \bullet \quad \alpha_z = f\oplus \mathsf{id}_z \ ,\ \text{lemma 5.29} \quad \} \\
 & \forall(g\colon g\in u\leftarrow v\colon \mathsf{id}_x\oplus g \circ f\oplus \mathsf{id}_v = f\oplus \mathsf{id}_u \circ \mathsf{id}_y\oplus g) \\
\equiv & \{ \quad \oplus \text{ is a binary functor: coherence requirement} \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

The coherence properties are straightforward.
□

Finally, we can give the main result

Theorem 5.33 Suppose $\oplus \in \mathcal{D}\leftarrow\mathcal{C}\times\mathcal{D}$ is a binary functor and for each object x in $\mathcal{C}$ the functor $x\oplus$ has an initial algebra denoted by $\mathsf{mu}(x\oplus)$. Then, the map operator $\varpi_{\oplus} \in \mathcal{D}\leftarrow\mathcal{C}$ is a functor defined in the following way:

$$(5.34) \quad \varpi_{\oplus}.x \ =\ \mu(x\oplus) \quad ,$$

and

$$(5.35) \quad \varpi_{\oplus}.f \ =\ (\!|y\oplus\,;\ \mathsf{mu}(x\oplus)\circ f\oplus \mathsf{id}_{\mu(x\oplus)} =\colon \mathsf{mu}(y\oplus)|\!) \quad ,$$

where $f\in x\leftarrow y$.

Proof Instantiate S in theorem 5.28 with the functor app .
□

From (5.35) we can immediately derive the following computation rule:

$$(5.36) \quad \varpi_{\oplus}.f \circ \mathsf{mu}(y\oplus) \ =\ \mathsf{mu}(x\oplus) \circ f \oplus \varpi_{\oplus}.f \quad ,$$

for all $f\in x\leftarrow y$. Let us now introduce the following functor.

Lemma 5.37 Let $F,G\in\mathcal{C}\leftarrow\mathcal{D}$ be functors. Then $F\dot{\oplus}G \in \mathcal{C}\leftarrow\mathcal{D}$ is a functor, where

$$(5.38) \quad (F\dot{\oplus}G).x = F.x \oplus G.x \quad ,$$

for objects x in $\mathcal{D}$ and

$$(5.39) \quad (F\dot{\oplus}G).f = F.f \oplus G.f. \quad ,$$

for arrows f in $\mathcal{D}$.

□

Then we can rewrite (5.36) as

$$\varpi_{\oplus}.\mathsf{f} \circ \mathsf{mu}(\mathsf{y}\oplus) = \mathsf{mu}(\mathsf{x}\oplus) \circ (\mathsf{Id} \dot{\oplus} \varpi_{\oplus}).\mathsf{f} \quad .$$

Similarly, using also that $\varpi_{\oplus}.\mathsf{x} = \mu(\mathsf{x}\oplus)$, we can give the typing of $\mathsf{mu}(\mathsf{x}\oplus)$ in the base category:

$$\mathsf{mu}(\mathsf{x}\oplus) \in \varpi_{\oplus}.\mathsf{x} \leftarrow (\mathsf{Id} \dot{\oplus} \varpi_{\oplus}).\mathsf{x} \quad .$$

In other words, we have a natural transformation

$$\varphi \in \varpi_{\oplus} \leftarrow \mathsf{Id} \dot{\oplus} \varpi_{\oplus} \tag{5.40}$$

defined on objects x by $\varphi_{\mathsf{x}} = \mathsf{mu}(\mathsf{x}\oplus)$.

Finally, using (5.20) we can derive a 'map-fusion rule': let $\mathsf{g} \in \mathsf{x} \leftarrow \mathsf{y}$ then

$$(\!|\mathsf{f} \;=:\; \mathsf{mu}(\mathsf{x}\oplus)|\!) \circ \varpi_{\oplus}.\mathsf{g} \;=\; (\!|\mathsf{f} \circ \mathsf{g} \oplus \mathsf{id}_{\mathsf{cod.f}} \;=:\; \mathsf{mu}(\mathsf{y}\oplus)|\!) \quad . \tag{5.41}$$

Chapter 6

Fixed Point Calculus

In this chapter we present four basic fixed point theorems, and two theorems that are each the result of combining two of the basic theorems. The basic fixed point theorems that we present are respectively: *the fusion rule*, *the abstraction theorem*, *the rolling rule* and *the diagonal rule*.

The fusion rule combines the concept of an initial algebra with the concept of an adjunction, and the abstraction theorem combines the concept of an initial algebra with the concept of parameterisation. Combining the fusion rule and the abstraction theorem enables us to prove in section 6.3 a "beautiful theorem" concerning the limit properties of fixed point constructions.

The rolling rule generalises the property (commonly known to category theoreticians as "Lambek's lemma" [13]) that initial F-algebras are fixed points of F. (Its namesake in lattice theory generalises the property that a least prefix point of monotonic function f is a fixed point of f.) The rolling rule is too elementary to be called "important" in its own right but it is extremely useful in combination with the other rules. As an illustration, we combine the rolling rule with the fusion rule in section 6.6 to prove another theorem, the *exchange rule*. The exchange rule is so called because it states when two lower adjoints may be exchanged in the construction of initial algebras.

The last basic fixed point rule, the diagonal rule, captures the basic principle of decomposing the construction of an initial algebra into the construction of a succession of such algebras.

Often, in lattice theory, the calculations concerning fixed points are performed within a complete lattice. The Knaster-Tarski theorem can then be used which states that for every monotonic function there is a least prefix point which coincides with its least fixed point. In this chapter a similar assumption about the categories we are working in will *not* be made. Each of the theorems has, instead, the form: "if functors F, G, ... have initial algebras α, β, ... then functor H has initial algebra γ defined as follows:".

In every section we first state the corresponding theorem in lattice theory. Although a category corresponds to a pre-ordered set we state the lattice-theoretic theorem within a complete partially ordered set, i.e. a set with a reflexive, transitive and anti-symmetric ordering. Presenting the theorem in a pre-ordered set introduces additional complications that we prefer to avoid.

6.1 The Fusion Rule

Given are two complete partially ordered sets $\mathcal{C} = (\mathsf{C}, \sqsupseteq)$ and $\mathcal{D} = (\mathsf{D}, \succeq)$. Suppose that $\mathsf{f} \in \mathcal{C} \leftarrow \mathcal{D}$, $\mathsf{g} \in \mathcal{D} \leftarrow \mathcal{D}$ and $\mathsf{h} \in \mathcal{C} \leftarrow \mathcal{C}$ are monotonic functions. We recall (5.5):

(6.1) $\forall(\mathsf{x}\colon \mathsf{x} \in \mathsf{Pre.g}\colon \mathsf{f.x} \in \mathsf{Pre.h}) \Leftarrow \mathsf{f}\bullet\mathsf{g} \mathrel{\dot{\sqsupseteq}} \mathsf{h}\bullet\mathsf{f}$.

In particular,

$$\mathsf{f}.\mu\mathsf{g} \in \mathsf{Pre.h} \Leftarrow \mathsf{f}\bullet\mathsf{g} \mathrel{\dot{\sqsupseteq}} \mathsf{h}\bullet\mathsf{f} \quad .$$

Furthermore, since $\mu\mathsf{h}$ is the least prefix point of h,

(6.2) $\mathsf{f}.\mu\mathsf{g} \sqsupseteq \mu\mathsf{h} \Leftarrow \mathsf{f}\bullet\mathsf{g} \mathrel{\dot{\sqsupseteq}} \mathsf{h}\bullet\mathsf{f}$.

This gives us the inspiration to look for a similar result but with as ordering equality.

Assume that f is a lower adjoint, i.e. f has an upper adjoint $\mathsf{f}^\sharp \in \mathcal{D} \leftarrow \mathcal{C}$. By instantiating $\mathsf{f,g,h} := \mathsf{f}^\sharp\mathsf{,h,g}$ in (6.1) we get

$$\forall(\mathsf{x}\colon \mathsf{x} \in \mathsf{Pre.h}\colon \mathsf{f}^\sharp.\mathsf{x} \in \mathsf{Pre.g}) \Leftarrow \mathsf{f}^\sharp\bullet\mathsf{h} \mathrel{\dot{\sqsupseteq}} \mathsf{g}\bullet\mathsf{f}^\sharp \quad .$$

But by the pseudo-inversality property (4.38):

$$\mathsf{h}\bullet\mathsf{f} \mathrel{\dot{\sqsupseteq}} \mathsf{f}\bullet\mathsf{g} \equiv \mathsf{f}^\sharp\bullet\mathsf{h} \mathrel{\dot{\sqsupseteq}} \mathsf{g}\bullet\mathsf{f}^\sharp \quad .$$

Hence, if $\mathsf{f}\bullet\mathsf{g} = \mathsf{h}\bullet\mathsf{f}$ then f and $\mathsf{f}^\sharp$ form a Galois connection between $\mathsf{Pre.h}$ and $\mathsf{Pre.g}$, since $\mathsf{Pre.h} \in \mathcal{C}$ and $\mathsf{Pre.g} \in \mathcal{D}$. Furthermore, since lower adjoints preserve least elements, we have $\mathsf{f}.\mu\mathsf{g} = \mu\mathsf{h}$. We have now proven the theorem we call the μ*-fusion rule*: if f is a lower adjoint then

$$\mathsf{f}.\mu\mathsf{g} = \mu\mathsf{h} \Leftarrow \mathsf{f}\bullet\mathsf{g} = \mathsf{h}\bullet\mathsf{f} \quad .$$

Our goal is to derive a similar result in category theory. We make the following assumptions: Given are two categories, $\mathcal{C}$ and $\mathcal{D}$, and two functors $\mathsf{G} \in \mathcal{D} \leftarrow \mathcal{D}$ and $\mathsf{H} \in \mathcal{C} \leftarrow \mathcal{C}$. Given also is an adjoint pair of functors $(\mathsf{F} \in \mathcal{C} \leftarrow \mathcal{D}\ ,\ \mathsf{F}^\sharp \in \mathcal{D} \leftarrow \mathcal{C})$ with unit and counit as unit and co-unit and $\lfloor\ \rfloor$ and $\lceil\ \rceil$ as left and right adjungate respectively. Finally, it is assumed that there is an isomorphism $\mathsf{swap} \in \mathsf{F}\bullet\mathsf{G} \cong \mathsf{H}\bullet\mathsf{F} \ni \mathsf{swap}^\cup$.

Recall from theorem 4.33 that the pair of functors $(\mathsf{F}\bullet\ ,\ (\mathsf{F}^\sharp)\bullet)$ forms an adjunction with adjungates $\llbracket\ \rrbracket$ and $\lceil\!\lceil\ \rceil\!\rceil$ defined by equation (4.36).

First, we have to construct an adjunction between the categories $\mathsf{Alg.H}$ and $\mathsf{Alg.G}$, i.e. an adjoint pair of functors $(\mathsf{K} \in \mathsf{Alg.H} \leftarrow \mathsf{Alg.G}\ ,\ \mathsf{K}^\sharp \in \mathsf{Alg.G} \leftarrow \mathsf{Alg.H})$. By theorem (4.39), there is a natural transformation $\mathsf{adjswap}$ such that:

(6.3) $\mathsf{adjswap} \in \mathsf{F}^\sharp\bullet\mathsf{H} \leftarrow \mathsf{G}\bullet\mathsf{F}^\sharp$.

More precisely,

(6.4) $\quad \mathsf{adjswap} = [\![\mathsf{F}^\sharp \bullet \ ;\ \mathsf{H}\bullet\mathsf{counit} \circ \mathsf{swap}^\cup \bullet \mathsf{F}^\sharp]\!]_{\mathsf{H,G}}$.

We're now going to use the given information to relate objects in the two categories $\mathsf{Alg.G}$ and $\mathsf{Alg.H}$ to each other. By lemma 5.6, K defined in the following way is a functor to $\mathsf{Alg.H}$ from $\mathsf{Alg.G}$.

(6.5) $\quad \mathsf{K}.g = \mathsf{F}.g \circ \mathsf{swap}_{\mathsf{cod}.g}$, where $g \in \mathsf{Alg.G}$,

(6.6) $\quad \mathsf{K}.\varphi = \mathsf{F}.\varphi$, where φ is an arrow in $\mathsf{Alg.G}$.

Lemma 5.6 can also be applied to construct a definition for $\mathsf{K}^\sharp$. The result of that construction is that $\mathsf{K}^\sharp$ defined in the following way is a functor to $\mathsf{Alg.G}$ from $\mathsf{Alg.H}$.

(6.7) $\quad \mathsf{K}^\sharp.h = \mathsf{F}^\sharp.h \circ \mathsf{adjswap}_{\mathsf{cod}.h}$, where $h \in \mathsf{Alg.H}$

(6.8) $\quad \mathsf{K}^\sharp.\psi = \mathsf{F}^\sharp.\psi$, where ψ is an arrow in $\mathsf{Alg.H}$.

Finally, we have to verify that $(\mathsf{K}, \mathsf{K}^\sharp)$, with K and $\mathsf{K}^\sharp$ defined in the previous way, is indeed an adjunction.

Similarity of K and $\mathsf{K}^\sharp$ with F and $\mathsf{F}^\sharp$ suggests that the adjunction $(\mathsf{K}, \mathsf{K}^\sharp)$ has left and right adjungates equal to $\lfloor\ \rfloor$ and $\lceil\ \rceil$ (with the domains limited to arrows of $\mathsf{Alg.H}$ and $\mathsf{Alg.G}$ respectively). First we verify that the types of $\lfloor\ \rfloor$ and $\lceil\ \rceil$ are correct, i.e. .

(6.9) $\quad \lfloor\psi\rfloor \in \mathsf{K}^\sharp.h \xleftarrow{\mathsf{Alg.G}} g \ \Leftarrow\ \psi \in h \xleftarrow{\mathsf{Alg.H}} \mathsf{K}.g$,

(6.10) $\quad \lceil\varphi\rceil \in h \xleftarrow{\mathsf{Alg.H}} \mathsf{K}.g \ \Leftarrow\ \varphi \in \mathsf{K}^\sharp.h \xleftarrow{\mathsf{Alg.G}} g$.

(6.9) and (6.10) are symmetric, so we only verify the first one. Suppose h is an H-algebra, g is a G-algebra, $\mathsf{cod}.h = x$ and $\mathsf{cod}.g = y$ then

$$
\begin{array}{ll}
 & \lfloor\psi\rfloor \in \mathsf{K}^\sharp.h \xleftarrow{\mathsf{Alg.G}} g \\
\equiv & \qquad \{ \qquad (6.7) \quad \} \\
 & \lfloor\psi\rfloor \ \in\ \mathsf{F}^\sharp.h \circ \mathsf{adjswap}_x \xleftarrow{\mathsf{Alg.G}} g \\
\equiv & \qquad \{ \qquad \text{definition 5.2, domains} \quad \} \\
 & \lfloor\psi\rfloor \in \mathsf{F}^\sharp.x \leftarrow y \ \wedge\ \mathsf{F}^\sharp.h \circ \mathsf{adjswap}_x \circ \mathsf{G}.\lfloor\psi\rfloor = \lfloor\psi\rfloor \circ g \\
\equiv & \qquad \{ \qquad (6.4) \quad \} \\
 & \quad \lfloor\psi\rfloor \in \mathsf{F}^\sharp.x \leftarrow y \\
 & \wedge\ \mathsf{F}^\sharp.h \circ \lfloor \mathsf{H}.\mathsf{counit}_x \circ \mathsf{swap}^\cup_{\mathsf{F}^\sharp.x} \rfloor \circ \mathsf{G}.\lfloor\psi\rfloor = \lfloor\psi\rfloor \circ g \\
\equiv & \qquad \{ \qquad (4.16d) \quad \} \\
 & \quad \lfloor\psi\rfloor \in \mathsf{F}^\sharp.x \leftarrow y
\end{array}
$$

$$\begin{array}{ll}
 & \wedge\ \lfloor h\circ H.\mathsf{counit}_x\circ \mathsf{swap}^{\cup}_{F^\sharp.x}\circ F.G.\lfloor\psi\rfloor\rfloor = \lfloor\psi\circ F.g\rfloor \\
\Leftarrow & \quad\{\quad \text{Leibniz}\quad\} \\
 & \lfloor\psi\rfloor\in F^\sharp.x\leftarrow y\ \wedge\ h\circ H.\mathsf{counit}_x\circ \mathsf{swap}^{\cup}_{F^\sharp.x}\circ F.G.\lfloor\psi\rfloor = \psi\circ F.g \\
\equiv & \quad\{\quad \mathsf{swap}^{\cup}\in H\bullet F\leftarrow F\bullet G\,,\ H \text{ is a functor}\quad\} \\
 & \lfloor\psi\rfloor\in F^\sharp.x\leftarrow y\ \wedge\ h\circ H.(\mathsf{counit}_x\circ F.\lfloor\psi\rfloor)\circ \mathsf{swap}^{\cup}_{y} = \psi\circ F.g \\
\equiv & \quad\{\quad \mathsf{counit}_x = \lceil \mathsf{id}_{F^\sharp.x}\rceil\,,\ \text{so}\ \mathsf{counit}_x\circ F.\lfloor\psi\rfloor = \psi\quad\} \\
 & \lfloor\psi\rfloor\in F^\sharp.x\leftarrow y\ \wedge\ h\circ H.\psi\circ \mathsf{swap}^{\cup}_{y} = \psi\circ F.g \\
\Leftarrow & \quad\{\quad \mathsf{swap}^{\cup} \text{ is an isomorphism, (4.16b)}\quad\} \\
 & \psi\in x\leftarrow F.y\ \wedge\ h\circ H.\psi = \psi\circ F.g\circ \mathsf{swap}_y \\
\equiv & \quad\{\quad \text{definition 5.2}\quad\} \\
 & \psi\in h \xleftarrow{\mathsf{Alg.H}} F.g\circ \mathsf{swap}_y \\
\equiv & \quad\{\quad (6.5)\quad\} \\
 & \psi\in h \xleftarrow{\mathsf{Alg.H}} K.g\quad .
\end{array}$$

The remaining coherence properties are

$$K^\sharp.\psi\circ\lfloor\varphi\rfloor\circ\gamma = \lfloor\psi\circ\varphi\circ K.\gamma\rfloor\quad ,$$

$$\psi\circ\lceil\varphi\rceil\circ K.\gamma = \lceil K^\sharp.\psi\circ\varphi\circ\gamma\rceil\quad ,$$

$$\lfloor\psi\rfloor = \varphi \equiv \psi = \lceil\varphi\rceil\quad .$$

All three follow immediately from (6.6) and (6.8) and the assumption that $(F, F^\sharp)$ is an adjoint pair of functors with $\lfloor\ \rfloor$ and $\lceil\ \rceil$ as left and right adjungate respectively.

In summary we have proven the following:

Theorem 6.11 Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. Let $G\in\mathcal{D}\leftarrow\mathcal{D}$ and $H\in\mathcal{C}\leftarrow\mathcal{C}$ be two functors and let $(F\in\mathcal{C}\leftarrow\mathcal{D}\ ,\ F^\sharp\in\mathcal{D}\leftarrow\mathcal{C})$ be an adjunction. Finally assume that there is an isomorphism $\mathsf{swap}\in F\bullet G\cong H\bullet F$. Then, there is an adjunction between the categories $\mathsf{Alg.H}$ and $\mathsf{Alg.G}$.

Specifics If we let unit and counit denote the unit and co-unit and $\lfloor\ \rfloor$ and $\lceil\ \rceil$ denote the left and right adjungate of the adjunction $(F\in\mathcal{C}\leftarrow\mathcal{D}\ ,\ F^\sharp\in\mathcal{D}\leftarrow\mathcal{C})$, respectively. Define $\mathsf{adjswap}$ by equation (6.4) and the adjungates $[\![\ ]\!]$ and $[\![\ ]\!]$ by equation (4.36). Then the functor $K\in\mathsf{Alg.H}\leftarrow\mathsf{Alg.G}$ defined by

$$K.g = F.g\circ\mathsf{swap}_{\mathsf{cod}.g}\quad \text{, where } g\in\mathsf{Alg.G},$$

$$K.\varphi = F.\varphi\quad \text{, where } \varphi \text{ is an arrow in } \mathsf{Alg.G}$$

is the lower adjoint and the functor $\mathsf{K}^\sharp \in \mathsf{Alg.G} \leftarrow \mathsf{Alg.H}$ defined by

$$\mathsf{K}^\sharp.h = \mathsf{F}^\sharp.h \circ \mathsf{adjswap}_{\mathsf{cod}.h} \quad , \text{ where } h \in \mathsf{Alg.H}$$

$$\mathsf{K}^\sharp.\psi = \mathsf{F}^\sharp.\psi \quad , \text{ where } \psi \text{ is an arrow in } \mathsf{Alg.H}$$

is the upper adjoint. The left adjungate of this adjunction is defined on arrows ψ in $\mathsf{Alg.H}$ by $\lfloor \psi \rfloor$. (Although, $\lfloor\ \rfloor$ is defined on arrows of $\mathcal{C}$, this definition is correct because the arrow ψ is *also* an arrow in the base category $\mathcal{C}$.) The right adjungate is defined similarly.
□

Corollary 6.12 (Fusion Rule) Given are three functors $\mathsf{F} \in \mathcal{C} \leftarrow \mathcal{D}$, $\mathsf{G} \in \mathcal{D} \leftarrow \mathcal{D}$ and $\mathsf{H} \in \mathcal{C} \leftarrow \mathcal{C}$. Given also is an isomorphism $\mathsf{swap} \in \mathsf{F} \bullet \mathsf{G} \cong \mathsf{H} \bullet \mathsf{F}$. Furthermore, the functor F has an upper adjoint $\mathsf{F}^\sharp$. Finally, it is assumed that $\mathsf{Alg.G}$ has an initial object muG. Then

$$\mathsf{F.muG} \circ \mathsf{swap}_{\mu\mathsf{G}}$$

is an initial object in the category $\mathsf{Alg.H}$. So, for every initial H-algebra muH

$$\mathsf{F.muG} \circ \mathsf{swap}_{\mu\mathsf{G}} \cong \mathsf{muH} \quad .$$

As a consequence we also have an isomorphism in the base category, i.e.

$$\mathsf{F}.\mu\mathsf{G} \cong \mu\mathsf{H} \quad .$$

Specifics In both isomorphisms

$$(\!|\mathsf{Alg.H}\,;\ \mathsf{F.muG} \circ \mathsf{swap}_{\mu\mathsf{G}} =:\ \mathsf{muH}|\!)$$

is the arrow to $\mathsf{F}.\mu\mathsf{G}$ from $\mu\mathsf{H}$ and

$$\lceil (\!|\mathsf{Alg.G}\,;\ \mathsf{F}^\sharp.\mathsf{muH} \circ \mathsf{adjswap}_{\mu\mathsf{H}} =:\ \mathsf{muG}|\!) \rceil_{\mathsf{muH},\mathsf{muG}}$$

is its inverse.

Proof That $\mathsf{F.muG} \circ \mathsf{swap}_{\mu\mathsf{G}}$ is an initial object in the category $\mathsf{Alg.H}$, follows immediately from our previous theorem and the fact that lower adjoints map initial objects to initial objects.

For the unique arrow to h from $\mathsf{F.muG} \circ \mathsf{swap}_{\mu\mathsf{G}}$ we have

$$\begin{array}{ll}
& \alpha \in h \xleftarrow{\mathsf{Alg.H}} \mathsf{F.muG} \circ \mathsf{swap}_{\mu\mathsf{G}} \\
\Leftarrow & \{ \quad \text{theorem 6.11}, \bullet \quad \alpha = \lceil \beta \rceil_{h,\mathsf{muG}}, \text{ definition of } \mathsf{K}^\sharp \quad \} \\
& \beta \in \mathsf{F}^\sharp.h \circ \mathsf{adjswap}_{\mu\mathsf{H}} \xleftarrow{\mathsf{Alg.G}} \mathsf{muG} \\
\Leftarrow & \{ \quad \text{initiality of } \mathsf{muG} \text{ in } \mathsf{Alg.G} \quad \} \\
& \beta = (\!|\mathsf{F}^\sharp.h \circ \mathsf{adjswap}_{\mu\mathsf{H}} =:\ \mathsf{muG}|\!) \quad .
\end{array}$$

The two arrows witnessing the isomorphism follow from the initiality of muH in $\mathsf{Alg.H}$ and instantiating muH for h in the above respectively.
□

6.2 The Abstraction Theorem

In lattice theory the *abstraction theorem* for least prefix points is the following. Given are three complete preordered sets $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$. Let $f \in \mathcal{C} \leftarrow \mathcal{E}$ be a monotonic function and let $\oplus \in \mathcal{D} \leftarrow \mathcal{C} \times \mathcal{D}$ be a monotonic binary function written as infix operator. For each element $x \in \mathcal{E}$ we denote the least prefix point of the function $(f.x)\oplus$ by $\mu((f.x)\oplus)$. Then the monotonic function $f\dot{\oplus} \in (\mathcal{D} \leftarrow \mathcal{E}) \leftarrow (\mathcal{D} \leftarrow \mathcal{E})$ defined by

$$((f\dot{\oplus}).g).x = (f\dot{\oplus}g).x = f.x \oplus g.x$$

has a least prefix point, specifically,

$$\mu(f\dot{\oplus}) = x \mapsto \mu((f.x)\oplus) \quad .$$

The justification for the name "abstraction theorem" comes from the following observation. Given the expression $f.x \oplus g.x$ we can either: first abstract from g then abstract from x and then take the least prefix point (which leads to $\mu(f\dot{\oplus})$) or we can first abstract from $g.x$ then take the least prefix point and then abstract from x (which leads to $x \mapsto \mu((f.x)\oplus)$). The equality says that in both cases we end up with the same expression. Moreover, if we allow extensionality then the equality says that we can define the least prefix point of $\mu(f\dot{\oplus})$ pointwise.

In this section we translate the above theorem to category theory. So, we make the following assumptions. Given are three categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$. Let $F \in \mathcal{C} \leftarrow \mathcal{E}$ be a functor and let $\oplus \in \mathcal{D} \leftarrow \mathcal{C} \times \mathcal{D}$ be a binary functor written as infix operator. Finally, it is assumed that for every object x in $\mathcal{C}$ an initial object $\mathsf{mu}((F.x)\oplus)$ exists in $\mathsf{Alg}.((F.x)\oplus)$. Guided by the above discussion, our first task is to define $F\dot{\oplus} \in \mathsf{Fun}(\mathcal{D},\mathcal{E}) \leftarrow \mathsf{Fun}(\mathcal{D},\mathcal{E})$ as a functor. On objects $F\dot{\oplus}$ maps a functor G to the functor $F\dot{\oplus}G$ defined in lemma 5.37.

Lemma 6.13 $F\dot{\oplus} \in \mathsf{Fun}(\mathcal{D},\mathcal{E}) \leftarrow \mathsf{Fun}(\mathcal{D},\mathcal{E})$ is a functor, where

(6.14) $(F\dot{\oplus}).G = F\dot{\oplus}G$,

for every functor $G \in \mathcal{D} \leftarrow \mathcal{E}$. For every natural transformation η in the functor category $\mathsf{Fun}(\mathcal{D},\mathcal{E})$, the natural transformation $(F\dot{\oplus}).\eta$ in $\mathsf{Fun}(\mathcal{D},\mathcal{E})$ is defined by:

(6.15) $((F\dot{\oplus}).\eta)_z = \mathsf{id}_{F.z} \oplus \eta_z$ for all $z \in \mathcal{E}$.

□

That $x \mapsto \mu((\mathsf{F}.x)\oplus)$ is a functor follows immediately from the following observation

(6.16) $x \mapsto \mu((\mathsf{F}.x)\oplus) = (x \mapsto \mu(x\oplus))\bullet\mathsf{F} = \varpi_\oplus\bullet\mathsf{F}$

where ϖ is the map operator of section 5.4.

We will show that if $\mathsf{Alg}.((\mathsf{F}.x)\oplus)$ has an initial object for all x (as our assumption states) then so does $\mathsf{Alg}.(\mathsf{F}\dot\oplus)$. We do this by constructing an initial object Θ in $\mathsf{Alg}.(\mathsf{F}\dot\oplus)$ in such a way that Θ has $\varpi_\oplus\bullet\mathsf{F}$ as its carrier. In theorem 2.6 we showed that all initial objects in a category are isomorphic, so Θ is isomorphic to all initial $(\mathsf{F}\dot\oplus)$-algebras. Thus, by lemma 5.3, an immediate consequence is that the carrier of Θ, i.e. $\varpi_\oplus\bullet\mathsf{F}$, is isomorphic (in the functor category) to the carriers of all initial $(\mathsf{F}\dot\oplus)$-algebras. Let $\mathsf{mu}(\mathsf{F}\dot\oplus)$ denote a particular initial $(\mathsf{F}\dot\oplus)$-algebra then we can conclude

$$\mu(\mathsf{F}\dot\oplus) \cong \varpi_\oplus\bullet\mathsf{F} \quad .$$

The construction of an initial $(\mathsf{F}\dot\oplus)$-algebra on the carrier $\varpi_\oplus\bullet\mathsf{F}$ goes as follows. First we construct a candidate:

$$\begin{array}{ll}
 & \Theta \in \varpi_\oplus\bullet\mathsf{F} \leftarrow \mathsf{F}\dot\oplus(\varpi_\oplus\bullet\mathsf{F}) \\
\equiv & \{ \quad \text{composition} \quad \} \\
 & \Theta \in \varpi_\oplus\bullet\mathsf{F} \leftarrow (\mathsf{Id}\dot\oplus\varpi_\oplus)\bullet\mathsf{F} \\
\Leftarrow & \{ \quad \text{Godement's rules,} \ \bullet \quad \Theta = \varphi\bullet\mathsf{F} \quad \} \\
 & \varphi \in \varpi_\oplus \leftarrow \mathsf{Id}\dot\oplus\varpi_\oplus \\
\Leftarrow & \{ \quad (5.40) \quad \} \\
 & \forall(x{::}\ \varphi_x = \mathsf{mu}(x\oplus)) \quad .
\end{array}$$

So, as a candidate we have the natural transformation Θ defined by $\Theta_x = \mathsf{mu}((\mathsf{F}.x)\oplus)$ for each object x.

It remains to prove that Θ is initial, i.e. has a unique arrow to any other $(\mathsf{F}\dot\oplus)$-algebra. Suppose $\eta \in \mathsf{G} \leftarrow \mathsf{F}\dot\oplus\mathsf{G}$ is an $(\mathsf{F}\dot\oplus)$-algebra. We have to construct a unique arrow to η from Θ. So, by construction of α:

$$\begin{array}{ll}
 & \alpha \in \eta \xleftarrow{\mathsf{Alg}.\mathsf{F}\dot\oplus} \Theta \\
\equiv & \{ \quad \text{arrow in } \mathsf{Alg}.(\mathsf{F}\dot\oplus) \quad \} \\
 & \eta \circ (\mathsf{F}\dot\oplus).\alpha = \alpha\circ\Theta \ \wedge\ \alpha \in \mathsf{G} \xleftarrow{\mathsf{Fun}(\mathcal{D},\mathcal{E})} \varpi_\oplus\bullet\mathsf{F} \\
\equiv & \{ \quad \text{extensionality, definition natural transformation} \quad \} \\
 & \quad \forall(x{::}\ \eta_x \circ \mathsf{id}_{\mathsf{F}.x}\oplus\alpha_x = \alpha_x\circ\Theta_x \ \wedge\ \alpha_x \in \mathsf{G}.x \leftarrow \varpi_\oplus.(\mathsf{F}.x)) \\
 & \wedge\ \forall(f{:}\ f \in x \leftarrow y{:}\ \mathsf{G}.f \circ \alpha_y = \alpha_x \circ \varpi_\oplus.(\mathsf{F}.f))
\end{array}$$

$$
\begin{array}{lll}
\equiv & \{ & (5.29):\ \mathsf{id}_{\mathsf{F.x}} \oplus \alpha_x = ((\mathsf{F.x})\oplus).\alpha_x\text{, definition of } \Theta_x \quad \} \\
& & \forall(x::\ \eta_x \circ ((\mathsf{F.x})\oplus).\alpha_x = \alpha_x \circ \Theta_x \ \wedge\ \alpha_x \in \mathsf{cod}.\eta_x \leftarrow \mathsf{cod}.\Theta_x) \\
& \wedge & \forall(f:\ f \in x \leftarrow y:\ \mathsf{G.f} \circ \alpha_y = \alpha_x \circ \varpi_{\oplus}.(\mathsf{F.f})) \\
\equiv & \{ & \text{definition natural transformation,} \\
& & \eta_x \text{ and } \Theta_x \text{ objects in } \mathsf{Alg}.((\mathsf{F.x})\oplus) \quad \} \\
& & \forall(x::\ \alpha_x \in \eta_x \xleftarrow{\mathsf{Alg}.((\mathsf{F.x})\oplus)} \Theta_x) \\
& \wedge & \forall(f:\ f \in x \leftarrow y:\ \mathsf{G.f} \circ \alpha_y = \alpha_x \circ \varpi_{\oplus}.(\mathsf{F.f})) \\
\equiv & \{ & \Theta_x = \mathsf{mu}((\mathsf{F.x})\oplus)\text{, } (5.18) \quad \} \\
(6.17) & & \forall(x::\ \alpha_x = (\!|\eta_x =:\ \mathsf{mu}((\mathsf{F.x})\oplus)|\!)) \\
& \wedge & \forall(f:\ f \in x \leftarrow y:\ \mathsf{G.f} \circ \alpha_y = \alpha_x \circ \varpi_{\oplus}.(\mathsf{F.f})) \\
\equiv & \{ & \text{see verification below} \quad \} \\
& & \forall(x::\ \alpha_x = (\!|\eta_x =:\ \mathsf{mu}((\mathsf{F.x})\oplus)|\!)) \quad .
\end{array}
$$

If we are now able to prove that the first conjunct in (6.17) implies the second conjunct then we have proven that the natural transformation α defined on objects x by $\alpha_x = (\!|\eta_x =:\ \mathsf{mu}((\mathsf{F.x})\oplus)|\!)$ is an arrow to η from Θ. Moreover, since in the above construction all the steps are equivalences, we have also verified the uniqueness of α. Let $f \in x \leftarrow y$. Then this verification goes as follows:

$$
\begin{array}{lll}
& & \mathsf{G.f} \circ (\!|\eta_y =:\ \mathsf{mu}((\mathsf{F.y})\oplus)|\!) = (\!|\eta_x =:\ \mathsf{mu}((\mathsf{F.x})\oplus)|\!) \circ \varpi_{\oplus}.(\mathsf{F.f}) \\
\equiv & \{ & \text{map-fusion: } (5.41) \quad \} \\
& & \mathsf{G.f} \circ (\!|\eta_y =:\ \mathsf{mu}((\mathsf{F.y})\oplus)|\!) = (\!|\eta_x \circ \mathsf{F.f} \oplus \mathsf{id}_{\mathsf{G.x}} =:\ \mathsf{mu}((\mathsf{F.y})\oplus)|\!) \\
\Leftarrow & \{ & \text{fusion: } (5.19e) \quad \} \\
& & \mathsf{G.f} \in \eta_x \circ \mathsf{F.f} \oplus \mathsf{id}_{\mathsf{G.x}} \xleftarrow{\mathsf{Alg}.((\mathsf{F.y})\oplus)} \eta_y \\
\equiv & \{ & \text{definition arrow in } \mathsf{Alg}.((\mathsf{F.y})\oplus)\text{,} \\
& & \text{(being an arrow in base category is trivial)} \quad \} \\
& & \eta_x \circ \mathsf{F.f} \oplus \mathsf{id}_{\mathsf{G.x}} \circ ((\mathsf{F.y})\oplus).\mathsf{G.f} = \mathsf{G.f} \circ \eta_y \\
\equiv & \{ & ((\mathsf{F.y})\oplus).(\mathsf{G.f}) = \mathsf{id}_{\mathsf{F.y}} \oplus \mathsf{G.f}\text{,} \\
& & \text{coherence property binary functor} \quad \} \\
& & \eta_x \circ \mathsf{F.f} \oplus \mathsf{G.f} = \mathsf{G.f} \circ \eta_y \\
\equiv & \{ & \eta \in \mathsf{G} \leftarrow \mathsf{F} \dot{\oplus} \mathsf{G} \quad \} \\
& & \mathit{true} \quad .
\end{array}
$$

We conclude that Θ is indeed an initial $(\mathsf{F}\dot{\oplus})$-algebra.

We have now reached the final part of the proof. Suppose $\mathsf{mu}(\mathsf{F}\dot{\oplus})$ denotes a particular initial $(\mathsf{F}\dot{\oplus})$-algebra. Then by theorem 2.6

$$\Theta \cong \mathsf{mu}(\mathsf{F}\dot{\oplus}) \quad .$$

The witnesses to this isomorphism are

$$(\!|\Theta =: \mathsf{mu}(\mathsf{F}\dot{\oplus})|\!) \in \Theta \xleftarrow{\mathsf{Alg}.\mathsf{F}\dot{\oplus}} \mathsf{mu}(\mathsf{F}\dot{\oplus}) \quad ,$$

which follows from the assumed initiality of $\mathsf{mu}(\mathsf{F}\dot{\oplus})$, and

$$\alpha \in \mathsf{mu}(\mathsf{F}\dot{\oplus}) \xleftarrow{\mathsf{Alg}.\mathsf{F}\dot{\oplus}} \Theta \quad ,$$

defined by $\alpha_x = (\!|(\mathsf{mu}(\mathsf{F}\dot{\oplus}))_x =: \mathsf{mu}((\mathsf{F}.x)\oplus)|\!)$ for each object x in $\mathcal{E}$, which follows from the above proof. That the composition of these arrows in both order gives the identity arrow follows from the assumed initiality of $\mathsf{mu}(\mathsf{F}\dot{\oplus})$, the proven initiality of Θ and, as a result, the uniqueness of the arrow to and from $\mathsf{mu}(\mathsf{F}\dot{\oplus})$ and to and from Θ. By lemma 5.3 also the carriers of Θ and $\mathsf{mu}(\mathsf{F}\dot{\oplus})$ are isomorphic so

$$x \mapsto \mu((\mathsf{F}.x)\oplus) \cong \mu(\mathsf{F}\dot{\oplus})$$

and the witnesses are the same. Finally we state the theorem in full.

Theorem 6.18 (Abstraction Theorem) Given are three categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$. Let $\mathsf{F} \in \mathcal{C} \leftarrow \mathcal{E}$ be a functor, and let $\oplus \in \mathcal{D} \leftarrow \mathcal{C} \times \mathcal{D}$ be a binary functor written as infix operator. Assume also that for every object x in $\mathcal{C}$ an initial object $\mathsf{mu}((\mathsf{F}.x)\oplus)$ exists in $\mathsf{Alg}.((\mathsf{F}.x)\oplus)$. Finally, let $\mathsf{F}\dot{\oplus}$ denote the functor $\mathsf{G} \mapsto (x \mapsto \mathsf{F}.x \oplus \mathsf{G}.x)$ of type $\mathsf{Fun}(\mathcal{D},\mathcal{E}) \leftarrow \mathsf{Fun}(\mathcal{D},\mathcal{E})$. Then the natural transformation Θ, defined pointwise by

$$\Theta_x = \mathsf{mu}((\mathsf{F}.x)\oplus) \quad ,$$

is an initial $(\mathsf{F}\dot{\oplus})$-algebra. Moreover, if $\mathsf{mu}(\mathsf{F}\dot{\oplus})$ denotes a particular initial $(\mathsf{F}\dot{\oplus})$-algebra having codomain $\mu(\mathsf{F}\dot{\oplus})$, then

$$x \mapsto \mu((\mathsf{F}.x)\oplus) \cong \mu(\mathsf{F}\dot{\oplus}) \quad .$$

Specifics In both isomorphisms, between Θ and $\mathsf{mu}(\mathsf{F}\dot{\oplus})$ and their carriers,

$$(\!|\Theta =: \mathsf{mu}(\mathsf{F}\dot{\oplus})|\!)$$

is the arrow to $x \mapsto \mu((\mathsf{F}.x)\oplus)$ from $\mu(\mathsf{F}\dot{\oplus})$ and the natural transformation α defined by

$$\alpha_z = (\!|(\mathsf{mu}(\mathsf{F}\dot{\oplus}))_z =: \mathsf{mu}((\mathsf{F}.z)\oplus)|\!) \quad \text{for each object } z \text{ in } \mathcal{E}$$

is its inverse.
□

6.3 The Beautiful Theorem

This section combines the abstraction theorem and fusion theorem to prove in category theory a theorem dubbed "beautiful" by Dijkstra and Scholten [6, p. 159].

Dijkstra and Scholten formulate the "beautiful theorem" in the context of the predicate calculus. In terms of lattice theory the theorem is stated as follows. Suppose that $\mathcal{A}=(A,\sqsupseteq)$ and $\mathcal{B}=(B,\succeq)$ are complete lattices and suppose $\oplus\in(\mathcal{A}\leftarrow\mathcal{A})\leftarrow\mathcal{B}$ is a monotonic function. Denote the least prefix point of $x\oplus$ by $\mu(x\oplus)$ and consider the function $x\mapsto\mu(x\oplus)$, which we denote by $\varpi_\oplus$. The "beautiful theorem" is that $\varpi_\oplus$ enjoys any type of supremum-preserving property that is enjoyed by the (uncurried binary) function $\oplus$. More precisely, letting $\mathsf{Sup}.f$ denote the supremum of f, then

$$(6.19)\quad \forall(f{::}\ \varpi_\oplus.(\mathsf{Sup}.f)=\mathsf{Sup}.(\varpi_\oplus\bullet f)) \Leftarrow \forall(f,g{::}\ (\mathsf{Sup}.f)\oplus(\mathsf{Sup}.g)=\mathsf{Sup}.(f\dot{\oplus}g)) \quad ,$$

where, by definition, $f\dot{\oplus}g$ is the function mapping x to $f.x\oplus g.x$ and $\bullet$ denotes function composition.

A proof of this theorem (in lattice theory) is a straightforward combination of the abstraction and fusion theorems (in lattice theory). Let us show how this is done. We begin our proof with a use of the abstraction theorem designed to get us into a position in which μ-fusion is applicable. For all $f\in\mathcal{B}\leftarrow\mathcal{C}$,

$$\begin{array}{ll}
 & \varpi_\oplus\bullet f \\
= & \quad\{\quad \text{extensionality, definition of composition} \quad\} \\
 & c\mapsto\varpi_\oplus.(f.c) \\
= & \quad\{\quad \text{definition of } \varpi_\oplus \quad\} \\
 & c\mapsto\mu((f.c)\oplus) \\
= & \quad\{\quad \text{abstraction theorem} \quad\} \\
 & \mu(f\dot{\oplus}) \quad .
\end{array}$$

Thus,

$$\begin{array}{ll}
 & \varpi_\oplus.(\mathsf{Sup}_\mathcal{B}.f)=\mathsf{Sup}_\mathcal{A}.(\varpi_\oplus\bullet f) \\
\equiv & \quad\{\quad \text{definition of } \varpi_\oplus \text{ applied to the lhs,} \\
 & \qquad\quad \text{above calculation applied to the rhs} \quad\} \\
 & \mu((\mathsf{Sup}_\mathcal{B}.f)\oplus)=\mathsf{Sup}_\mathcal{A}.\mu(f\dot{\oplus}) \\
\Leftarrow & \quad\{\quad \mu\text{-fusion (which is applicable since } \mathsf{Sup}_\mathcal{A} \text{ is} \\
 & \qquad\quad \text{a lower adjoint) and extensionality} \quad\} \\
 & \forall(g{::}\ (\mathsf{Sup}_\mathcal{B}.f)\oplus(\mathsf{Sup}_\mathcal{A}.g)=\mathsf{Sup}_\mathcal{A}.(f\dot{\oplus}g)) \quad .
\end{array}$$

We have thus proved that

$$\forall(f{::}\ \varpi_\oplus.(\mathsf{Sup}_\mathcal{B}.f)=\mathsf{Sup}_\mathcal{A}.(\varpi_\oplus\bullet f) \Leftarrow \forall(g{::}\ (\mathsf{Sup}_\mathcal{B}.f)\oplus(\mathsf{Sup}_\mathcal{A}.g)=\mathsf{Sup}_\mathcal{A}.(f\dot{\oplus}g))) \quad .$$

The "beautiful theorem" follows by elementary calculus.

A remarkable aspect of that proof is that no use is made whatsoever of the details of the Galois connection defining suprema. Only the fact that the supremum operator is the lower adjoint in a Galois connection is needed. In this section we formulate and prove the categorical version of the theorem.

Suppose $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are categories and suppose that $\mathcal{A}$ and $\mathcal{B}$ are both $\mathcal{C}$-cocomplete. Let $\mathsf{Col}_{\mathcal{A}}$ and $\mathsf{Col}_{\mathcal{B}}$ denote colimit operators of type $\mathcal{A}\leftarrow\mathsf{Fun}(\mathcal{A},\mathcal{C})$ and $\mathcal{B}\leftarrow\mathsf{Fun}(\mathcal{B},\mathcal{C})$, respectively. The only fact that we will use about these two operators is that $\mathsf{Col}_{\mathcal{A}}$ is a lower adjoint in an adjunction between the categories $\mathcal{A}$ and $\mathsf{Fun}(\mathcal{A},\mathcal{C})$.

Suppose $\oplus\in(\mathcal{A}\leftarrow\mathcal{A})\leftarrow\mathcal{B}$. We adopt the same notational conventions and assumptions as in section 6.2. Thus we assume that, for all $\mathsf{b}\in\mathcal{B}$, the functor $\mathsf{b}\oplus$ has an initial object denoted by $\mathsf{mu}(\mathsf{b}\oplus)$. The codomain of $\mathsf{mu}(\mathsf{b}\oplus)$ is again denoted by $\mu(\mathsf{b}\oplus)$. Let $\varpi_{\oplus}$ denote the functor whose action on objects is to map b to $\mu(\mathsf{b}\oplus)$. More precisely, define

$$\varpi_{\oplus} = \mathsf{car}\bullet\mathsf{Mu}\bullet\mathsf{app} \quad .$$

(See section 5.4 for the definitions of the three functors on the right side of this equation.) Our goal is to prove that $\varpi_{\oplus}$ "commutes" with Col if $\oplus$ "commutes" with Col. To be precise, we prove that

$$\begin{array}{ll} & \forall(\mathsf{F}::\varpi_{\oplus}.(\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\cong\mathsf{Col}_{\mathcal{A}}.(\varpi_{\oplus}\bullet\mathsf{F})) \\ \Leftarrow & \forall(\mathsf{F},\mathsf{G}::\mathsf{Col}_{\mathcal{A}}.(\mathsf{F}\dot{\oplus}\mathsf{G})\cong(\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus(\mathsf{Col}_{\mathcal{A}}.\mathsf{G})) \quad . \end{array}$$

Here and elsewhere the dummy F ranges over functors of type $\mathcal{B}\leftarrow\mathcal{C}$ and the dummy G over functors of type $\mathcal{A}\leftarrow\mathcal{C}$.

Let us consider the premise. Eliminating the quantification over G, we are given that, for all F,

$$\mathsf{Col}_{\mathcal{A}}\bullet(\mathsf{F}\dot{\oplus}) \cong (\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus\bullet\,\mathsf{Col}_{\mathcal{A}} \quad .$$

Suppose the witnessing isomorphism is τ. Then, by the fusion theorem (and the fact that $\mathsf{Col}_{\mathcal{A}}$ is a lower adjoint),

$$\mathsf{Col}_{\mathcal{A}}.\mathsf{mu}(\mathsf{F}\dot{\oplus})\circ\tau_{\mu(\mathsf{F}\dot{\oplus})}$$

is an initial $((\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus)$-algebra whenever $\mathsf{mu}(\mathsf{F}\dot{\oplus})$ is an initial $(\mathsf{F}\dot{\oplus})$-algebra. But, by the abstraction theorem, π, defined by $\pi_{\mathsf{x}}=\mathsf{mu}((\mathsf{F}.\mathsf{x})\oplus)$, is an initial $(\mathsf{F}\dot{\oplus})$-algebra. Thus,

$$\begin{array}{ll} & \varpi_{\oplus}.(\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\cong\mathsf{Col}_{\mathcal{A}}.(\varpi_{\oplus}\bullet\mathsf{F}) \\ \equiv & \{ \quad \varpi_{\oplus}.(\mathsf{Col}_{\mathcal{B}}.\mathsf{F}) \text{ is by definition the codomain of an initial} \\ & \quad ((\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus)\text{-algebra; by definition, } \varpi_{\oplus}\bullet\mathsf{F}=\mathsf{cod}.\pi \quad \} \\ & \mathsf{cod}.(\mathsf{Col}_{\mathcal{A}}.\pi\circ\tau_{\mu(\mathsf{F}\dot{\oplus})})\cong\mathsf{Col}_{\mathcal{A}}.(\mathsf{cod}.\pi) \\ \equiv & \{ \quad \text{codomain calculus} \quad \} \\ & \mathit{true} \quad . \end{array}$$

We have thus proved that, for all F,

$$\varpi_{\oplus}.(\mathsf{Col}_{\mathcal{B}}.\mathsf{F}) \cong \mathsf{Col}_{\mathcal{A}}.(\varpi_{\oplus}\bullet\mathsf{F}) \Leftarrow \forall(\mathsf{G}::\mathsf{Col}_{\mathcal{A}}.(\mathsf{F}\dot{\oplus}\mathsf{G}) \cong (\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus(\mathsf{Col}_{\mathcal{A}}.\mathsf{G})) \quad .$$

The categorical "beautiful theorem" follows by elementary calculus.

For completeness we should of course provide details of the witnesses to the constructed isomorphism. The details are complicated but entirely mechanical. Let $\tau^{\cup}$ denote the inverse of τ. Let $\lfloor\ \rfloor$ and $\lceil\ \rceil$ denote the lower and upper adjungate and counit the counit of the adjunction $(\mathsf{Col}_{\mathcal{A}}, \mathsf{K}_{\mathcal{A},\mathcal{C}})$. Let f be a $((\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus)$-algebra and let $\xi_{\mathsf{f}} = \lfloor(\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus\mathsf{counit}_{\mathsf{cod.f}} \circ \tau^{\cup}_{\mathsf{K}_{\mathcal{A},\mathcal{C}}.(\mathsf{cod.f})}\rfloor$. Then $\lceil(\!|\mathsf{K}_{\mathcal{A},\mathcal{C}}.\mathsf{f} \circ \xi_{\mathsf{f}} =:\ \pi|\!)\rceil$ is the unique arrow from $\mathsf{Col}_{\mathcal{A}}.\pi{\circ}\tau_{\mu(\mathsf{F}\dot{\oplus})}$ to f. Furthermore,

$$\varpi_{\oplus}.(\mathsf{Col}_{\mathcal{B}}.\mathsf{F}) = (\mathsf{car}\bullet\mathsf{Mu}\bullet\mathsf{app}).(\mathsf{Col}_{\mathcal{B}}.\mathsf{F}) = \mathsf{cod}.(\mathsf{mu}((\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus)) \quad .$$

Thus,

$$\lceil(\!|\mathsf{K}_{\mathcal{A},\mathcal{C}}.\mathsf{mu}((\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus) \circ \xi_{\mathsf{mu}((\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus)} =:\ \pi|\!)\rceil$$

is the unique arrow to $\varpi_{\oplus}.(\mathsf{Col}_{\mathcal{B}}.\mathsf{F})$ from $\mathsf{cod}.(\mathsf{Col}_{\mathcal{A}}.\pi{\circ}\tau_{\mu(\mathsf{F}\dot{\oplus})})$. (Recall that an arrow between two algebras in an algebra category is also an arrow between their carriers in the base category.) Finally,

$$\mathsf{cod}.(\mathsf{Col}_{\mathcal{A}}.\pi{\circ}\tau_{\mu(\mathsf{F}\dot{\oplus})}) = \mathsf{Col}_{\mathcal{A}}.(\mathsf{cod}.\pi) = \mathsf{Col}_{\mathcal{A}}.(\varpi_{\oplus}\bullet\mathsf{F}) \quad .$$

So, defining

$$\eta = \lceil(\!|\mathsf{K}_{\mathcal{A},\mathcal{C}}.\mathsf{mu}((\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus) \circ \xi_{\mathsf{mu}((\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus)} =:\ \pi|\!)\rceil$$

we have:

$$\eta \in \varpi_{\oplus}.(\mathsf{Col}_{\mathcal{B}}.\mathsf{F}) \cong \mathsf{Col}_{\mathcal{A}}.(\varpi_{\oplus}\bullet\mathsf{F}) \ \Leftarrow\ \tau \in \mathsf{Col}_{\mathcal{A}}\bullet(\mathsf{F}\dot{\oplus}) \cong (\mathsf{Col}_{\mathcal{B}}.\mathsf{F})\oplus\bullet\,\mathsf{Col}_{\mathcal{A}} \quad .$$

An important special instance of the "beautiful theorem" is that ω-cocontinuity is preserved by the process of constructing initial algebras [19, p.289]. A striking feature of our proof is that no use is made whatsoever of the structure of the adjunction defining colimits; we have used the fact that $\mathcal{B}$-cocompleteness of a category is equivalent to the existence of an adjunction but the structure of the colimit functor or its upper adjoint has not entered the picture. This is in marked contrast to proofs given by others of the preservation of ω-cocontinuity [16, 17] where details of the construction of a category of cocones plays a prominent rôle. This emphasises the more fundamental nature of the abstraction and fusion theorems.

6.4 The Rolling Rule

Given two complete partially ordered sets $\mathcal{C}$ and $\mathcal{D}$ and two monotonic functions $\mathsf{f} \in \mathcal{C} \leftarrow \mathcal{D}$ and $\mathsf{g} \in \mathcal{D} \leftarrow \mathcal{C}$, the so-called *rolling rule* in lattice theory states that

$$\mathsf{f}.\mu(\mathsf{g}\bullet\mathsf{f}) = \mu(\mathsf{f}\bullet\mathsf{g}) \quad .$$

In this section we give a corresponding theorem in category theory. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $\mathsf{F} \in \mathcal{C} \leftarrow \mathcal{D}$ and $\mathsf{G} \in \mathcal{D} \leftarrow \mathcal{C}$ be functors. Suppose that $\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})$ is an initial $(\mathsf{G}\bullet\mathsf{F})$-algebra. We will show how to construct an initial $(\mathsf{F}\bullet\mathsf{G})$-algebra.

Since $\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})$ is a $(\mathsf{G}\bullet\mathsf{F})$-algebra by assumption, we can use corollary 5.12 to conclude that $\mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})$ is an $(\mathsf{F}\bullet\mathsf{G})$-algebra. This is our candidate initial $(\mathsf{F}\bullet\mathsf{G})$-algebra. So, we have to show that there is a unique arrow from $\mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})$ to any other $(\mathsf{F}\bullet\mathsf{G})$-algebra.

First we prove the existence of such an arrow. Suppose f is an $(\mathsf{F}\bullet\mathsf{G})$-algebra. We construct an arrow α in the category $\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G})$ to f from $\mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})$ as follows.

$$\begin{array}{cl}
 & \alpha \in \mathsf{f} \xleftarrow{\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G})} \mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}) \\
\Leftarrow & \qquad \{ \qquad \text{by lemma 5.17 , } \ \mathsf{f} \in \mathsf{f} \xleftarrow{\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G})} \mathsf{F}.\mathsf{G}.\mathsf{f} \ \ , \\
 & \qquad\qquad\quad \text{constructive transitivity, } \bullet \quad \alpha = \mathsf{f}\circ\beta \quad \} \\
 & \beta \in \mathsf{F}.\mathsf{G}.\mathsf{f} \xleftarrow{\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G})} \mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}) \\
\Leftarrow & \qquad \{ \qquad \text{corollary 5.12, } \bullet \quad \beta = \mathsf{F}.\gamma \quad \} \\
 & \gamma \in \mathsf{G}.\mathsf{f} \xleftarrow{\mathsf{Alg}.(\mathsf{G}\bullet\mathsf{F})} \mathsf{mu}(\mathsf{G}\bullet\mathsf{F}) \\
\Leftarrow & \qquad \{ \qquad \mathsf{mu}(\mathsf{G}\bullet\mathsf{F}) \text{ is an initial } (\mathsf{G}\bullet\mathsf{F})\text{-algebra} \quad \} \\
 & \gamma = (\!|\mathsf{G}.\mathsf{f} =: \mathsf{mu}(\mathsf{G}\bullet\mathsf{F})|\!) \quad .
\end{array}$$

We have thus constructed the arrow

(6.20) $\mathsf{f} \circ \mathsf{F}.(\!|\mathsf{G}.\mathsf{f} =: \mathsf{mu}(\mathsf{G}\bullet\mathsf{F})|\!) \ \in \ \mathsf{f} \xleftarrow{\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G})} \mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}) \quad .$

We now have to show that the above constructed arrow is unique. Suppose that

(6.21) $\varphi \ \in \ \mathsf{f} \xleftarrow{\mathsf{Alg}.(\mathsf{F}\bullet\mathsf{G})} \mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}) \quad .$

That is, by definition 5.2,

(6.22) $\varphi \ \in \ \mathsf{cod}.\mathsf{f} \leftarrow \mathsf{F}.\mu(\mathsf{G}\bullet\mathsf{F})$ and

(6.23) $\mathsf{f} \circ \mathsf{F}.\mathsf{G}.\varphi \ = \ \varphi \circ \mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}) \quad .$

We have to verify that $\alpha = \varphi$ (where α is constructed as above). The key insight in the proof is that there is an arrow $(\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}))^{\cup}$ such that $\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})$ and $(\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}))^{\cup}$ are witnesses to the fact that $\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})$ and $\mathsf{G}.\mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})$ are isomorphic $\mathsf{G}\bullet\mathsf{F}$-algebras (see theorem 5.21). Although we know specifically how to construct $(\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}))^{\cup}$ it will not be necessary to exploit the details of that construction.

$$
\begin{array}{ll}
& \alpha = \varphi \\
\equiv & \{ \quad \mathsf{F.mu(G{\bullet}F)} \circ \mathsf{F.(mu(G{\bullet}F))}^{\cup} = \mathsf{F.id}_{\mathsf{mu(G{\bullet}F)}} = \mathsf{F.id}_{\mu(\mathsf{G{\bullet}F})} \quad \} \\
& \alpha = \varphi \circ \mathsf{F.mu(G{\bullet}F)} \circ \mathsf{F.(mu(G{\bullet}F))}^{\cup} \\
\equiv & \{ \quad \alpha = \mathsf{f} \circ \mathsf{F.}(\!|\mathsf{G.f} =: \mathsf{mu(G{\bullet}F)}|\!) \text{ (see above) , (6.23)} \quad \} \\
& \mathsf{f} \circ \mathsf{F.}(\!|\mathsf{G.f} =: \mathsf{mu(G{\bullet}F)}|\!) = \mathsf{f} \circ \mathsf{F.G.}\varphi \circ \mathsf{F.(mu(G{\bullet}F))}^{\cup} \\
\Leftarrow & \{ \quad \mathsf{F} \text{ is a functor, Leibnitz} \quad \} \\
& (\!|\mathsf{G.f} =: \mathsf{mu(G{\bullet}F)}|\!) = \mathsf{G.}\varphi \circ \mathsf{(mu(G{\bullet}F))}^{\cup} \\
\equiv & \{ \quad (5.18) \quad \} \\
& \mathsf{G.}\varphi \circ \mathsf{(mu(G{\bullet}F))}^{\cup} \in \mathsf{G.f} \xleftarrow{\mathsf{Alg.(G{\bullet}F)}} \mathsf{mu(G{\bullet}F)} \\
\Leftarrow & \{ \quad \mathsf{(mu(G{\bullet}F))}^{\cup} \in \mathsf{G.F.mu(G{\bullet}F)} \xleftarrow{\mathsf{Alg.(G{\bullet}F)}} \mathsf{mu(G{\bullet}F)} \quad \} \\
& \mathsf{G.}\varphi \in \mathsf{G.f} \xleftarrow{\mathsf{Alg.(G{\bullet}F)}} \mathsf{G.F.mu(G{\bullet}F)} \\
\Leftarrow & \{ \quad (5.12) \quad \} \\
& \varphi \in \mathsf{f} \xleftarrow{\mathsf{Alg.(F{\bullet}G)}} \mathsf{F.mu(G{\bullet}F)} \\
\equiv & \{ \quad (6.21) \quad \} \\
& \mathit{true} \quad .
\end{array}
$$

This concludes our proof that $\mathsf{F.mu(G{\bullet}F)}$ is an initial $(\mathsf{F{\bullet}G})$-algebra.

Now suppose that $\mathsf{mu(F{\bullet}G)}$ is an initial $(\mathsf{F{\bullet}G})$-algebra. Then by theorem 2.6

$$\mathsf{F.mu(G{\bullet}F)} \cong \mathsf{mu(F{\bullet}G)} \quad .$$

The witnesses to the isomorphism are

$$(\!|\mathsf{F.mu(G{\bullet}F)} =: \mathsf{mu(F{\bullet}G)}|\!) \in \mathsf{F.mu(G{\bullet}F)} \xleftarrow{\mathsf{Alg.(F{\bullet}G)}} \mathsf{mu(F{\bullet}G)} \quad ,$$

which follows from the assumed initiality of $\mathsf{mu(F{\bullet}G)}$ and

$$\mathsf{mu(F{\bullet}G)} \circ \mathsf{F.}(\!|\mathsf{G.mu(F{\bullet}G)} =: \mathsf{mu(G{\bullet}F)}|\!) \in \mathsf{mu(F{\bullet}G)} \xleftarrow{\mathsf{Alg.(F{\bullet}G)}} \mathsf{F.mu(G{\bullet}F)} \quad ,$$

which follows from the above proof. By lemma 5.3, we also have an isomorphism between the carriers, i.e.

$$\mathsf{F.}\mu(\mathsf{G{\bullet}F}) \cong \mu(\mathsf{F{\bullet}G})$$

and the witnesses to this isomorphism are the same. We finally state the theorem in full.

Theorem 6.24 (Rolling Rule) Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $\mathsf{F} \in \mathcal{C} \leftarrow \mathcal{D}$ and $\mathsf{G} \in \mathcal{D} \leftarrow \mathcal{C}$ be functors. Suppose that $\mathsf{mu(G{\bullet}F)}$ is an initial $\mathsf{G{\bullet}F}$-algebra. Then

$$\mathsf{F.mu(G{\bullet}F)}$$

is an initial $(\mathsf{F}\bullet\mathsf{G})$-algebra. So, for every initial $(\mathsf{F}\bullet\mathsf{G})$-algebra $\mathsf{mu}(\mathsf{F}\bullet\mathsf{G})$

$$\mathsf{F.mu}(\mathsf{G}\bullet\mathsf{F}) \cong \mathsf{mu}(\mathsf{F}\bullet\mathsf{G}) \quad .$$

As a consequence we also have an isomorphism in the base category, i.e.

$$\mathsf{F}.\mu(\mathsf{G}\bullet\mathsf{F}) \cong \mu(\mathsf{F}\bullet\mathsf{G}) \quad .$$

Specifics In both isomorphisms

$$(\!|\mathsf{F.mu}(\mathsf{G}\bullet\mathsf{F}) \;=:\; \mathsf{mu}(\mathsf{F}\bullet\mathsf{G})|\!)$$

is the arrow to $\mathsf{F}.\mu(\mathsf{G}\bullet\mathsf{F})$ from $\mu(\mathsf{F}\bullet\mathsf{G})$ and

$$\mathsf{mu}(\mathsf{F}\bullet\mathsf{G}) \circ \mathsf{F}.(\!|\mathsf{G.mu}(\mathsf{F}\bullet\mathsf{G}) \;=:\; \mathsf{mu}(\mathsf{G}\bullet\mathsf{F})|\!)$$

is its inverse.
$\square$
This theorem is stated and proved in [10].

6.5 The Square Theorem

Suppose f is a monotonic endofunction on a partially ordered set $\mathcal{C} = (\mathsf{C}, \sqsubseteq)$. (We assume nothing else about $\mathcal{C}$.) Suppose that $\mathsf{f}\bullet\mathsf{f}$ has a least prefix point $\mu(\mathsf{f}\bullet\mathsf{f})$. Then f also has a least prefix point (indeed $\mu(\mathsf{f}\bullet\mathsf{f})$ is itself the least prefix point of f). The proof is straightforward. Specifically, suppose x is a prefix point of f. Then x is a prefix point of $\mathsf{f}\bullet\mathsf{f}$ as shown by the following elementary calculation:

$$\begin{array}{ll}
 & \mathsf{x} \sqsupseteq \mathsf{f.f.x} \\
\Leftarrow & \quad \{ \qquad \text{transitivity} \quad \} \\
 & \mathsf{x} \sqsupseteq \mathsf{f.x} \wedge \mathsf{f.x} \sqsupseteq \mathsf{f.f.x} \\
\Leftarrow & \quad \{ \qquad \mathsf{f} \text{ is monotonic} \quad \} \\
 & \mathsf{x} \sqsupseteq \mathsf{f.x} \quad .
\end{array}$$

The prefix points of f thus form a subset of the prefix points of $\mathsf{f}\bullet\mathsf{f}$ and $\mu(\mathsf{f}\bullet\mathsf{f})$, being a lower bound on all prefix points of $\mathsf{f}\bullet\mathsf{f}$, is a lower bound on all prefix points of f. It follows that it is a *least* prefix point of f if it is a prefix point of f. But, by the rolling rule $\mu(\mathsf{f}\bullet\mathsf{f}) = \mathsf{f}\,.\,\mu(\mathsf{f}\bullet\mathsf{f})$ and so this is indeed the case.

In this section we prove the categorical version of this theorem. The theorem is due to Freyd [10] who called it the *iterated square* theorem. Our proof follows the lines of the above lattice-theoretic argument, in particular exploiting the rolling rule. (Freyd does not use the rolling rule; instead he uses the square theorem to prove the mutual recursion theorem from which he derives the rolling rule. See the discussion of the mutual recursion theorem.)

Suppose F is an endofunctor and suppose $\mathsf{F}\bullet\mathsf{F}$ has an initial algebra $\mathsf{mu}(\mathsf{F}\bullet\mathsf{F})$. The goal is to prove that F has an initial algebra.

One might suppose from the lattice-theoretic proof that the first step is to establish that any F-algebra is an (F•F)-algebra. This however is not the case. (It is the case that any *carrier* of an F-algebra is a *carrier* of an (F•F)-algebra.) What is the case is that there is a functor mapping F-algebras to (F•F)-algebras. Proving this property is our first task.

Suppose f is an F-algebra with carrier x. (I.e. $f \in x \leftarrow \mathsf{F}.x$.) We construct an (F•F)-algebra with carrier x as follows:

$$
\begin{array}{ll}
 & \alpha \in x \leftarrow \mathsf{F}.\mathsf{F}.x \\
\Leftarrow & \{ \qquad \bullet \quad \alpha = \beta\circ\gamma \quad \} \\
 & \beta \in x \leftarrow \mathsf{F}.x \ \wedge\ \gamma \in \mathsf{F}.x \leftarrow \mathsf{F}.\mathsf{F}.x \\
\Leftarrow & \{ \qquad \mathsf{F} \text{ is a functor; } \bullet \quad \gamma = \mathsf{F}.\beta \quad \} \\
 & \beta \in x \leftarrow \mathsf{F}.x \\
\Leftarrow & \{ \qquad \text{assumption} \quad \} \\
 & \beta = f \ .
\end{array}
$$

We conclude that $f \mapsto f\circ\mathsf{F}.f$ is a mapping from the objects of $\mathsf{Alg.F}$ to the objects of $\mathsf{Alg.(F\bullet F)}$. For the arrow part of the functor we are trying to construct it is a simple matter to derive that

$$
\alpha \ \in\ f\circ\mathsf{F}.f \xleftarrow{\mathsf{Alg.(F\bullet F)}} g\circ\mathsf{F}.g \quad \Leftarrow \quad \alpha \in f \xleftarrow{\mathsf{Alg.F}} g \ .
$$

(We leave this calculation to the reader.) Thus the identity function is what we are looking for (and because it is the identity function we know immediately that we have constructed a functor).

Let us denote the functor we have constructed by Sq. By definition, $\mathsf{Sq}.f = f\circ\mathsf{F}.f$ for all F-algebras f, and $\mathsf{Sq}.\alpha = \alpha$ for all arrows $\alpha \in f \xleftarrow{\mathsf{Alg.F}} g$.

We next remark that $\mathsf{mu}(\mathsf{F}\bullet\mathsf{F})$ is in the range of the (object part of the) functor Sq. Specifically, by construction of Θ,

$$
\begin{array}{ll}
 & \mathsf{mu}(\mathsf{F}\bullet\mathsf{F}) = \mathsf{Sq}.\Theta \\
\equiv & \{ \qquad \text{Lambek's lemma (5.21) and lemma (5.17);} \\
 & \qquad\ \ \mathsf{mu}(\mathsf{F}\bullet\mathsf{F}) \text{ is the unique arrow in } \mathsf{Alg.(F\bullet F)} \\
 & \qquad\ \ \text{to } \mathsf{mu}(\mathsf{F}\bullet\mathsf{F}) \text{ from } (\mathsf{F}\bullet\mathsf{F}).\mathsf{mu}(\mathsf{F}\bullet\mathsf{F}) \quad \} \\
 & \mathsf{Sq}.\Theta \ \in\ \mathsf{mu}(\mathsf{F}\bullet\mathsf{F}) \xleftarrow{\mathsf{Alg.(F\bullet F)}} (\mathsf{F}\bullet\mathsf{F}).\mathsf{mu}(\mathsf{F}\bullet\mathsf{F}) \\
\Leftarrow & \{ \qquad \mathsf{Sq}.\Theta = \Theta\circ\mathsf{F}.\Theta \text{ by definition;} \\
 & \qquad\ \ \text{straightforward calculation using the definition} \\
 & \qquad\ \ \text{of arrows in } \mathsf{Alg.(F\bullet F)} \quad \}
\end{array}
$$

$$\begin{array}{ll}
& \Theta \in \mathsf{mu(F{\bullet}F)} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{F.mu(F{\bullet}F)} \\
\Leftarrow & \quad \{ \quad \text{rolling rule} \quad \} \\
& \Theta = \mathsf{mu(F{\bullet}F)} \circ \mathsf{F.}(\!|\mathsf{F.mu(F{\bullet}F)} \;=:\; \mathsf{mu(F{\bullet}F)}|\!) \quad .
\end{array}$$

Note that, although we have given the details of the construction of Θ, we shall have no need of those details. The property we will use of Θ is that it is an isomorphism between $\mathsf{mu(F{\bullet}F)}$ and $\mathsf{F.mu(F{\bullet}F)}$.

We now want to verify that Θ is an initial F-algebra. That is, we must show that there is a unique arrow in $\mathsf{Alg.F}$ to each F-algebra from Θ. Now, supposing f is an F-algebra,

$$\begin{array}{ll}
& \alpha \in \mathsf{f} \xleftarrow{\mathsf{Alg.F}} \Theta \\
\Rightarrow & \quad \{ \quad \text{definition of } \mathsf{Sq} \quad \} \\
& \alpha \in \mathsf{Sq.f} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{Sq.}\Theta \\
\equiv & \quad \{ \quad \mathsf{Sq.}\Theta = \mathsf{mu(F{\bullet}F)} \quad \} \\
& \alpha = (\!|\mathsf{Alg.(F{\bullet}F)}\,;\,\mathsf{Sq.f} \;=:\; \mathsf{mu(F{\bullet}F)}|\!) \quad .
\end{array}$$

Initiality of Θ demands that we establish an equivalence rather than an implication. Thus the proof is completed by showing that

$$(\!|\mathsf{Alg.(F{\bullet}F)}\,;\,\mathsf{Sq.f} \;=:\; \mathsf{mu(F{\bullet}F)}|\!) \in \mathsf{f} \xleftarrow{\mathsf{Alg.F}} \Theta \quad .$$

Let us introduce the abbreviation Ψ for $(\!|\mathsf{Alg.(F{\bullet}F)}\,;\,\mathsf{Sq.f} \;=:\; \mathsf{mu(F{\bullet}F)}|\!)$. Then,

$$\begin{array}{ll}
& \Psi \in \mathsf{f} \xleftarrow{\mathsf{Alg.F}} \Theta \\
\equiv & \quad \{ \quad \text{definition of } \mathsf{Alg.F} \quad \} \\
& \Psi \in \mathsf{cod.f} \leftarrow \mathsf{cod.}\Theta \;\wedge\; \mathsf{f} \circ \mathsf{F.}\Psi = \Psi \circ \Theta \\
\equiv & \quad \{ \quad \Psi \in \mathsf{Sq.f} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{Sq.}\Theta\text{, codomain calculus} \quad \} \\
& \mathsf{f} \circ \mathsf{F.}\Psi = \Psi \circ \Theta \\
\equiv & \quad \{ \quad \Theta \text{ is an isomorphism} \quad \} \\
& \mathsf{f} \circ \mathsf{F.}\Psi \circ \Theta^{\cup} = \Psi \\
\equiv & \quad \{ \quad \text{unicity of } \Psi \quad \} \\
& \mathsf{f} \circ \mathsf{F.}\Psi \circ \Theta^{\cup} \in \mathsf{Sq.f} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{mu(F{\bullet}F)} \\
\Leftarrow & \quad \{ \quad \Theta^{\cup} \in \mathsf{F.mu(F{\bullet}F)} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{mu(F{\bullet}F)} \quad \} \\
& \mathsf{f} \circ \mathsf{F.}\Psi \in \mathsf{Sq.f} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{F.mu(F{\bullet}F)} \\
\equiv & \quad \{ \quad \Psi \in \mathsf{Sq.f} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{mu(F{\bullet}F)}\text{, corollary 5.12} \quad \} \\
& \mathsf{f} \in \mathsf{Sq.f} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{F.(Sq.f)} \\
\equiv & \quad \{ \quad \mathsf{F.(Sq.f)} = \mathsf{F.(f \circ F.f)} = \mathsf{F.f \circ F.(F.f)} = \mathsf{Sq.(F.f)} \quad \}
\end{array}$$

$$\begin{array}{ll}
 & \mathsf{f} \in \mathsf{Sq.f} \xleftarrow{\mathsf{Alg.(F{\bullet}F)}} \mathsf{Sq.(F.f)} \\
\Leftarrow & \{ \quad \mathsf{f} = \mathsf{Sq.f} \text{ (for } \textit{arrow} \ \mathsf{f}\text{)}, \ \mathsf{Sq} \text{ is a functor} \quad \} \\
 & \mathsf{f} \in \mathsf{f} \xleftarrow{\mathsf{Alg.F}} \mathsf{F.f} \\
\equiv & \{ \quad \text{lemma 5.17} \quad \} \\
 & \textit{true} \quad .
\end{array}$$

This completes the proof. Summarising what we have shown, we have:

Theorem 6.25 (Square Theorem) If F is an endofunctor and $\mathsf{mu(F{\bullet}F)}$ is an initial $(\mathsf{F{\bullet}F})$-algebra then $\Theta = \mathsf{mu(F{\bullet}F)} \circ \mathsf{F}.(\!|\mathsf{F.mu(F{\bullet}F)} \ =: \ \mathsf{mu(F{\bullet}F)}|\!)$ is an initial F-algebra. Moreover, for all F-algebras f,

$$(\!|\mathsf{Alg.(F{\bullet}F)}\,;\ \mathsf{f} \circ \mathsf{F.f} =: \mathsf{mu(F{\bullet}F)}|\!) = (\!|\mathsf{Alg.F}\,;\mathsf{f} =: \Theta|\!) \quad .$$

□

6.6 The Exchange Rule

In lattice theory, we can combine the rolling and fusion rule to establish yet another useful equality between least prefix points. More specifically: Given are two complete partially ordered sets $\mathcal{C}$ and $\mathcal{D}$ and three monotonic functions $\mathsf{f} \in \mathcal{C} \leftarrow \mathcal{D}$, $\mathsf{g} \in \mathcal{D} \leftarrow \mathcal{C}$ and $\mathsf{h} \in \mathcal{D} \leftarrow \mathcal{C}$ such that both g and h have an upper adjoint. Then the theorem we call the *exchange rule* states

(6.26) $\quad \mu(\mathsf{f{\bullet}g}) = \mu(\mathsf{f{\bullet}h}) \Leftarrow \mathsf{h{\bullet}f{\bullet}g} = \mathsf{g{\bullet}f{\bullet}h} \quad .$

A possible proof of this theorem is the following. We first calculate as follows: (for this calculation suppose f and g are monotonic endofunctions)

$$\begin{array}{ll}
 & \mu\mathsf{f} = \mu\mathsf{g} \\
\equiv & \{ \quad \sqsupseteq \text{ is anti-symmetric} \quad \} \\
 & \mu\mathsf{f} \sqsupseteq \mu\mathsf{g} \wedge \mu\mathsf{g} \sqsupseteq \mu\mathsf{f} \\
\Leftarrow & \{ \quad \text{least prefix point } \mu\mathsf{f} \text{ and } \mu\mathsf{g} \quad \} \\
 & \mu\mathsf{f} \sqsupseteq \mathsf{g}.\mu\mathsf{f} \ \wedge \ \mu\mathsf{g} \sqsupseteq \mathsf{f}.\mu\mathsf{g} \\
\Leftarrow & \{ \quad \sqsupseteq \text{ is reflexive} \quad \} \\
 & \mu\mathsf{f} = \mathsf{g}.\mu\mathsf{f} \ \wedge \ \mu\mathsf{g} = \mathsf{f}.\mu\mathsf{g} \quad .
\end{array}$$

Summarising,

(6.27) $\quad \mu\mathsf{f} = \mu\mathsf{g} \ \Leftarrow \ \mu\mathsf{f} = \mathsf{g}.\mu\mathsf{f} \ \wedge \ \mu\mathsf{g} = \mathsf{f}.\mu\mathsf{g} \quad .$

Using this calculation we prove (6.26):

$$
\begin{array}{ll}
 & \mu(\mathsf{f}\bullet\mathsf{g}) = \mu(\mathsf{f}\bullet\mathsf{h}) \\
\Leftarrow & \{ \quad (6.27) \quad \} \\
 & \mu(\mathsf{f}\bullet\mathsf{g}) = \mathsf{f}.\mathsf{h}.\mu(\mathsf{f}\bullet\mathsf{g}) \ \wedge\ \mu(\mathsf{f}\bullet\mathsf{h}) = \mathsf{f}.\mathsf{g}.\mu(\mathsf{f}\bullet\mathsf{h}) \\
\equiv & \{ \quad \mu\text{-fusion rule (twice), } \mathsf{g} \text{ and } \mathsf{h} \text{ have an upper adjoint,} \\
 & \quad\ \ \mathsf{h}\bullet\mathsf{f}\bullet\mathsf{g} = \mathsf{g}\bullet\mathsf{f}\bullet\mathsf{h} \quad \} \\
 & \mu(\mathsf{f}\bullet\mathsf{g}) = \mathsf{f}.\mu(\mathsf{g}\bullet\mathsf{f}) \ \wedge\ \mu(\mathsf{f}\bullet\mathsf{h}) = \mathsf{f}.\mu(\mathsf{h}\bullet\mathsf{f}) \\
\equiv & \{ \quad \text{rolling rule (twice)} \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

In this section we are going to prove a similar result in category theory.

We begin by constructing a similar result to (6.27). Given is the category $\mathcal{C}$ and two endofunctors F and G on $\mathcal{C}$ which both have initial algebras denoted by muF and muG respectively. We want to show that we can construct an initial G-algebra on the carrier $\mu\mathsf{F}$ and conversely construct an initial F-algebra on the carrier $\mu\mathsf{G}$. As a consequence we then have $\mu\mathsf{F} \cong \mu\mathsf{G}$.

We introduce the following assumption: there is an isomorphism $\eta \in \mathsf{F}\bullet\mathsf{G} \cong \mathsf{G}\bullet\mathsf{F}$. Then, by lemma 5.6, $\mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}$ is a G-algebra and $\mathsf{G.muF} \circ \eta^{\cup}_{\mu\mathsf{F}}$ is an F-algebra. Thus, (6.27) corresponds to the assumption that $\mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}$ is an *initial* G-algebra and $\mathsf{G.muF} \circ \eta^{\cup}_{\mu\mathsf{F}}$ is an *initial* F-algebra. We calculate as follows. $\mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}$ has a unique arrow to any other G-algebra, in particular to muG. I.e.

(6.28) $(\!|\mathsf{muG} =: \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}|\!) \in \mathsf{muG} \xleftarrow{\mathsf{Alg.G}} \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}$

Let us use Θ to denote $(\!|\mathsf{muG} =: \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}|\!)$. This abbreviation is used throughout the rest of this section. So, we now have

(6.29) $\Theta = (\!|\mathsf{muG} =: \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}|\!)$.

Furthermore, (6.28) can now be restated as:

(6.30) $\Theta \in \mathsf{muG} \xleftarrow{\mathsf{Alg.G}} \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}$.

Finally, from (6.30) and the definition of an arrow in $\mathsf{Alg.G}$ the following computation rule follows immediately:

(6.31) $\mathsf{muG} \circ \mathsf{G}.\Theta = \Theta \circ \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}$.

To see that Θ is an F-algebra we observe that it is also an arrow in the base category $\mathcal{C}$. More specifically:

(6.32) $\Theta \in \mu\mathsf{G} \xleftarrow{\mathcal{C}} \mathsf{F}.\mu\mathsf{G}$.

We now show that Θ is an initial F-algebra, by showing that it is isomorphic to muF. By definition we have

(6.33) $\quad (\!|\Theta =: \mathsf{muF}|\!) \in \Theta \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{muF}$.

This gives us half of the required isomorphism. For the arrow in the other direction observe that, symmetrically, by interchanging F and G and η and $\eta^{\cup}$ in (6.28) we also have an arrow Ψ with the following properties:

(6.34) $\quad \Psi = (\!|\mathsf{muF} =: \mathsf{G.muF} \circ \eta^{\cup}_{\mu\mathsf{F}}|\!)$,

(6.35) $\quad \Psi \in \mathsf{muF} \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{G.muF} \circ \eta^{\cup}_{\mu\mathsf{F}}$,

(6.36) $\quad \mathsf{muF} \circ \mathsf{F}.\Psi = \Psi \circ \mathsf{G.muF} \circ \eta^{\cup}_{\mu\mathsf{F}}$,

and finally:

$$\Psi \in \mu\mathsf{F} \overset{\mathcal{C}}{\longleftarrow} \mathsf{G}.\mu\mathsf{F} \quad .$$

So by definition we also have:

(6.37) $\quad (\!|\Psi =: \mathsf{muG}|\!) \in \Psi \overset{\mathsf{Alg.G}}{\longleftarrow} \mathsf{muG}$.

Thus, in the base category $(\!|\Psi =: \mathsf{muG}|\!)$ is an arrow to $\mathsf{cod}.\Psi$ from $\mu\mathsf{G}$, i.e. to $\mu\mathsf{F}$ from $\mathsf{cod}.\Theta$. We conjecture that $(\!|\Psi =: \mathsf{muG}|\!)$ is the inverse to $(\!|\Theta =: \mathsf{muF}|\!)$. Using theorem 5.19 it suffices to show that

$$\mathsf{muF} \circ \mathsf{F}.(\!|\Psi =: \mathsf{muG}|\!) = (\!|\Psi =: \mathsf{muG}|\!) \circ \Theta \quad .$$

The verification goes as follows:

$$\begin{array}{cl}
 & \mathsf{muF} \circ \mathsf{F}.(\!|\Psi =: \mathsf{muG}|\!) = (\!|\Psi =: \mathsf{muG}|\!) \circ \Theta \\
\equiv & \qquad \{ \quad \text{lemma 5.20, definition of } \Theta\text{: (6.29)} \quad \} \\
 & \mathsf{muF} \circ \mathsf{F}.(\!|\Psi =: \mathsf{muG}|\!) = (\!|\Psi =: \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}}|\!) \\
\equiv & \qquad \{ \quad \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}} \text{ is an initial G-algebra: (5.18)} \quad \} \\
 & \mathsf{muF} \circ \mathsf{F}.(\!|\Psi =: \mathsf{muG}|\!) \in \Psi \overset{\mathsf{Alg.G}}{\longleftarrow} \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}} \\
\equiv & \qquad \{ \quad \eta \in \mathsf{F} \bullet \mathsf{G} \cong \mathsf{G} \bullet \mathsf{F}\text{, (5.6):} \\
 & \qquad\qquad \mathsf{F}.(\!|\Psi =: \mathsf{muG}|\!) \in \mathsf{F}.\Psi \circ \eta_{\mu\mathsf{F}} \overset{\mathsf{Alg.G}}{\longleftarrow} \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}} \quad \} \\
 & \mathsf{muF} \in \Psi \overset{\mathsf{Alg.G}}{\longleftarrow} \mathsf{F}.\Psi \circ \eta_{\mu\mathsf{F}} \\
\equiv & \qquad \{ \quad \text{definition arrow in } \mathsf{Alg.G}\text{, typing base category trivial} \quad \} \\
 & \Psi \circ \mathsf{G.muF} = \mathsf{muF} \circ \mathsf{F}.\Psi \circ \eta_{\mu\mathsf{F}} \\
\equiv & \qquad \{ \quad (6.36) \quad \} \\
 & \mathit{true} \quad .
\end{array}$$

In summary we have proven the following:

$$(\!|\Theta =: \mathsf{muF}|\!) \ \in \ \Theta \xleftarrow{\mathsf{Alg.F}} \mathsf{muF} \quad ,$$

and

$$(\!|\Psi =: \mathsf{muG}|\!) \in \mathsf{muF} \xleftarrow{\mathsf{Alg.F}} \Theta \quad .$$

From the assumed initiality of muF we know that all the arrows in $\mathsf{Alg.F}$ from muF to another F-algebra are unique, in particular the identity arrow $\mathsf{id}_{\mathsf{muF}}$ is the unique arrow to muF from muF. But the composition of $(\!|\Psi =: \mathsf{muG}|\!)$ and $(\!|\Theta =: \mathsf{muF}|\!)$ is also an arrow in $\mathsf{Alg.F}$ to muF from muF, so we immediately have:

$$(\!|\Psi =: \mathsf{muG}|\!) \circ (\!|\Theta =: \mathsf{muF}|\!) \ = \ \mathsf{id}^{\mathsf{Alg.F}}_{\mathsf{muF}} \quad .$$

Symmetrically we also have:

$$(\!|\Theta =: \mathsf{muF}|\!) \circ (\!|\Psi =: \mathsf{muG}|\!) \ = \ \mathsf{id}^{\mathsf{Alg.G}}_{\mathsf{muG}} \quad .$$

But, the identity arrows in an algebra are the identity arrows in the base category, so we have:

$$\begin{array}{ll} & \mathsf{id}^{\mathsf{Alg.G}}_{\mathsf{muG}} \\ = & \mathsf{id}^{\mathcal{C}}_{\mu\mathsf{G}} \\ = & \mathsf{id}^{\mathcal{C}}_{\mathsf{cod}.\Theta} \\ = & \mathsf{id}^{\mathsf{Alg.F}}_{\Theta} \quad . \end{array}$$

I.e. muF and Θ are isomorphic in $\mathsf{Alg.F}$. Symmetrically, muG and Ψ are isomorphic in $\mathsf{Alg.G}$. Finally we have that $\mu\mathsf{F}$ and $\mu\mathsf{G}(=\mathsf{cod}.\Theta)$ are isomorphic in the base category $\mathcal{C}$.

In summary, we have proven:

Lemma 6.38 Given is the category $\mathcal{C}$ and two endofunctors F and G on $\mathcal{C}$ which both have an initial algebra denoted by muF and muG respectively. Given also is an isomorphism $\eta \in \mathsf{F}\bullet\mathsf{G} \cong \mathsf{G}\bullet\mathsf{F}$. Then:

$$\begin{array}{ll} & \sigma \in \mathsf{muG} \cong_{\mathsf{G}} \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}} \ \wedge \ \tau \in \mathsf{muF} \cong_{\mathsf{F}} \mathsf{G.muF} \circ \eta^{\cup}_{\mu\mathsf{F}} \\ \Rightarrow & \quad (\!|\sigma =: \mathsf{muF}|\!) \in \sigma \cong_{\mathsf{F}} \mathsf{muF} \wedge (\!|\tau =: \mathsf{muG}|\!) \in \tau \cong_{\mathsf{G}} \mathsf{muG} \\ & \wedge \ (\!|\sigma =: \mathsf{muF}|\!)^{\cup} = (\!|\tau =: \mathsf{muG}|\!) \quad . \end{array}$$

Moreover, σ is an F-algebra with $\mu\mathsf{G}$ as carrier, so for the base category we have:

$$\begin{array}{ll} & \sigma \in \mathsf{muG} \cong_{\mathsf{G}} \mathsf{F.muG} \circ \eta_{\mu\mathsf{G}} \ \wedge \ \tau \in \mathsf{muF} \cong_{\mathsf{F}} \mathsf{G.muF} \circ \eta^{\cup}_{\mu\mathsf{F}} \\ \Rightarrow & (\!|\sigma =: \mathsf{muF}|\!) \in \mu\mathsf{G} \cong_{\mathcal{C}} \mu\mathsf{F} \wedge (\!|\sigma =: \mathsf{muF}|\!)^{\cup} = (\!|\tau =: \mathsf{muG}|\!) \quad . \end{array}$$

□

We can now continue with the proof of the exchange rule. Given are two categories, $\mathcal{C}$ and $\mathcal{D}$, and three functors $\mathsf{F}\in\mathcal{C}\leftarrow\mathcal{D}$, $\mathsf{G}\in\mathcal{D}\leftarrow\mathcal{C}$ and $\mathsf{H}\in\mathcal{D}\leftarrow\mathcal{C}$ such that G is the lower adjoint in an adjunction and H is also the lower adjoint in an adjunction. (Details of the two adjunctions will not be needed in this section. We will simply make use of the results of the fusion rule.) Futhermore, an initial $(\mathsf{F}\bullet\mathsf{G})$-algebra and an initial $(\mathsf{F}\bullet\mathsf{H})$-algebra exist denoted by $\mathsf{mu}(\mathsf{F}\bullet\mathsf{G})$ and $\mathsf{mu}(\mathsf{F}\bullet\mathsf{H})$ respectively. We will show that

$$\mu(\mathsf{F}\bullet\mathsf{G})\cong\mu(\mathsf{F}\bullet\mathsf{H}) \;\Leftarrow\; \mathsf{mirror}\in\mathsf{H}\bullet\mathsf{F}\bullet\mathsf{G}\cong\mathsf{G}\bullet\mathsf{F}\bullet\mathsf{H} \quad .$$

We recall that the rolling rule and fusion rule give the existence of two isomorphisms of the following type: (for appropriate F, G, H and swap)

$$\mathsf{roll}_{\mathsf{F},\mathsf{G}} \in \mathsf{mu}(\mathsf{F}\bullet\mathsf{G}) \cong_{\mathsf{F}\bullet\mathsf{G}} \mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F}) \quad ,$$

and, for F a lower adjoint,

$$\mathsf{fuse}_{\mathsf{F},\mathsf{G},\mathsf{H}}.\mathsf{swap} \in \mathsf{muH} \cong_{\mathsf{H}} \mathsf{F}.\mathsf{muG}\circ\mathsf{swap}_{\mu\mathsf{G}} \quad .$$

We prove the isomorphism of $\mu(\mathsf{F}\bullet\mathsf{G})$ and $\mu(\mathsf{F}\bullet\mathsf{H})$ and simultaneously construct the witnesses α and β.

$$\begin{array}{ll}
 & \alpha\in\mu(\mathsf{F}\bullet\mathsf{G})\cong_{\mathcal{C}}\mu(\mathsf{F}\bullet\mathsf{H}) \;\wedge\; \beta=\alpha^{\cup} \\
\Leftarrow & \{\quad \text{lemma 6.38:}\;\; \mathsf{F},\mathsf{G} := (\mathsf{F}\bullet\mathsf{H}),(\mathsf{F}\bullet\mathsf{G}) \\
 & \quad\;\; \text{above,} \;\bullet\quad \alpha=(\!|\gamma =: \mathsf{mu}(\mathsf{F}\bullet\mathsf{H})|\!)\wedge\beta=(\!|\delta =: \mathsf{mu}(\mathsf{F}\bullet\mathsf{G})|\!) \quad\} \\
 & \quad\gamma \in \mathsf{mu}(\mathsf{F}\bullet\mathsf{G}) \cong_{\mathsf{F}\bullet\mathsf{G}} (\mathsf{F}\bullet\mathsf{H}).\mathsf{mu}(\mathsf{F}\bullet\mathsf{G})\circ\eta_{\mu(\mathsf{F}\bullet\mathsf{G})} \\
 & \wedge\;\delta \in \mathsf{mu}(\mathsf{F}\bullet\mathsf{H}) \cong_{\mathsf{F}\bullet\mathsf{H}} (\mathsf{F}\bullet\mathsf{G}).\mathsf{mu}(\mathsf{F}\bullet\mathsf{H})\circ\eta^{\cup}_{\mu(\mathsf{F}\bullet\mathsf{H})} \\
 & \wedge\;\eta\in\mathsf{F}\bullet\mathsf{H}\bullet\mathsf{F}\bullet\mathsf{G}\cong\mathsf{F}\bullet\mathsf{G}\bullet\mathsf{F}\bullet\mathsf{H} \quad . \\
\Leftarrow & \{\quad\bullet\quad \eta=\mathsf{F}\bullet\mathsf{mirror} \quad\} \\
 & \quad\gamma \in \mathsf{mu}(\mathsf{F}\bullet\mathsf{G}) \cong_{\mathsf{F}\bullet\mathsf{G}} (\mathsf{F}\bullet\mathsf{H}).\mathsf{mu}(\mathsf{F}\bullet\mathsf{G})\circ(\mathsf{F}\bullet\mathsf{mirror})_{\mu(\mathsf{F}\bullet\mathsf{G})} \\
 & \wedge\;\delta \in \mathsf{mu}(\mathsf{F}\bullet\mathsf{H}) \cong_{\mathsf{F}\bullet\mathsf{H}} (\mathsf{F}\bullet\mathsf{G}).\mathsf{mu}(\mathsf{F}\bullet\mathsf{H})\circ(\mathsf{F}\bullet\mathsf{mirror})^{\cup}_{\mu(\mathsf{F}\bullet\mathsf{H})} \quad .
\end{array}$$

Further,

$$\begin{array}{ll}
 & (\mathsf{F}\bullet\mathsf{H}).\mathsf{mu}(\mathsf{F}\bullet\mathsf{G})\circ(\mathsf{F}\bullet\mathsf{mirror})_{\mu(\mathsf{F}\bullet\mathsf{G})} \\
= & \{\quad \text{Godement's rules,}\;\mathsf{F}\;\text{is a functor}\quad\} \\
 & \mathsf{F}.(\mathsf{H}.\mathsf{mu}(\mathsf{F}\bullet\mathsf{G})\circ\mathsf{mirror}_{\mu(\mathsf{F}\bullet\mathsf{G})}) \\
\cong_{\mathsf{F}\bullet\mathsf{G}} & \{\quad \text{fusion rule,}\;\bullet\quad \mathsf{F}.(\mathsf{fuse}_{\mathsf{H},(\mathsf{F}\bullet\mathsf{G}),(\mathsf{G}\bullet\mathsf{F})}.\mathsf{mirror})\;, \\
 & \quad\;\; \text{and lemma 5.12}\quad\} \\
 & \mathsf{F}.\mathsf{mu}(\mathsf{G}\bullet\mathsf{F})
\end{array}$$

$\cong_{F\bullet G}$ { rolling rule, • $\mathsf{roll}_{F,G}$ }

$\mathsf{mu}(F\bullet G)$

I.e.

$$(\!|\mathsf{roll}_{F,G} \circ F.(\mathsf{fuse}_{H,(F\bullet G),(G\bullet F)}.\mathsf{mirror}) \;=:\; \mathsf{mu}(F\bullet H)|\!) \in \mu(F\bullet G) \overset{\mathcal{C}}{\longleftarrow} \mu(F\bullet H) \quad ,$$

Symmetrically,

$$(\!|\mathsf{roll}_{F,H} \circ F.(\mathsf{fuse}_{G,(F\bullet H),(H\bullet F)}.(\mathsf{mirror}^{\cup})) \;=:\; \mathsf{mu}(F\bullet G)|\!) \in \mu(F\bullet H) \overset{\mathcal{C}}{\longleftarrow} \mu(F\bullet G) \quad .$$

Moreover these arrows are each others' inverses, so $\mu(F\bullet G)$ and $\mu(F\bullet H)$ are isomorphic. Finally, we can state the theorem in full.

Theorem 6.39 (Exchange Rule) Given are two categories, $\mathcal{C}$ and $\mathcal{D}$, and three functors $F \in \mathcal{C} \leftarrow \mathcal{D}$, $G \in \mathcal{D} \leftarrow \mathcal{C}$ and $H \in \mathcal{D} \leftarrow \mathcal{C}$ such that G is the lower adjoint in an adjunction and H is also the lower adjoint in an adjunction. Furthermore, we have the natural isomorphism $\mathsf{mirror} \in H\bullet F\bullet G \cong G\bullet F\bullet H$. Finally, we assume that an initial $(F\bullet G)$-algebra, $\mathsf{mu}(F\bullet G)$, and an initial $(F\bullet H)$-algebra, $\mathsf{mu}(F\bullet H)$, exist. Then

$$\mu(F\bullet G) \cong \mu(F\bullet H) \quad .$$

Specifics In this isomorphism

$$(\!|\mathsf{roll}_{F,G} \circ F.(\mathsf{fuse}_{H,(F\bullet G),(G\bullet F)}.\mathsf{mirror}) \;=:\; \mathsf{mu}(F\bullet H)|\!)$$

is the witness to $\mu(F\bullet G)$ from $\mu(F\bullet H)$ and

$$(\!|\mathsf{roll}_{F,H} \circ F.(\mathsf{fuse}_{G,(F\bullet H),(H\bullet F)}.(\mathsf{mirror}^{\cup})) \;=:\; \mathsf{mu}(F\bullet G)|\!)$$

is its inverse.
□

In the same way we can construct a similar result for $\mu(G\bullet F)$ and $\mu(H\bullet F)$. Specifically,

$$(\!|\mathsf{fuse}_{H,(F\bullet G),(G\bullet F)}.\mathsf{mirror} \circ H.\mathsf{roll}_{F,G} \;=:\; \mathsf{mu}(H\bullet F)|\!) \in \mu(G\bullet F) \overset{\mathcal{C}}{\longleftarrow} \mu(H\bullet F) \quad ,$$

and

$$(\!|\mathsf{fuse}_{G,(F\bullet H),(H\bullet F)}.(\mathsf{mirror}^{\cup}) \circ G.\mathsf{roll}_{F,H} \;=:\; \mathsf{mu}(G\bullet F)|\!) \in \mu(H\bullet F) \overset{\mathcal{C}}{\longleftarrow} \mu(G\bullet F) \quad .$$

Moreover these arrows are each others' inverses, so $\mu(G\bullet F)$ and $\mu(H\bullet F)$ are isomorphic.

Exercise 6.40 The two exchange rules can be combined into one. Specifically, suppose given are three categories, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, and four functors $F \in \mathcal{C} \leftarrow \mathcal{D}$, $G \in \mathcal{D} \leftarrow \mathcal{E}$, $H \in \mathcal{D} \leftarrow \mathcal{E}$ and $K \in \mathcal{E} \leftarrow \mathcal{C}$ such that G is the lower adjoint in an adjunction and H is also the lower adjoint in an adjunction. Furthermore, suppose we have the natural isomorphism $\mathsf{mirror} \in H\bullet K\bullet F\bullet G \cong G\bullet K\bullet F\bullet H$. Finally, we assume that an initial $(F\bullet G\bullet K)$-algebra, $\mathsf{mu}(F\bullet G\bullet K)$, and an initial $(F\bullet H\bullet K)$-algebra, $\mathsf{mu}(F\bullet H\bullet K)$, exist. Then

$$\mu(F\bullet G\bullet K) \cong \mu(F\bullet H\bullet K) \quad .$$

Establish this theorem.

□

6.7 The Diagonal Rule

Let $\mathcal{C}$ be a complete partially ordered set and $\oplus \in \mathcal{C} \leftarrow \mathcal{C} \times \mathcal{C}$ a monotonic binary function. Then the lattice theoretic *diagonal rule* states:

$$\mu(x \mapsto x \oplus x) = \mu(x \mapsto \mu(y \mapsto x \oplus y)) \quad .$$

In this section we derive the corresponding theorem in category theory. The unusual length of the section is due to the fact that we prove the existence of an initial algebra in two directions. More specifically, given an initial $(x \mapsto x \oplus x)$-algebra we construct an initial $(x \mapsto \mu(y \mapsto x \oplus y))$-algebra and,vice versa, given an initial $(x \mapsto \mu(y \mapsto x \oplus y))$-algebra we construct an initial $(x \mapsto x \oplus x)$-algebra.

Given are a category $\mathcal{C}$ and a binary functor $\oplus \in \mathcal{C} \leftarrow \mathcal{C} \times \mathcal{C}$. Furthermore, we assume that for each element x in $\mathcal{C}$ an initial object $\mathsf{mu}(x\oplus)$ exists in $\mathsf{Alg}.(x\oplus)$. We begin with introducing the following notation. With the unary functor $\hat{\oplus}$ we denote the functor defined on objects x and arrows f by respectively

(6.41) $\hat{\oplus} x = x \oplus x \quad ,$

(6.42) $\hat{\oplus} f = f \oplus f \quad .$

Note that the functor $x \mapsto \mu(y \mapsto x \oplus y)$ corresponds to the map operator ϖ of section 5.4. With these assumptions, we prove that the following two statements are equivalent:

(a) an initial object $\mathsf{mu}\hat{\oplus}$ exists in $\mathsf{Alg}.\hat{\oplus}$,
(b) an initial object $\mathsf{mu}\varpi$ exists in $\mathsf{Alg}.\varpi$.

More specifically, given an initial algebra in one category, we will construct an initial algebra in the other category on the same carrier. We may then derive an isomorphism between the carriers of initial $\hat{\oplus}$-algebras and initial ϖ-algebras.

6.7.1 One Half

We begin with the proof that (a) implies (b). We make the assumption that $\mathsf{mu}\hat{\oplus}$ exists in $\mathsf{Alg}.\hat{\oplus}$ and we have to show that an initial object α exists in $\mathsf{Alg}.\varpi$ on the carrier $\mu\hat{\oplus}$.

The first step is to construct α such that it is an object of $\mathsf{Alg}.\varpi$, i.e. an arrow in the base category $\mathcal{C}$ to $\mu\hat{\oplus}$ from $\varpi.\mu\hat{\oplus}$.

$$\begin{array}{ll} & \alpha \in \mu\hat{\oplus} \leftarrow \varpi.\mu\hat{\oplus} \\ \equiv & \quad \{ \quad \text{definition of } \varpi \quad \} \\ & \alpha \in \mu\hat{\oplus} \leftarrow \mu((\mu\hat{\oplus})\oplus) \\ \Leftarrow & \quad \{ \quad \mathsf{mu}(x\oplus) \text{ is initial in } \mathsf{Alg}.(x\oplus), \end{array}$$

$$\bullet \quad \alpha = (\!|\beta \ =: \ \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!) \ . \quad \}$$

$$\beta \ \in \ \mu\hat{\oplus} \leftarrow \mu\hat{\oplus} \oplus \mu\hat{\oplus}$$

$$\Leftarrow \quad \{ \quad \text{initial } \hat{\oplus}\text{-algebra} \quad \}$$

$$\beta = \mathsf{mu}\hat{\oplus} \ .$$

We conclude from this calculation that $(\!|\mathsf{mu}\hat{\oplus} \ =: \ \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!) \in \mu\hat{\oplus} \leftarrow \varpi.\mu\hat{\oplus}$ is an object of $\mathsf{Alg}.\varpi$. For brevity let Θ denote $(\!|\mathsf{mu}\hat{\oplus} \ =: \ \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!)$. As computation rule we then immediately have

(6.43) $\quad \Theta \circ \mathsf{mu}((\mu\hat{\oplus})\oplus) = \mathsf{mu}\hat{\oplus} \circ \mathsf{id}_{\mu\hat{\oplus}} \oplus \Theta \ .$

Now suppose that $\mathsf{f} \in \mathsf{x} \leftarrow \varpi.\mathsf{x}$. We now have to construct a unique arrow (in $\mathsf{Alg}.\varpi$) to f from Θ. By construction of α:

$$\alpha \in \mathsf{f} \overset{\mathsf{Alg}.\varpi}{\longleftarrow} \Theta$$

$$\Leftarrow \quad \{ \quad (5.17)\text{: } \mathsf{f} \in \mathsf{f} \overset{\mathsf{Alg}.\varpi}{\longleftarrow} \varpi.\mathsf{f} \ , \ \bullet \quad \alpha = \mathsf{f} \circ \beta \quad \}$$

$$\beta \in \varpi.\mathsf{f} \overset{\mathsf{Alg}.\varpi}{\longleftarrow} \Theta$$

$$\equiv \quad \{ \quad \text{definition arrow in } \mathsf{Alg}.\varpi \quad \}$$

$$\varpi.(\mathsf{f} \circ \beta) = \beta \circ \Theta \ \wedge \ \beta \in \varpi.\mathsf{x} \leftarrow \mu\hat{\oplus}$$

$$\equiv \quad \{ \quad \mathsf{f} \circ \beta \in \mathsf{x} \leftarrow \mu\hat{\oplus}\text{: definition } \varpi \text{ (5.35), definition } \Theta \quad \}$$

$$(\!|\mathsf{mu}(\mathsf{x}\oplus) \circ (\mathsf{f} \circ \beta) \oplus \mathsf{id}_{\varpi.\mathsf{x}} \ =: \ \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!)$$

$$= \ \beta \circ (\!|\mathsf{mu}\hat{\oplus} \ =: \ \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!) \ \wedge \ \beta \in \varpi.\mathsf{x} \leftarrow \mu\hat{\oplus}$$

$$\Leftarrow \quad \{ \quad \text{fusion: (5.19e)} \quad \}$$

$$\mathsf{mu}(\mathsf{x}\oplus) \circ (\mathsf{f} \circ \beta) \oplus \mathsf{id}_{\varpi.\mathsf{x}} \circ \mathsf{id}_{\mu\hat{\oplus}} \oplus \beta \ = \ \beta \circ \mathsf{mu}\hat{\oplus} \ \wedge \ \beta \in \varpi.\mathsf{x} \leftarrow \mu\hat{\oplus}$$

$$\equiv \quad \{ \quad \text{coherence requirement binary functor } \oplus \quad \}$$

$$\mathsf{mu}(\mathsf{x}\oplus) \circ \mathsf{f} \oplus \mathsf{id}_{\varpi.\mathsf{x}} \circ \beta \oplus \beta \ = \ \beta \circ \mathsf{mu}\hat{\oplus} \ \wedge \ \beta \in \varpi.\mathsf{x} \leftarrow \mu\hat{\oplus}$$

$$\equiv \quad \{ \quad \text{uep: (5.19b)} \quad \}$$

$$\beta = (\!|\mathsf{mu}(\mathsf{x}\oplus) \circ \mathsf{f} \oplus \mathsf{id}_{\varpi.\mathsf{x}} \ =: \ \mathsf{mu}\hat{\oplus}|\!)$$

Thus, for all $\mathsf{f} \in \mathsf{Alg}.\varpi$ (where $\mathsf{cod.f} = \mathsf{x}$),

(6.44) $\quad \mathsf{f} \circ (\!|\mathsf{mu}(\mathsf{x}\oplus) \circ \mathsf{f} \oplus \mathsf{id}_{\varpi.\mathsf{x}} \ =: \ \mathsf{mu}\hat{\oplus}|\!) \in \mathsf{f} \overset{\mathsf{Alg}.\varpi}{\longleftarrow} \Theta \ .$

We now have to show that $\mathsf{f} \circ (\!|\mathsf{mu}(\mathsf{x}\oplus) \circ \mathsf{f} \oplus \mathsf{id}_{\varpi.\mathsf{x}} \ =: \ \mathsf{mu}\hat{\oplus}|\!)$ is the *unique* arrow in $\mathsf{Alg}.\varpi$ to f from Θ. Assume that $\varphi \in \mathsf{f} \overset{\mathsf{Alg}.\varpi}{\longleftarrow} \Theta$. That is, assume

(6.45) $\quad \varphi \in \mathsf{x} \leftarrow \mu\hat{\oplus}$ and

(6.46) $\quad \mathsf{f} \circ \varpi.\varphi \ = \ \varphi \circ \Theta \ .$

We want to show that $\varphi = \mathsf{f}\circ(\!|\mathsf{mu}(\mathsf{x}\oplus)\circ\mathsf{f}\oplus\mathsf{id}_{\varpi.\mathsf{x}} \ =:\ \mathsf{mu}\widehat{\oplus}|\!)$.

In order to exploit (6.46) we need first to find some arrow β such that

$$\Theta\circ\beta = \mathsf{id}_{\mu\widehat{\oplus}} \quad .$$

The calculation of β is carried out as follows:

$$\begin{array}{cl}
& \Theta\circ\beta = \mathsf{id}_{\mu\widehat{\oplus}} \\
\Leftarrow & \{ \quad \text{by (5.19d):} \quad \mathsf{id}_{\mu\widehat{\oplus}} = \mathsf{id}_{\mathsf{mu}\widehat{\oplus}} = (\!|\mathsf{mu}\widehat{\oplus} \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \quad , \\
& \qquad \bullet \quad \beta = (\!|\gamma \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \ , \ \text{aiming for fusion} \quad \} \\
& \Theta\circ(\!|\gamma \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \ = \ (\!|\mathsf{mu}\widehat{\oplus} \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \\
\Leftarrow & \{ \quad \text{fusion, } \Theta \in \mu\widehat{\oplus} \leftarrow \varpi.\mu\widehat{\oplus} \quad \} \\
& \mathsf{mu}\widehat{\oplus}\circ\Theta\oplus\Theta = \Theta\circ\gamma \ \wedge\ \mathsf{cod}.\gamma = \varpi.\mu\widehat{\oplus} \\
\Leftarrow & \{ \quad \bullet \quad \gamma \ = \ \mathsf{mu}((\mu\widehat{\oplus})\oplus)\circ\delta \ , \ \text{(6.43), Leibniz} \quad \} \\
& \Theta\oplus\Theta \ = \ \mathsf{id}_{\mu\widehat{\oplus}}\oplus\Theta\circ\delta \\
\Leftarrow & \{ \quad \oplus \text{ is a binary functor} \quad \} \\
& \delta \ = \ \Theta\oplus\mathsf{id}_{\varpi.\mu\widehat{\oplus}} \quad .
\end{array}$$

To summarise the latter calculation:

$$\Theta\circ(\!|\mathsf{mu}((\mu\widehat{\oplus})\oplus)\circ\Theta\oplus\mathsf{id}_{\varpi.\mu\widehat{\oplus}} \ =:\ \mathsf{mu}\widehat{\oplus}|\!) = \mathsf{id}_{\mu\widehat{\oplus}} \quad .$$

Now we go back to our main calculation.

$$\begin{array}{cl}
& \varphi = \mathsf{f}\circ(\!|\mathsf{mu}(\mathsf{x}\oplus)\circ\mathsf{f}\oplus\mathsf{id}_{\varpi.\mathsf{x}} \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \\
\equiv & \{ \quad \varphi \ = \ \varphi\circ\mathsf{id}_{\mu\widehat{\oplus}} \ , \ \text{above} \quad \} \\
& \quad \varphi\circ\Theta\circ(\!|\mathsf{mu}((\mu\widehat{\oplus})\oplus)\circ\Theta\oplus\mathsf{id}_{\varpi.\mu\widehat{\oplus}} \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \\
& = \ \mathsf{f}\circ(\!|\mathsf{mu}(\mathsf{x}\oplus)\circ\mathsf{f}\oplus\mathsf{id}_{\varpi.\mathsf{x}} \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \\
\Leftarrow & \{ \quad \text{assumption (6.46), Leibnitz to remove ``}\mathsf{f}\circ\text{''} \quad \} \\
& \quad \varpi.\varphi\circ(\!|\mathsf{mu}((\mu\widehat{\oplus})\oplus)\circ\Theta\oplus\mathsf{id}_{\varpi.\mu\widehat{\oplus}} \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \\
& = \ (\!|\mathsf{mu}(\mathsf{x}\oplus)\circ\mathsf{f}\oplus\mathsf{id}_{\varpi.\mathsf{x}} \ =:\ \mathsf{mu}\widehat{\oplus}|\!) \\
\Leftarrow & \{ \quad \text{fusion, } \varpi.\varphi \in \mu(\mathsf{x}\oplus) \leftarrow \mu((\mu\widehat{\oplus})\oplus) \quad \} \\
& \mathsf{mu}(\mathsf{x}\oplus)\circ\mathsf{f}\oplus\mathsf{id}_{\varpi.\mathsf{x}}\circ\varpi.\varphi\oplus\varpi.\varphi \ = \ \varpi.\varphi\circ\mathsf{mu}((\mu\widehat{\oplus})\oplus)\circ\Theta\oplus\mathsf{id}_{\varpi.\mu\widehat{\oplus}} \\
\equiv & \{ \quad \varphi \in \mathsf{x} \leftarrow \mu\widehat{\oplus} : \ \text{computation rule map (5.36)} \quad \} \\
& \mathsf{mu}(\mathsf{x}\oplus)\circ\mathsf{f}\oplus\mathsf{id}_{\varpi.\mathsf{x}}\circ\varpi.\varphi\oplus\varpi.\varphi \ = \ \mathsf{mu}(\mathsf{x}\oplus)\circ\varphi\oplus\varpi.\varphi\circ\Theta\oplus\mathsf{id}_{\varpi.\mu\widehat{\oplus}} \\
\Leftarrow & \{ \quad \text{coherence requirement for binary functors, Leibniz} \quad \} \\
& \mathsf{f}\circ\varpi.\varphi \ = \ \varphi\circ\Theta
\end{array}$$

$$
\begin{array}{ll}
\equiv & \{ \quad \text{assumption: (6.46)} \quad \} \\
& \mathit{true} \quad .
\end{array}
$$

This concludes the proof of (a) implies (b).

6.7.2 The Other Half

We now begin the proof that (b) implies (a). The assumption this time is that the category $\mathsf{Alg}.\varpi$ has initial object $\mathsf{mu}\varpi$. We have to show that there is an initial object α in $\mathsf{Alg}.\hat{\oplus}$ on the carrier $\mu\varpi$.

The steps in the proof are the same as in the first part. We first construct a candidate arrow α to $\mu\varpi$ from $\mu\varpi \oplus \mu\varpi$ in the base category. Then we assume that f is an object of $\mathsf{Alg}.\hat{\oplus}$ and construct a unique arrow to f from α in the category $\mathsf{Alg}.\hat{\oplus}$. For the first step we have the following calculation:

$$
\begin{array}{ll}
& \alpha \in \mu\varpi \leftarrow \mu\varpi \oplus \mu\varpi \\
\Leftarrow & \{ \quad \mathsf{mu}\varpi \in \mu\varpi \leftarrow \varpi.\mu\varpi \;\;, \bullet \quad \alpha = \mathsf{mu}\varpi \circ \beta \;, \quad \} \\
& \beta \in \varpi.\mu\varpi \leftarrow \mu\varpi \oplus \mu\varpi \\
\equiv & \{ \quad \text{definition of } \varpi \quad \} \\
& \beta \in \mu((\mu\varpi)\oplus) \leftarrow \mu\varpi \oplus \mu\varpi \\
\Leftarrow & \{ \quad \mathsf{mu}(x\oplus) \in \mu(x\oplus) \leftarrow x \oplus \mu(x\oplus) \;\;, \text{ for all } x\,. \\
& \qquad \bullet \quad \beta = \mathsf{mu}((\mu\varpi)\oplus) \circ \gamma \;. \quad \} \\
& \gamma \in \mu\varpi \oplus \mu((\mu\varpi)\oplus) \leftarrow \mu\varpi \oplus \mu\varpi \\
\Leftarrow & \{ \quad \text{definition of } \varpi\text{, theorem 5.21: } \mathsf{mu}\varpi \text{ has an} \\
& \qquad \text{inverse denoted by } (\mathsf{mu}\varpi)^{\cup} \in \varpi.\mu\varpi \leftarrow \mu\varpi \quad \} \\
& \gamma \;=\; \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \quad .
\end{array}
$$

Thus,

$$
\mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \in \mu\varpi \leftarrow \mu\varpi \oplus \mu\varpi \quad .
$$

Now suppose $f \in x \leftarrow x \oplus x$. We construct an arrow in $\mathsf{Alg}.\hat{\oplus}$ to f from $\mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup}$ as follows.

$$
\begin{array}{ll}
& \alpha \in f \xleftarrow{\mathsf{Alg}.\hat{\oplus}} \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \\
\equiv & \{ \quad \text{definition of an arrow in } \mathsf{Alg}.\hat{\oplus} \quad \} \\
& \quad \alpha \in x \leftarrow \mu\varpi \\
& \wedge \; f \circ \alpha \oplus \alpha \;=\; \alpha \circ \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \\
\equiv & \{ \quad \text{inverses, coherence property binary functor} \quad \}
\end{array}
$$

$$\begin{array}{ll}
& \alpha \in x \leftarrow \mu\varpi \;\wedge\; \mathsf{f} \circ \alpha \oplus (\alpha \circ \mathsf{mu}\varpi) \;=\; \alpha \circ \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \\
\equiv & \qquad \{ \qquad \text{coherence requirement binary functor} \quad \} \\
& \quad \alpha \in x \leftarrow \mu\varpi \\
& \wedge\; \mathsf{f} \circ \alpha \oplus \mathsf{id}_x \circ \mathsf{id}_{\mu\varpi} \oplus (\alpha \circ \mathsf{mu}\varpi) = (\alpha \circ \mathsf{mu}\varpi) \circ \mathsf{mu}((\mu\varpi)\oplus) \\
\equiv & \qquad \{ \qquad \text{uep: (5.19b)} \quad \} \\
& \alpha \in x \leftarrow \mu\varpi \;\wedge\; \alpha \circ \mathsf{mu}\varpi = (\!|\mathsf{f} \circ \alpha \oplus \mathsf{id}_x \;\; =: \; \mathsf{mu}((\mu\varpi)\oplus)|\!) \\
\equiv & \qquad \{ \qquad \text{definition } \varpi \text{ and theorem 5.41} \quad \} \\
& \alpha \in x \leftarrow \mu\varpi \;\wedge\; \alpha \circ \mathsf{mu}\varpi = (\!|\mathsf{f} \; =: \; \mathsf{mu}(x\oplus)|\!) \circ \varpi.\alpha \\
\equiv & \qquad \{ \qquad \text{uep: (5.19b)} \quad \} \\
& \alpha = (\!|(\!|\mathsf{f} \; =: \; \mathsf{mu}(x\oplus)|\!) \; =: \; \mathsf{mu}\varpi|\!)
\end{array}$$

Thus,

$$\begin{array}{ll}
(6.47) & (\!|(\!|\mathsf{f} \; =: \; \mathsf{mu}(x\oplus)|\!) \; =: \; \mathsf{mu}\varpi|\!) \\
\in & \mathsf{f} \xleftarrow{\mathsf{Alg}.\hat{\oplus}} \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup}
\end{array}$$

In the above construction all the steps are equivalences, so at the same time we also have proven the uniqueness of this arrow. This completes the proof that (b) implies (a).

We have now reached the last part of the proof. We assume that $\mathsf{mu}\hat{\oplus}$ is an initial object in $\mathsf{Alg}.\hat{\oplus}$, and that $\mathsf{mu}\varpi$ is an initial object in $\mathsf{Alg}.\varpi$. In the previous part we have proven that under these assumptions

$$(\!|\mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!)$$

is an initial ϖ-algebra and

$$\mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup}$$

is an initial $\hat{\oplus}$-algebra. So, by theorem 2.6, we have the following two isomorphisms

$$(\!|\mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!) \cong \mathsf{mu}\varpi \quad ,$$

$$\mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \;\cong\; \mathsf{mu}\hat{\oplus} \quad .$$

Furthermore, $(\!|\mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!)$ has $\mu\hat{\oplus}$ as carrier, so we also have an isomorphism in the base category

$$\mu\hat{\oplus} \cong \mu\varpi \quad .$$

(Note, that this also follows from the fact that $\mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup}$ has $\mu\varpi$ as carrier) We will now show the construction of the witnesses.

Substituting $\mathsf{mu}\varpi$ for f in (6.44) we obtain:

$$
\begin{array}{ll}
 & \mathsf{mu}\varpi \circ \llparenthesis \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{mu}\varpi \oplus \mathsf{id}_{\varpi.\mu\varpi} \;\; =: \; \mathsf{mu}\hat{\oplus} \rrparenthesis \\
\in & \mathsf{mu}\varpi \xleftarrow{\mathsf{Alg}.\varpi} \llparenthesis \mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus) \rrparenthesis \quad .
\end{array}
$$

Similarly, substituting $\mathsf{mu}\hat{\oplus}$ for f in (6.47) we obtain:

$$
\begin{array}{ll}
 & \llparenthesis \llparenthesis \mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus) \rrparenthesis \; =: \; \mathsf{mu}\varpi \rrparenthesis \\
\in & \mathsf{mu}\hat{\oplus} \xleftarrow{\mathsf{Alg}.\hat{\oplus}} \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \quad .
\end{array}
$$

From the assumed initiality of $\mathsf{mu}\varpi$ in $\mathsf{Alg}.\varpi$ we can conclude that

$$
\begin{array}{ll}
 & \llparenthesis \llparenthesis \mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus) \rrparenthesis \; =: \; \mathsf{mu}\varpi \rrparenthesis \\
\in & \llparenthesis \mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus) \rrparenthesis \xleftarrow{\mathsf{Alg}.\varpi} \mathsf{mu}\varpi \quad .
\end{array}
$$

Symmetry suggests that also

$$
\begin{array}{ll}
 & \mathsf{mu}\varpi \circ \llparenthesis \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{mu}\varpi \oplus \mathsf{id}_{\varpi.\mu\varpi} \;\; =: \; \mathsf{mu}\hat{\oplus} \rrparenthesis \\
\in & \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \xleftarrow{\mathsf{Alg}.\hat{\oplus}} \mathsf{mu}\hat{\oplus} \quad .
\end{array}
$$

Because of the initiality of $\mathsf{mu}\hat{\oplus}$ in $\mathsf{Alg}.\hat{\oplus}$ and (5.18) we can verify this as follows:

$$
\begin{array}{ll}
 & \quad \mathsf{mu}\varpi \circ \llparenthesis \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{mu}\varpi \oplus \mathsf{id}_{\varpi.\mu\varpi} \;\; =: \; \mathsf{mu}\hat{\oplus} \rrparenthesis \\
 & = \; \llparenthesis \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \;\; =: \; \mathsf{mu}\hat{\oplus} \rrparenthesis \\
\equiv & \qquad \{ \qquad \text{fusion, typing base category trivial} \quad \} \\
 & \quad \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{mu}\varpi \oplus \mathsf{id}_{\varpi.\mu\varpi} \\
 & = \; \mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \circ \mathsf{mu}\varpi \oplus \mathsf{mu}\varpi \\
\equiv & \qquad \{ \qquad \text{coherence requirements } \oplus \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

Arrows in an algebra category are also arrows in the base category, so we can conclude that the arrows

$$
\llparenthesis \llparenthesis \mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus) \rrparenthesis \; =: \; \mathsf{mu}\varpi \rrparenthesis
$$

and

$$
\mathsf{mu}\varpi \circ \llparenthesis \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{mu}\varpi \oplus \mathsf{id}_{\varpi.\mu\varpi} \;\; =: \; \mathsf{mu}\hat{\oplus} \rrparenthesis
$$

are the witnesses to all three isomorphisms. Finally, we state the theorem in full.

Theorem 6.48 (Diagonal Rule) Let $\mathcal{C}$ be a category and $\oplus \in \mathcal{C} \leftarrow \mathcal{C} \times \mathcal{C}$ be a binary functor. Furthermore, we assume that for each element x in $\mathcal{C}$ an initial object, $\mathsf{mu}(x\oplus)$, exists in $\mathsf{Alg}.(x\oplus)$. If $\mathsf{mu}\hat{\oplus}$ is an initial object in $\mathsf{Alg}.\hat{\oplus}$, then

$$
\llparenthesis \mathsf{mu}\hat{\oplus} \; =: \; \mathsf{mu}((\mu\hat{\oplus})\oplus) \rrparenthesis
$$

is an initial object in the category $\mathsf{Alg}.\varpi$. So, for every initial ϖ-algebra $\mathsf{mu}\varpi$

$$(\!|\mathsf{mu}\hat{\oplus} =: \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!) \cong \mathsf{mu}\varpi \quad .$$

Conversely, if $\mathsf{mu}\varpi$ is an initial object in $\mathsf{Alg}.\varpi$ then

$$\mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup}$$

is an initial object in the category $\mathsf{Alg}.\hat{\oplus}$. So, for every initial $\hat{\oplus}$-algebra $\mathsf{mu}\hat{\oplus}$,

$$\mathsf{mu}\varpi \circ \mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{id}_{\mu\varpi} \oplus (\mathsf{mu}\varpi)^{\cup} \;\cong\; \mathsf{mu}\hat{\oplus} \quad .$$

As a consequence we also have an isomorphism in the base category: $\mu\hat{\oplus} \cong \mu\varpi$, i.e.

$$\mu(x \mapsto x\oplus x) \cong \mu(x \mapsto \mu(y \mapsto x\oplus y)) \quad .$$

Specifics In all three isomorphisms

$$(\!|(\!|\mathsf{mu}\hat{\oplus} =: \mathsf{mu}((\mu\hat{\oplus})\oplus)|\!) =: \mathsf{mu}\varpi|\!)$$

is the arrow to $\mu\hat{\oplus}$ from $\mu\varpi$ and

$$\mathsf{mu}\varpi \circ (\!|\mathsf{mu}((\mu\varpi)\oplus) \circ \mathsf{mu}\varpi \oplus \mathsf{id}_{\varpi.\mu\varpi} =: \mathsf{mu}\hat{\oplus}|\!)$$

is its inverse.
□

6.8 Mutual Recursion

For our last fixed point rule we consider the problem of computing initial algebras in a product category. That is, supposing $\mathcal{C}$ and $\mathcal{D}$ are categories and F is an endofunctor on the product category $\mathcal{C}\times\mathcal{D}$, we determine what information is needed about initial algebras of endofunctors on $\mathcal{C}$ and on $\mathcal{D}$ in order to be able to compute an initial F-algebra. As we shall see the main tool in this analysis is the diagonal rule.

For this particular problem being able to determine a solution in lattice theory really pays off. (We will discuss why at the end of the section.) So let us discuss that solution first.

Suppose $\mathcal{C} = (\mathsf{C}, \sqsupseteq)$ and $\mathcal{D} = (\mathsf{D}, \succeq)$ are partially ordered sets. Then $\mathcal{C}\times\mathcal{D}$ is partially ordered by the relation $\sqsupseteq \times \succeq$. Any function to $\mathcal{C}\times\mathcal{D}$ can, of course, be broken down into two functions, one to $\mathcal{C}$ and one to $\mathcal{D}$. Any endofunction on $\mathcal{C}\times\mathcal{D}$ can thus be broken down into two binary functions of types $\mathcal{C} \leftarrow \mathcal{D}\times\mathcal{C}$ and $\mathcal{D} \leftarrow \mathcal{C}\times\mathcal{D}$. We may, therefore, state our problem in the following way. Suppose $\odot \in \mathcal{C} \leftarrow \mathcal{D}\times\mathcal{C}$ and $\otimes \in \mathcal{D} \leftarrow \mathcal{C}\times\mathcal{D}$ are monotonic. What conditions on the existence of least prefix points of endofunctors on $\mathcal{C}$ and on $\mathcal{D}$ are sufficient to guarantee the existence of a least prefix point of the (monotonic endofunction) $(x, y) \mapsto (y\odot x, x\otimes y)$? The answer is given in lemma 6.50 below. (The proof we give using the diagonal rule appears to be original to the authors and their colleagues in the Eindhoven University Mathematics of Program Construction Group [20, lemma 14]. The lemma is often attributed to Bekic [3].). First two simple lemmas dealing with two special cases of the general problem.

Lemma 6.49

(a) $\mu((x,y)\mapsto(f.x,g.y)) \;=\; (\mu f\,,\mu g)$

(b) $\mu((x,y)\mapsto(f.y,g.x)) \;=\; (\mu(f\circ g)\,,\mu(g\circ f))$.

□

The proof of lemma 6.49 is straightforward: use fusion for the first (noting that the two projection functions are lower adjoints), and use the square theorem (theorem 6.25) for the second.

Lemma 6.50

$$\mu((x,y)\mapsto(y\odot x, x\otimes y)) \;=\; (\mu(x\mapsto \varpi_{\otimes}.x\odot x)\;,\;\mu(y\mapsto \varpi_{\odot}.y\otimes y)) \quad ,$$

where $\varpi_{\otimes}.x = \mu(v\mapsto x\otimes v)$ and $\varpi_{\odot}.y = \mu(u\mapsto y\odot u)$, i.e. $\varpi_{\otimes}.x$ and $\varpi_{\odot}.y$ are the least fixed points of the individual equations.

Proof

$$
\begin{array}{cl}
 & \mu((x,y)\mapsto(y\odot x\,,\,x\otimes y)) \\
= & \{\quad \text{diagonal rule, with } x := (x,y) \text{ and} \\
 & \quad\;\; \oplus \text{ defined by } (u,v)\oplus(x,y)=(y\odot u, x\otimes v) \quad \} \\
 & \mu((x,y)\mapsto\mu((u,v)\mapsto(y\odot u\,,\,x\otimes v))) \\
= & \{\quad \text{lemma 6.49(a) and definition of } \varpi_{\otimes} \text{ and } \varpi_{\odot} \quad \} \\
 & \mu((x,y)\mapsto(\varpi_{\odot}.y\,,\,\varpi_{\otimes}.x)) \\
= & \{\quad \text{square theorem (specifically lemma 6.49(b))} \quad \} \\
 & (\mu(x\mapsto\varpi_{\odot}.(\varpi_{\otimes}.x))\,,\,\mu(y\mapsto\varpi_{\otimes}.(\varpi_{\odot}.y))) \\
= & \{\quad \text{definition } \varpi_{\odot}\,,\;\varpi_{\otimes} \quad \} \\
 & (\mu(x\mapsto\mu(u\mapsto\varpi_{\otimes}.x\odot u))\,,\,\mu(y\mapsto\mu(v\mapsto\varpi_{\odot}.y\otimes v))) \\
= & \{\quad \text{diagonal rule twice: } u := x\,,\; v := y \quad \} \\
 & (\mu(x\mapsto\varpi_{\otimes}.x\odot x)\,,\,\mu(y\mapsto\varpi_{\odot}.y\otimes y)) \quad .
\end{array}
$$

□

With this proof as guide we shall now show how to formulate and prove the theorem in category theory.

$$
\begin{array}{lll}
& \alpha \text{ is an initial } ((x,y)\mapsto(y\odot x\,,\,x\otimes y)) \text{ algebra} & \\
\Leftarrow & \{ \quad \text{diagonal rule: theorem 6.48, with } x := (x,y) \text{ and} & \\
& \quad\ \oplus \text{ defined by } (u,v)\oplus(x,y) = (y\odot u, x\otimes v) & \\
& \quad\ \bullet \quad \alpha = \beta\circ \mathsf{mu}((\mathsf{cod}.\beta)\oplus)\circ \mathsf{id}_{\mathsf{cod}.\beta}\oplus\beta\cup \quad \} & \\
& \quad \beta \text{ is an initial } F\text{-algebra} & \\
& \wedge\ F.(x,y) \text{ is an initial } (u,v)\mapsto(y\odot u, x\otimes v) \text{ algebra} & \\
\equiv & \{ \quad \text{lemma 6.49(a) and definitions of } \varpi_\otimes \text{ and } \varpi_\odot \quad \} & \\
& \beta \text{ is an initial } (x,y)\mapsto(\varpi_\odot.y\,,\,\varpi_\otimes.x) \text{ algebra} & \\
\Leftarrow & \{ \quad \text{square theorem: theorem 6.25} & \\
& \quad\ \bullet \quad \beta = \gamma\circ\varpi_\oplus.(\!|\varpi_\oplus^2\,;\,\varpi_\oplus.\gamma =: \gamma|\!) \quad \} & \\
& \gamma \text{ is an initial } \varpi_\oplus^2 \text{ algebra} & \\
\Leftarrow & \{ \quad \varpi_\oplus^2 = (x\mapsto\varpi_\odot.(\varpi_\otimes.x)\ ,\ y\mapsto\varpi_\otimes.(\varpi_\odot.y)) & \\
& \quad\ \bullet \quad \gamma = (\delta,\varepsilon) \quad \} & \\
& \quad \delta \text{ is an initial } x\mapsto\varpi_\odot.(\varpi_\otimes.x) \text{ algebra} & \\
& \wedge\ \varepsilon \text{ is an initial } y\mapsto\varpi_\otimes.(\varpi_\odot.y) \text{ algebra} & \\
\equiv & \{ \quad \text{definition } \varpi_\odot\,,\ \varpi_\otimes \quad \} & \\
& \quad \delta \text{ is an initial } x\mapsto\mu(u\mapsto\varpi_\otimes.x\odot u) \text{ algebra} & \\
& \wedge\ \varepsilon \text{ is an initial } y\mapsto\mu(v\mapsto\varpi_\odot.y\otimes v) \text{ algebra} & \\
\Leftarrow & \{ \quad \text{diagonal rule (theorem 6.48) twice: } u := x\,,\ v := y & \\
& \quad\ \bullet \quad \delta = (\!|u\mapsto\varpi_\otimes.(\mathsf{cod}.\xi)\odot u;\ \xi =: \mathsf{mu}(u\mapsto\varpi_\otimes.(\mathsf{cod}.\xi)\odot u)|\!) & \\
& \quad\ \bullet \quad \varepsilon = (\!|v\mapsto\varpi_\odot.(\mathsf{cod}.\zeta)\otimes v;\ \zeta =: \mathsf{mu}(v\mapsto\varpi_\odot.(\mathsf{cod}.\zeta)\otimes v)|\!) & \\
& \quad\ \} & \\
& \quad \xi \text{ is an initial } x\mapsto\varpi_\otimes.x\odot x \text{ algebra} & \\
& \wedge\ \zeta \text{ is an initial } y\mapsto\varpi_\odot.y\otimes y \text{ algebra} \quad . &
\end{array}
$$

Theorem 6.51 (Mutual recursion) Let $\odot\in\mathcal{C}\leftarrow\mathcal{D}\times\mathcal{C}$ and $\otimes\in\mathcal{D}\leftarrow\mathcal{C}\times\mathcal{D}$ be binary functors such that, for all $x\in\mathcal{C}$ and $y\in\mathcal{D}$, $x\otimes$, $y\odot$, $\varpi_\otimes\dot{\odot}\mathsf{Id}_\mathcal{C}$ and $\varpi_\odot\dot{\otimes}\mathsf{Id}_\mathcal{D}$ have initial algebras. Then the functor F defined by $F.(x,y) = (y\odot x, x\otimes y)$ has an initial algebra.

□

Freyd [10] defines a category to be *algebraically complete* if all its endofunctors have initial algebras. Relaxing the conditions on existence of initial algebras in theorem 6.51 we obtain the following corollary due to Freyd.

Corollary 6.52 (Algebraic Completeness) The product of two algebraically complete categories is algebraically complete.

□

Chapter 7

Monads

7.1 Introduction

In this chapter we generalise the basic theory of closure operators in lattice theory to monads in category theory.

There are several equivalent definitions of a closure operator. The closest to the usual definition of a monad is a reflexive, idempotent, monotonic endofunction on a partially ordered set.

Given a monotonic endofunction f on a complete lattice there is a unique minimal closure operator $\mathsf{f}^\star$ that is at least f. Moreover, $\mathsf{f}^\star.\perp\!\!\perp$ is the least prefix point of f. The proof is a good example of the use of Galois connections. Consider $\mathsf{Pre.f}$, the set of prefix points of f. Observe that the infimum in the base set of a subset of $\mathsf{Pre.f}$ is a prefix point of f. Thus $\mathsf{Pre.f}$ is a complete lattice and, in particular, the function embedding $\mathsf{Pre.f}$ in the base lattice distributes universally over infima. (It is just the identity function!) It thus has a lower adjoint, and the composition of the embedding function after the lower adjoint is a closure operator. Finally, since adjoint functions preserve limits the lower adjoint applied to the least element of the base set is the least prefix point of f.

Remark It is a short step from the above argument to the well-known Knaster-Tarski theorem that every monotonic function on a complete lattice has a least fixed point: it suffices to note that a least prefix point of a monotonic function is a fixed point of the function. It is also a short step to Tarski's theorem that the fixed points of a monotonic function on a complete lattice form a complete lattice: We have shown that the prefix points form a complete lattice. Dually the postfix points also form a complete lattice. So it suffices to observe that a fixed point is a point that is simultaneously a prefix point and a postfix point and a monotonic function maps its prefix points to prefix points (i.e. monotonic function f is an endofunction on $\mathsf{Pre.f}$). *End of Remark*

We use the notation $\mathsf{f}^\star$ for the least closure operator above f because the operation of mapping f to $\mathsf{f}^\star$ has several properties in common with the Kleene star in the theory of regular languages. An elementary example is that $\mathsf{f}^\star$ is above all powers of f (including the zeroth power, i.e. the identity function). More substantial examples are that $\mathsf{f}^\star$ is the least prefix point of the function $\mathcal{F}$ mapping function g to $\mathsf{Id}\sqcup\mathsf{f}\bullet\mathsf{g}$ (where Id is the identity function) and the star decomposition rule

$$(\mathsf{f}\sqcup\mathsf{g})^\star = \mathsf{g}^\star \bullet (\mathsf{f} \bullet \mathsf{g}^\star)^\star \quad .$$

In this chapter we begin from the (generalisation of the) premise that the function embedding $\mathsf{Pre.f}$ in the base set has a lower adjoint. Theorem 7.15 generalises the construction of a least prefix point, and theorem 7.19 the fact that $\mathsf{f}^\star$ is itself a least prefix point. The *monad decomposition* rule generalising the star decomposition rule is proved in section 7.5.

7.2 Monads and Adjunctions

A closure operator is a function from a preorder to a preorder that is monotonic, reflexive and idempotent. A monad is a "coherently constructive" closure operator.

Definition 7.1 A *monad* in a category $\mathcal{C}$ consists of an endofunctor F on $\mathcal{C}$ and two natural transformations $\mathsf{unit}\in\mathsf{F}\leftarrow\mathsf{id}$ and $\mathsf{mul}\in\mathsf{F}\leftarrow\mathsf{F}\bullet\mathsf{F}$ such that

(7.2) $\quad \mathsf{mul}\circ\mathsf{F}\bullet\mathsf{mul} = \mathsf{mul}\circ\mathsf{mul}\bullet\mathsf{F}$

and

(7.3) $\quad \mathsf{mul}\circ\mathsf{F}\bullet\mathsf{unit} = \mathsf{F}\bullet\mathsf{id} = \mathsf{mul}\circ\mathsf{unit}\bullet\mathsf{F} = \mathsf{id}\bullet\mathsf{F} \quad .$

As suggested by the notation, unit is called the *unit* of the monad; mul is called the *multiplication* of the monad. It is also common to refer to the functor F as "a monad" although strictly it is but one component of a monad.

□

Starting from the natural transformations $\mathsf{unit}\in\mathsf{F}\leftarrow\mathsf{id}$ and $\mathsf{mul}\in\mathsf{F}\leftarrow\mathsf{F}\bullet\mathsf{F}$ together with the identity transformations $\mathsf{F}\bullet\mathsf{id}\in\mathsf{F}\leftarrow\mathsf{F}$ and $\mathsf{id}\bullet\mathsf{F}\in\mathsf{F}\leftarrow\mathsf{F}$ it is straightforward to construct natural transformations $\alpha_i\in\mathsf{F}\leftarrow\mathsf{F}^i$ for all $i\geq 0$ (where F^i denotes the i-fold composition of F with itself). Indeed, with the exception of α_0, each can be constructed in an unbounded number of ways. The two requirements (7.2) and (7.3) guarantee the "coherence" of all such constructions in the sense that any two ways of constructing one such transformation give rise to equal transformations. The two terms of (7.2) correspond to the most elementary constructions of natural transformations to F from F^3 as shown in the following diagram.

$$\begin{tikzcd}
\mathsf{F} & \mathsf{F}\bullet\mathsf{F} \arrow[l, "\mathsf{mul}"'] \\
\mathsf{F}\bullet\mathsf{F} \arrow[u, "\mathsf{mul}"] & \mathsf{F}\bullet\mathsf{F}\bullet\mathsf{F} \arrow[l, "\mathsf{mul}\bullet\mathsf{F}"] \arrow[u, "\mathsf{F}\bullet\mathsf{mul}"']
\end{tikzcd}$$

The four terms in (7.3) correspond to the most elementary constructions of natural transformations to F from F ($=\mathsf{F}^1$). (The second and fourth terms are of course always equal.) The inductive proof that these two properties are sufficient to guarantee uniqueness of α_i , for all i , is left to the reader.

A standard theorem in lattice theory is that the composition of the lower adjoint of a Galois connection after the upper adjoint is a closure operator. The corresponding theorem in category theory is also standard [15].

Theorem 7.4 Suppose ($\mathsf{F}\in\mathcal{C}\leftarrow\mathcal{D}$, $\mathsf{G}\in\mathcal{D}\leftarrow\mathcal{C}$) is an adjunction. Let $\mathsf{unit}\in\mathsf{G}{\bullet}\mathsf{F}\leftarrow\mathsf{id}$ be its unit and $\mathsf{counit}\in\mathsf{id}\leftarrow\mathsf{F}{\bullet}\mathsf{G}$ its counit. Then the triple consisting of $\mathsf{G}{\bullet}\mathsf{F}$, unit and $\mathsf{G}{\bullet}\mathsf{counit}{\bullet}\mathsf{F}$ is a monad on $\mathcal{C}$.

□

The details of the proof of theorem 7.4 are left to the reader. (Verification of the coherence property (7.2) is a straightforward application of Godement's interchange law (2.19), and (7.3) is obtained by applying theorem 4.25 in combination with Godement's laws.)

7.3 Basic Adjunction

Recall our (sketched) proof of the Knaster-Tarski theorem. Assuming a complete lattice, the embedding of the prefix points of monotonic endofunction f in the lattice has as lower adjoint $f^\star$, the least closure operator above f, and $f^\star$ applied to the bottom element of the lattice is the least prefix point (and thus least fixed point) of f. The key theorem in this argument states that a function with domain a complete lattice that distributes universally over all infima in the lattice has a lower adjoint. The corresponding theorem in category theory, known as Freyd's adjoint functor theorem [15, 11], involves conditions that we do not want go into. We begin this section, therefore, with the assumption that an adjunction exists between a category of algebras and a base category.

Let F denote an endofunctor on the category $\mathcal{C}$. We use α, β, ... to denote objects of $\mathsf{Alg.F}$ and φ, ϕ, ... to denote arrows of $\mathsf{Alg.F}$. (Note that both objects and arrows of $\mathsf{Alg.F}$ are also arrows of the category $\mathcal{C}$.) We use x, y, ... to denote objects of $\mathcal{C}$ and f, g, ... to denote arrows of $\mathcal{C}$ that are not necessarily arrows of $\mathsf{Alg.F}$.

Let U denote the functor in $\mathcal{C}\leftarrow\mathsf{Alg.F}$ defined by $\mathsf{U}.\alpha=\mathsf{cod}.\alpha$ and $\mathsf{U}.\varphi=\varphi$. This functor is called a *forgetful functor*: it forgets about the structure of an algebra by mapping it to its carrier. Note that U is dependent on F although the notation does not make the dependency explicit. For the moment that is unnecessary; we will make the dependency explicit when it does become so.

Suppose U has a lower adjoint denoted by $\mathsf{U}^\flat$. The lower and upper adjungates of this adjunction will be denoted by $\lfloor\ \rfloor$ and $\lceil\ \rceil$ respectively. We introduce a special notation for the composition of U and $\mathsf{U}^\flat$: let $\mathsf{F}^\star$ denote $\mathsf{U}{\bullet}\mathsf{U}^\flat$. The lower and upper adjungates of the assumed adjunction must, by definition, satisfy the following 6 rules.

(7.5) $\quad \lfloor\varphi\rfloor_{\alpha,x}\in\mathsf{cod}.\alpha\overset{\mathcal{C}}{\longleftarrow}x \;\equiv\; \varphi\in\alpha\overset{\mathsf{Alg.F}}{\longleftarrow}\mathsf{U}^\flat.x$, (7.6) $\lceil f \rceil_{\alpha,x} \in \alpha \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{U}^\flat.x \;\equiv\; f \in \mathsf{cod}.\alpha \overset{\mathcal{C}}{\longleftarrow} x$,

(7.7) $\lceil \lfloor \varphi \rfloor_{\alpha,x} \rceil_{\alpha,x} = \varphi \;\Leftarrow\; \varphi \in \alpha \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{U}^\flat.x$,

(7.8) $\lfloor \lceil f \rceil_{\alpha,x} \rfloor_{\alpha,x} = f \;\Leftarrow\; f \in \mathsf{cod}.\alpha \overset{\mathcal{C}}{\longleftarrow} x$,

(7.9)
$$\begin{aligned} & \psi \circ \lfloor \varphi \rfloor_{\alpha,x} \circ g \;=\; \lfloor \psi \circ \varphi \circ \mathsf{U}^\flat.g \rfloor_{\beta,y} \\ \Leftarrow\quad & \psi \in \beta \overset{\mathsf{Alg.F}}{\longleftarrow} \alpha \;\wedge\; \varphi \in \alpha \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{U}^\flat.x \;\wedge\; g \in x \overset{\mathcal{C}}{\longleftarrow} y \quad , \end{aligned}$$

(7.10)
$$\begin{aligned} & \psi \circ \lceil f \rceil_{\alpha,x} \circ \mathsf{U}^\flat.g \;=\; \lceil \psi \circ f \circ g \rceil_{\beta,y} \\ \Leftarrow\quad & \psi \in \beta \overset{\mathsf{Alg.F}}{\longleftarrow} \alpha \wedge f \in \mathsf{cod}.\alpha \overset{\mathcal{C}}{\longleftarrow} x \wedge g \in x \overset{\mathcal{C}}{\longleftarrow} y \quad . \end{aligned}$$

Let τ_x denote $\lfloor \mathsf{id}_{\mathsf{U}^\flat.x} \rfloor_{\mathsf{U}^\flat.x,x}$. Then, from (7.5),

$$\tau_x \;\in\; \mathsf{F}^\star.x \overset{\mathcal{C}}{\longleftarrow} x \quad .$$

(Recall, $\mathsf{cod}.(\mathsf{U}^\flat.x) = \mathsf{U}.(\mathsf{U}^\flat.x) = \mathsf{F}^\star.x$.) Note, by theorem 7.4, $\mathsf{F}^\star$ is (the functor-part of) a *monad* with τ as its unit.

With the above we can easily derive the following computation rules. From (7.9),

$$\psi \circ \tau_x = \lfloor \psi \rfloor_{\beta,x} \;\Leftarrow\; \psi \in \beta \overset{\mathsf{Alg.F}}{\longleftarrow} \mathsf{U}^\flat.x \quad .$$

In particular, from (7.6) and (7.8),

(7.11) $\lceil f \rceil_{\alpha,x} \circ \tau_x = f \;\Leftarrow\; f \in \mathsf{cod}.\alpha \overset{\mathcal{C}}{\longleftarrow} x$.

Also, from (7.6) and the definition of an arrow in $\mathsf{Alg.F}$,

(7.12) $\alpha \circ \mathsf{F}.\lceil f \rceil_{\alpha,x} = \lceil f \rceil_{\alpha,x} \circ \mathsf{U}^\flat.x \;\Leftarrow\; f \in \mathsf{cod}.\alpha \overset{\mathcal{C}}{\longleftarrow} x$.

From (7.7),

(7.13) $\lceil \tau_x \rceil_{\mathsf{U}^\flat.x,x} = \mathsf{id}_{\mathsf{U}^\flat.x}$.

Lemma 7.14

$$\iota \bullet \mathsf{U}^\flat \;\in\; \mathsf{F}^\star \leftarrow \mathsf{F} \bullet \mathsf{F}^\star \quad ,$$

where $\iota \in \mathsf{U} \leftarrow \mathsf{F} \bullet \mathsf{U}$ is defined by $\iota_\alpha = \alpha$.

Proof

$$
\begin{array}{ll}
 & \iota \bullet \mathsf{U}^\flat \in \mathsf{F}^\star \leftarrow \mathsf{F} \bullet \mathsf{F}^\star \\
\equiv & \{ \quad \mathsf{F}^\star = \mathsf{U} \bullet \mathsf{U}^\flat \quad \} \\
 & \iota \bullet \mathsf{U}^\flat \in \mathsf{U} \bullet \mathsf{U}^\flat \leftarrow \mathsf{F} \bullet \mathsf{U} \bullet \mathsf{U}^\flat \\
\Leftarrow & \{ \quad \bullet \mathsf{U}^\flat \text{ is a functor} \quad \} \\
 & \iota \in \mathsf{U} \leftarrow \mathsf{F} \bullet \mathsf{U} \\
\equiv & \{ \quad \text{natural transformation} \quad \} \\
 & \quad \forall(\alpha \colon \alpha \in \mathsf{Alg.F} \colon \iota_\alpha \in \mathsf{U}.\alpha \leftarrow \mathsf{F}.(\mathsf{U}.\alpha)) \\
 & \wedge\ \forall(\varphi \colon \varphi \in \alpha \xleftarrow{\mathsf{Alg.F}} \beta \colon \mathsf{U}.\varphi \circ \iota_\beta = \iota_\alpha \circ \mathsf{F}.(\mathsf{U}.\varphi)) \\
\equiv & \{ \quad \text{definition } \mathsf{U} \quad \} \\
 & \quad \forall(\alpha \colon \alpha \in \mathsf{Alg.F} \colon \iota_\alpha \in \mathsf{cod}.\alpha \leftarrow \mathsf{F}.(\mathsf{cod}.\alpha)) \\
 & \wedge\ \forall(\varphi \colon \varphi \in \alpha \xleftarrow{\mathsf{Alg.F}} \beta \colon \varphi \circ \iota_\beta = \iota_\alpha \circ \mathsf{F}.\varphi) \\
\equiv & \{ \quad \iota_\alpha = \alpha, \text{ definition of an arrow in } \mathsf{Alg.F} \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

□

Let $\perp\!\!\perp$ denote an initial object in category $\mathcal{C}$. Then,

$$\mathsf{U}^\flat.\perp\!\!\perp \ \in \ \mathsf{F}^\star.\perp\!\!\perp \ \leftarrow \ \mathsf{F}.(\mathsf{F}^\star.\perp\!\!\perp) \quad .$$

Moreover, since lower adjoints map an initial object to an initial object we immediately have the following theorem.

Theorem 7.15 $\mathsf{U}^\flat.\perp\!\!\perp$ is an initial F-algebra with $\mathsf{F}^\star.\perp\!\!\perp$ as its carrier.

□

From the proof that lower adjoints preserve initial objects we see that the unique arrow in $\mathsf{Alg.F}$ to an object α from $\mathsf{U}^\flat.\perp\!\!\perp$ is given by $\lceil (\!|\mathcal{C}\,;\mathsf{cod}.\alpha =\colon \perp\!\!\perp|\!) \rceil_{\alpha,\perp\!\!\perp}$. I.e. given catamorphisms for the base category $\mathcal{C}$, this defines catamorphisms for the category $\mathsf{Alg.F}$: letting muF denote $\mathsf{U}^\flat.\perp\!\!\perp$ we define

$$(7.16) \quad (\!|\mathsf{Alg.F}\,;\, \alpha =\colon \mathsf{muF}|\!) \quad \widehat{=} \quad \lceil (\!|\mathcal{C}\,;\mathsf{cod}.\alpha =\colon \perp\!\!\perp|\!) \rceil_{\alpha,\perp\!\!\perp} \quad .$$

Note that the initiality of $\mathsf{U}^\flat.\perp\!\!\perp$ in $\mathsf{Alg.F}$ immediately gives the *catamorphism fusion* rule:

$$\psi \circ (\!|\mathsf{Alg.F}\,;\, \alpha =\colon \mathsf{muF}|\!) = (\!|\mathsf{Alg.F}\,;\, \beta =\colon \mathsf{muF}|\!) \ \Leftarrow\ \psi \in \beta \xleftarrow{\mathsf{Alg.F}} \alpha \quad .$$

Also, instantiating $\mathsf{f},\mathsf{x} := (\!|\mathcal{C}\,;\mathsf{cod}.\alpha =\colon \perp\!\!\perp|\!),\perp\!\!\perp$ in (7.12) gives the computation rule:

$$\alpha \circ \mathsf{F}.(\!|\mathsf{Alg.F}\,;\, \alpha =\colon \mathsf{muF}|\!) = (\!|\mathsf{Alg.F}\,;\, \alpha =\colon \mathsf{muF}|\!) \circ \mathsf{muF} \quad .$$

7.4 Lifted Adjunction

In lattice theory a useful technique in the study of $F^\star$ is to "lift" the Galois connection between the base set and $\mathsf{Pre}.F$ to a Galois connection between the monotonic endofunctions on the base set and $\mathsf{Pre}.(F\bullet)$. Adopting this strategy we are lead to the following lemma:

Lemma 7.17 The adjunction $(U^\flat, U)$ between $\mathcal{C}$ and $\mathsf{Alg}.F$ induces an adjunction between $\mathsf{End}.\mathcal{C}$ and $\mathsf{Alg}.(F\bullet)$. Specifically, with dummy α ranging over $\mathsf{Alg}.(F\bullet)$ and dummy G ranging over $\mathsf{End}.\mathcal{C}$, we have:

$$[\alpha, G :: \ \alpha \xleftarrow{\mathsf{Alg}.(F\bullet)} \iota\bullet U^\flat\bullet G] \cong [\alpha, G :: \ \mathsf{cod}.\alpha \xleftarrow{\mathsf{End}.\mathcal{C}} G] \quad .$$

Proof First, note the following calculation. Let $\alpha,\beta \in \mathsf{Alg}.(F\bullet)$ then,

$$\begin{array}{cl}
 & \varphi \in \alpha \xleftarrow{\mathsf{Alg}.(F\bullet)} \beta \\
\equiv & \quad \{ \quad \text{definition} \quad \} \\
 & \varphi \in \mathsf{cod}.\alpha \xleftarrow{\mathsf{End}.\mathcal{C}} \mathsf{cod}.\beta \ \wedge \ \alpha \circ F\bullet\varphi = \varphi \circ \beta \\
\equiv & \quad \{ \quad \text{definition nat. trans., pointwise and} \\
 & \qquad\quad \text{some rearrangement} \quad \} \\
 & \quad \forall(x{::}\ \ \varphi_x \in (\mathsf{cod}.\alpha).x \xleftarrow{\mathcal{C}} (\mathsf{cod}.\beta).x \ \wedge \ \alpha_x \circ F.\varphi_x = \varphi_x \circ \beta_x) \\
 & \wedge \ \forall(f{:}\ \ f \in x \xleftarrow{\mathcal{C}} y{:}\ \ (\mathsf{cod}.\alpha).f \circ \varphi_y = \varphi_x \circ (\mathsf{cod}.\beta).f) \\
\equiv & \quad \{ \quad (\mathsf{cod}.\alpha).x = \mathsf{cod}.\alpha_x \text{, definition } F\text{-algebra} \quad \} \\
 & \quad \forall(x{::}\ \ \varphi_x \in \alpha_x \xleftarrow{\mathsf{Alg}.F} \beta_x) \\
 & \wedge \ \forall(f{:}\ \ f \in x \xleftarrow{\mathcal{C}} y{:}\ \ (\mathsf{cod}.\alpha).f \circ \varphi_y = \varphi_x \circ (\mathsf{cod}.\beta).f) \quad .
\end{array}$$

Our proof obligations are: construct lower and upper adjungates and prove that they satisfy certain coherence requirements.

Let $\alpha{\in}\mathsf{Alg}.(F\bullet)$ and $G{\in}\mathsf{End}.\mathcal{C}$. Suppose $\eta \in \mathsf{cod}.\alpha \xleftarrow{\mathsf{End}.\mathcal{C}} G$. By construction of φ we derive a candidate for the upper adjungate.

$$\begin{array}{cl}
 & \varphi \in \alpha \xleftarrow{\mathsf{Alg}.(F\bullet)} \iota\bullet U^\flat\bullet G \\
\equiv & \quad \{ \quad \text{above, } \iota\bullet U^\flat\bullet G \in F^\star\bullet G \leftarrow F\bullet F^\star\bullet G \quad \} \\
 & \quad \forall(x{::}\ \ \varphi_x \in \alpha_x \xleftarrow{\mathsf{Alg}.F} U^\flat.G.x) \\
 & \wedge \ \forall(f{:}\ \ f \in x \xleftarrow{\mathcal{C}} y{:}\ \ (\mathsf{cod}.\alpha).f \circ \varphi_y = \varphi_x \circ F^\star.G.f) \\
\Leftarrow & \quad \{ \quad \text{assumption: } \ \varphi_x = \lceil \sigma_x \rceil_{\alpha_x, G.x}\,, \\
 & \qquad\quad F^\star.G.f = U.U^\flat.G.f = U^\flat.G.f\text{, fusion rules} \quad \} \\
 & \quad \forall(x{::}\ \ \sigma_x \in \mathsf{cod}.\alpha_x \xleftarrow{\mathcal{C}} G.x) \\
 & \wedge \ \forall(f{:}\ \ f \in x \xleftarrow{\mathcal{C}} y{:}\ \ (\mathsf{cod}.\alpha).f \circ \sigma_y = \sigma_x \circ G.f)
\end{array}$$

$$\Leftarrow \qquad \{ \qquad \eta \in \mathsf{cod}.\alpha \xleftarrow{\mathsf{End}.\mathcal{C}} \mathsf{G} \quad \}$$

$$\sigma_x = \eta_x \quad .$$

So the upper adjungate $[\![\ ]\!]$ is defined by $([\![\eta]\!]_{\alpha,\mathsf{G}})_x = \lceil \eta_x \rceil_{\alpha_x,\mathsf{G}.x}$.

Suppose $\varphi \in \alpha \xleftarrow{\mathsf{Alg}.(\mathsf{F}\bullet)} \iota\bullet\mathsf{U}^\flat\bullet\mathsf{G}$. By construction of η we derive a candidate for the lower adjungate.

$$\eta \in \mathsf{cod}.\alpha \xleftarrow{\mathsf{End}.\mathcal{C}} \mathsf{G}$$

$$\equiv \qquad \{ \qquad \text{natural transformation} \quad \}$$

$$\forall(x{::}\ \eta_x \in \mathsf{cod}.\alpha_x \xleftarrow{\mathcal{C}} \mathsf{G}.x)$$

$$\wedge\ \forall(f{:}\ f \in x \xleftarrow{\mathcal{C}} y{:}\ (\mathsf{cod}.\alpha).f \circ \eta_y = \eta_x \circ \mathsf{G}.f)$$

$$\Leftarrow \qquad \{ \qquad \text{assumption: } \eta_x = \lfloor \psi_x \rfloor_{\alpha_x,\mathsf{G}.x}\ ,$$

$$\text{fusion rules, } \mathsf{U}^\flat.\mathsf{G}.f = \mathsf{U}.\mathsf{U}^\flat.\mathsf{G}.f = \mathsf{F}^\star.\mathsf{G}.f \quad \}$$

$$\forall(x{::}\ \psi_x \in \alpha_x \xleftarrow{\mathsf{Alg}.\mathsf{F}} \mathsf{U}^\flat.\mathsf{G}.x)$$

$$\wedge\ \forall(f{:}\ f \in x \xleftarrow{\mathcal{C}} y{:}\ (\mathsf{cod}.\alpha).f \circ \psi_y = \psi_x \circ \mathsf{F}^\star.\mathsf{G}.f)$$

$$\Leftarrow \qquad \{ \qquad \text{above, } \iota\bullet\mathsf{U}^\flat\bullet\mathsf{G} \in \mathsf{F}^\star\bullet\mathsf{G} \leftarrow \mathsf{F}\bullet\mathsf{F}^\star\bullet\mathsf{G} \quad \}$$

$$\psi_x = \varphi_x \quad .$$

So the lower adjungate $\lfloor\!\lfloor\ \rfloor\!\rfloor$ is defined by $(\lfloor\!\lfloor\varphi\rfloor\!\rfloor_{\alpha,\mathsf{G}})_x = \lfloor \varphi_x \rfloor_{\alpha_x,\mathsf{G}.x}$. The coherence requirements follow immediately by the pointwise definition of the adjungates.

□

Thus as a result we have the following family of isomorphisms: for each $\alpha{\in}\mathsf{Alg}.(\mathsf{F}\bullet)$,

$$\alpha \xleftarrow{\mathsf{Alg}.(\mathsf{F}\bullet)} \iota\bullet\mathsf{U}^\flat \cong \mathsf{cod}.\alpha \xleftarrow{\mathsf{End}.\mathcal{C}} \mathsf{Id} \quad ,$$

with $[\![\ ]\!]_{\alpha,\mathsf{Id}}$ as witness from right to left and $\lfloor\!\lfloor\ \rfloor\!\rfloor_{\alpha,\mathsf{Id}}$ as its inverse. Furthermore, we had previously defined τ_x by $\lfloor \mathsf{id}_{\mathsf{U}^\flat.x} \rfloor_{\mathsf{U}^\flat.x,x}$. In terms of this newly established adjunction we can reexpress τ as:

(7.18) $\quad \tau = \lfloor\!\lfloor \mathsf{id}\bullet\mathsf{U}^\flat \rfloor\!\rfloor_{\iota\bullet\mathsf{U}^\flat,\mathsf{Id}} (\in \mathsf{F}^\star \xleftarrow{\mathsf{End}.\mathcal{C}} \mathsf{Id}) \quad .$

We are now ready for the theorem that justifies our “Kleene star” notation.

Theorem 7.19 Define the functor $\mathcal{F} \in \mathsf{End}.\mathcal{C} \leftarrow \mathsf{End}.\mathcal{C}$ by $\mathcal{F}.\mathsf{H} = \mathsf{G}{+}(\mathsf{F}\bullet\mathsf{H})$. Then, $\tau \triangledown (\iota\bullet\mathsf{U}^\flat) \bullet \mathsf{G}$ is initial in $\mathsf{Alg}.\mathcal{F}$ with $\mathsf{F}^\star\bullet\mathsf{G}$ as carrier.

Proof First, $\tau \in \mathsf{F}^\star \xleftarrow{\mathsf{End}.\mathcal{C}} \mathsf{Id}$ and $\iota\bullet\mathsf{U}^\flat \in \mathsf{F}^\star \xleftarrow{\mathsf{End}.\mathcal{C}} \mathsf{F}\bullet\mathsf{F}^\star$. Thus,

$$\tau \triangledown (\iota\bullet\mathsf{U}^\flat) \bullet \mathsf{G} \in \mathsf{F}^\star\bullet\mathsf{G} \xleftarrow{\mathsf{End}.\mathcal{C}} \mathsf{Id}{+}(\mathsf{F}\bullet\mathsf{F}^\star) \bullet \mathsf{G} \quad .$$

Since $\mathsf{Id}{+}(\mathsf{F}\bullet\mathsf{F}^\star) \bullet \mathsf{G} = \mathsf{G}{+}(\mathsf{F}\bullet\mathsf{F}^\star\bullet\mathsf{G})$, by definition of addition, $\tau \triangledown (\iota\bullet\mathsf{U}^\flat) \bullet \mathsf{G}$ is an $\mathcal{F}$-algebra with $\mathsf{F}^\star\bullet\mathsf{G}$ as carrier. Suppose $\alpha{\in}\mathsf{Alg}.\mathcal{F}$. We have to construct a *unique* arrow in $\mathsf{Alg}.\mathcal{F}$ to α from $\tau \triangledown (\iota\bullet\mathsf{U}^\flat) \bullet \mathsf{G}$.

$$\begin{array}{ll}
& \varphi \in \alpha \overset{\mathsf{Alg}.\mathcal{F}}{\longleftarrow} \tau\triangledown(\iota\bullet \mathsf{U}^\flat) \bullet \mathsf{G} \\
\equiv & \qquad \{ \qquad \text{definition arrow in } \mathsf{Alg}.\mathcal{F} \quad \} \\
& \varphi \in \mathsf{cod}.\alpha \overset{\mathsf{End}.\mathcal{C}}{\longleftarrow} \mathsf{F}^\star\bullet\mathsf{G} \ \wedge\ \alpha\circ(\mathsf{Id}\bullet\mathsf{G})+(\mathsf{F}\bullet\varphi) = \varphi\circ(\tau\triangledown(\iota\bullet\mathsf{U}^\flat)\bullet\mathsf{G}) \\
\equiv & \qquad \{ \qquad \eta+\sigma = (\mathsf{inl}\circ\eta)\triangledown(\mathsf{inr}\circ\sigma) \quad \} \\
& \varphi \in \mathsf{cod}.\alpha \overset{\mathsf{End}.\mathcal{C}}{\longleftarrow} \mathsf{F}^\star\bullet\mathsf{G} \ \wedge\ \alpha\circ(\mathsf{inl}\bullet\mathsf{G})\triangledown(\mathsf{inr}\circ\mathsf{F}\bullet\varphi) = \varphi\circ(\tau\triangledown(\iota\bullet\mathsf{U}^\flat)\bullet\mathsf{G}) \\
\equiv & \qquad \{ \qquad \triangledown\,/\,+ \text{ - computation rules} \quad \} \\
& \varphi \in \mathsf{cod}.\alpha \overset{\mathsf{End}.\mathcal{C}}{\longleftarrow} \mathsf{F}^\star\bullet\mathsf{G} \ \wedge\ \alpha\circ(\mathsf{inl}\bullet\mathsf{G}) = \varphi\circ\tau \ \wedge\ \alpha\circ\mathsf{inr}\circ(\mathsf{F}\bullet\varphi) = \varphi\circ(\iota\bullet\mathsf{U}^\flat\bullet\mathsf{G}) \\
\equiv & \qquad \{ \qquad \text{definition arrow in } \mathsf{Alg}.(\mathsf{F}\bullet)\,,\ \mathsf{F}^\star = \mathsf{U}\bullet\mathsf{U}^\flat = \mathsf{cod}.(\iota\bullet\mathsf{U}^\flat) \quad \} \\
& \varphi \in \alpha\circ\mathsf{inr} \overset{\mathsf{Alg}.(\mathsf{F}\bullet)}{\longleftarrow} \iota\bullet\mathsf{U}^\flat\bullet\mathsf{G} \ \wedge\ \alpha\circ(\mathsf{inl}\bullet\mathsf{G}) = \varphi\circ\tau \\
\equiv & \qquad \{ \qquad \tau = (\!|\mathsf{id}\bullet\mathsf{U}^\flat|\!)_{\iota\bullet\mathsf{U}^\flat,\mathsf{Id}}\,, \text{ fusion} \quad \} \\
& \varphi \in \alpha\circ\mathsf{inr} \overset{\mathsf{Alg}.(\mathsf{F}\bullet)}{\longleftarrow} \iota\bullet\mathsf{U}^\flat\bullet\mathsf{G} \ \wedge\ \alpha\circ(\mathsf{inl}\bullet\mathsf{G}) = (\!|\varphi|\!)_{\alpha\circ\mathsf{inr}\,,\,\mathsf{G}} \\
\equiv & \qquad \{ \qquad \text{lemma 7.17, inverse} \quad \} \\
& \varphi = [\![\alpha\circ(\mathsf{inl}\bullet\mathsf{G})]\!]_{\alpha\circ\mathsf{inr}\,,\,\mathsf{G}} \quad .
\end{array}$$

In the above proof all the steps are equivalences so we have at the same time also proven uniqueness.
□

7.5 Decomposition Theorem

In this section we prove the categorical generalisation of the theorem that

$$(7.20) \quad (\mathsf{f}+\mathsf{g})^\star = \mathsf{g}^\star\bullet(\mathsf{f}\bullet\mathsf{g}^\star)^\star \quad ,$$

where $\mathsf{f}^\star$ denotes the least closure operator above f and $+$ denotes the binary supremum operator lifted pointwise to functions.

A proof in lattice theory is a straightforward calculation:

$$\begin{array}{ll}
& (\mathsf{f}+\mathsf{g})^\star = \mathsf{g}^\star\bullet(\mathsf{f}\bullet\mathsf{g}^\star)^\star \\
\equiv & \qquad \{ \qquad \text{definition of } {}^\star \quad \} \\
& \mu(\mathsf{h}\mapsto\mathsf{id}+(\mathsf{f}+\mathsf{g}\bullet\mathsf{h})) \ = \ \mathsf{g}^\star\bullet\mu(\mathsf{h}\mapsto\mathsf{id}+(\mathsf{f}\bullet\mathsf{g}^\star\bullet\mathsf{h})) \\
\equiv & \qquad \{ \qquad \text{diagonal rule: } \mathsf{id}+(\mathsf{f}+\mathsf{g}\bullet\mathsf{h}) \ = \ (\mathsf{id}+(\mathsf{f}\bullet\mathsf{h}))+(\mathsf{g}\bullet\mathsf{h}) \\
& \qquad\qquad\quad \text{rolling rule} \quad \} \\
& \mu(\mathsf{h}\mapsto\mu(\mathsf{j}\mapsto(\mathsf{id}+(\mathsf{f}\bullet\mathsf{h}))+(\mathsf{g}\bullet\mathsf{j}))) \ = \ \mu(\mathsf{h}\mapsto\mathsf{g}^\star\bullet\mathsf{id}+(\mathsf{f}\bullet\mathsf{h})) \\
\equiv & \qquad \{ \qquad \text{definition of } {}^\star \quad \} \\
& \mu(\mathsf{h}\mapsto\mathsf{g}^\star\bullet\mathsf{id}+(\mathsf{f}\bullet\mathsf{h})) \ = \ \mu(\mathsf{h}\mapsto\mathsf{g}^\star\bullet\mathsf{id}+(\mathsf{f}\bullet\mathsf{h}))
\end{array}$$

$\equiv$ { reflexivity }

true .

The categorical theorem we want to prove is that, for functors F and G, the monad $(\mathsf{F}{+}\mathsf{G})^\star$ is isomorphic to the monad $\mathsf{G}^\star{\bullet}(\mathsf{F}{\bullet}\mathsf{G}^\star)^\star$. The proof has the same ingredients as the lattice theoretic proof.

In order to apply theorem 7.19 we need to differentiate between forgetful functors: in general, we shall use U_F to denote the forgetful functor from $\mathsf{Alg}.\mathsf{F}$ into the base category $\mathcal{C}$. Similarly, we use τ_F and ι_F for the unit of the monad $\mathsf{F}^\star$ and the natural transformation to U_F from $\mathsf{F}{\bullet}\mathsf{U}_\mathsf{F}$, respectively.

We present the proof twice, once in a "wordy" style and once in a calculational style. The reader may choose which is most appealing.

The "wordy" proof has five steps. First, by the rolling rule, the functor $(\mathsf{G}^\star){\bullet}$ maps an initial object of $\mathsf{Alg}.(\mathsf{H} \mapsto \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{G}^\star{\bullet}\mathsf{H}))$ to an initial object of $\mathsf{Alg}.(\mathsf{H} \mapsto \mathsf{G}^\star \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H}))$. Thus, by theorem 7.19, $\mathsf{G}^\star \bullet \tau_{\mathsf{F}\bullet\mathsf{G}^\star}\triangledown(\iota_{\mathsf{F}\bullet\mathsf{G}^\star}{\bullet}(\mathsf{U}_{\mathsf{F}\bullet\mathsf{G}^\star})^\flat)$ is an initial $\mathsf{H} \mapsto \mathsf{G}^\star \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})$-algebra.

Second, observe by theorem 7.19 that $\tau_\mathsf{G}\triangledown(\iota_\mathsf{G}{\bullet}(\mathsf{U}_\mathsf{G})^\flat) \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})$ is an initial object of the functor mapping J to $(\mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})){+}(\mathsf{G}{\bullet}\mathsf{J})$ with the functor $\mathsf{G}^\star \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})$ as carrier.

Third, defining the binary functor $\oplus$ by

$$\mathsf{H}{\oplus}\mathsf{J} = (\mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})){+}(\mathsf{G}{\bullet}\mathsf{J})$$

and the map functor $\varpi_\oplus$ as in theorem 5.33, we note, by applying the diagonal rule (theorem 6.48), that $\tau_{\mathsf{F}+\mathsf{G}}\triangledown(\iota_{\mathsf{F}+\mathsf{G}}{\bullet}(\mathsf{U}_{\mathsf{F}+\mathsf{G}})^\flat)$ is an initial object of $\mathsf{Alg}.\varpi_\oplus$. There is thus an initial object in $\mathsf{Alg}.\varpi_\oplus$ with carrier $(\mathsf{F}{+}\mathsf{G})^\star$. (A detail in the application of the diagonal rule is that $\hat{\oplus}\mathsf{H}$ is isomorphic to $\mathsf{Id}{+}(\mathsf{F}{+}\mathsf{G} \bullet \mathsf{H})$. This follows from the associativity of $+$ and the definition of (the lifting of definition of $+$ to) $\mathsf{F}{+}\mathsf{G}$.)

Fourth, we note that, by theorem 7.19, $\mathsf{G}^\star \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})$ is the carrier of the initial algebra $\tau_\mathsf{G}\triangledown(\iota_\mathsf{G}{\bullet}(\mathsf{U}_\mathsf{G})^\flat) \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})$ and thus, by definition of the map functor $\varpi_\oplus$ on objects, $\varpi_\oplus.\mathsf{H}$ equals $\mathsf{G}^\star \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})$ (for all functors H).

Combining the third and fourth observations, the functor $\mathsf{H} \mapsto \mathsf{G}^\star \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})$ has initial algebra $\tau_{\mathsf{F}+\mathsf{G}}\triangledown(\iota_{\mathsf{F}+\mathsf{G}}{\bullet}(\mathsf{U}_{\mathsf{F}+\mathsf{G}})^\flat)$ with carrier $(\mathsf{F}{+}\mathsf{G})^\star$. Combining this with the first observation, $\mathsf{G}^\star \bullet \tau_{\mathsf{F}\bullet\mathsf{G}^\star}\triangledown(\iota_{\mathsf{F}\bullet\mathsf{G}^\star}{\bullet}(\mathsf{U}_{\mathsf{F}\bullet\mathsf{G}^\star})^\flat)$ and $\tau_{\mathsf{F}+\mathsf{G}}\triangledown(\iota_{\mathsf{F}+\mathsf{G}}{\bullet}(\mathsf{U}_{\mathsf{F}+\mathsf{G}})^\flat)$ are isomorphic. Since the first has carrier $\mathsf{G}^\star{\bullet}(\mathsf{F}{\bullet}\mathsf{G}^\star)^\star$ and the second $(\mathsf{F}{+}\mathsf{G})^\star$ these two functors are also isomorphic.

The calculational proof also has five steps:

$\mathsf{G}^\star{\bullet}(\mathsf{F}{\bullet}\mathsf{G}^\star)^\star \cong (\mathsf{F}{+}\mathsf{G})^\star$

$\Leftarrow$ { taking carriers preserves isomorphism }

$\mathsf{G}^\star \bullet \tau_{\mathsf{F}\bullet\mathsf{G}^\star}\triangledown(\iota_{\mathsf{F}\bullet\mathsf{G}^\star}{\bullet}(\mathsf{U}_{\mathsf{F}\bullet\mathsf{G}^\star})^\flat) \cong \tau_{\mathsf{F}+\mathsf{G}}\triangledown(\iota_{\mathsf{F}+\mathsf{G}}{\bullet}(\mathsf{U}_{\mathsf{F}+\mathsf{G}})^\flat)$

$\Leftarrow$ { $\tau_{\mathsf{F}+\mathsf{G}}\triangledown(\iota_{\mathsf{F}+\mathsf{G}}{\bullet}(\mathsf{U}_{\mathsf{F}+\mathsf{G}})^\flat)$ is an initial
$\mathsf{H} \mapsto \mathsf{Id}{+}(\mathsf{F}{+}\mathsf{G} \bullet \mathsf{H})$ algebra }

$\mathsf{G}^\star \bullet \tau_{\mathsf{F}\bullet\mathsf{G}^\star}\triangledown(\iota_{\mathsf{F}\bullet\mathsf{G}^\star}{\bullet}(\mathsf{U}_{\mathsf{F}\bullet\mathsf{G}^\star})^\flat)$ is an initial $\mathsf{H} \mapsto \mathsf{Id}{+}(\mathsf{F}{+}\mathsf{G} \bullet \mathsf{H})$ algebra

$\Leftarrow$ { by the diagonal rule, an initial $\mathsf{H} \mapsto \mathsf{G}^\star \bullet \mathsf{Id}{+}(\mathsf{F}{\bullet}\mathsf{H})$

$$
\begin{array}{ll}
 & \qquad \{ \quad \text{algebra is an initial } \mathsf{H} \mapsto \mathsf{Id}{+}(\mathsf{F}{+}\mathsf{G} \bullet \mathsf{H}) \text{ algebra} \quad \} \\
 & \mathsf{G}^\star \bullet \tau_{\mathsf{F}\bullet\mathsf{G}^\star} \triangledown (\iota_{\mathsf{F}\bullet\mathsf{G}^\star} \bullet (\mathsf{U}_{\mathsf{F}\bullet\mathsf{G}^\star})^\flat) \text{ is an initial } \mathsf{H} \mapsto \mathsf{G}^\star \bullet \mathsf{Id}{+}(\mathsf{F}\bullet\mathsf{H}) \text{ algebra} \\
\Leftarrow & \qquad \{ \quad \text{rolling rule} \quad \} \\
 & \tau_{\mathsf{F}\bullet\mathsf{G}^\star} \triangledown (\iota_{\mathsf{F}\bullet\mathsf{G}^\star} \bullet (\mathsf{U}_{\mathsf{F}\bullet\mathsf{G}^\star})^\flat) \text{ is an initial } \mathsf{H} \mapsto \mathsf{Id}{+}(\mathsf{F}\bullet\mathsf{G}^\star\bullet\mathsf{H}) \text{ algebra} \\
\equiv & \qquad \{ \quad \text{theorem 7.19} \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

Chapter 8

Applications to Lists

In this chapter we use the categorical fixed point theorems of the previous chapter to prove isomorphisms between certain list structures and simultaneously construct its witnesses. The isomorphisms we are going to prove correspond to well-known properties of the regular algebra. Before we can start with these proofs we first have to give categorical definitions corresponding to the basic elements of regular algebra and give some elementary properties.

8.1 Preliminaries

**** Needs to be revised to exploit previous material on limits and in particular parameterised limit theorem. *****

We begin by giving the definition of a *sum* and a *product* of two objects. Sum is frequently introduced as an instance of a so-called *colimit*. However, this would involve giving full explanation of what a colimit is – a colimit corresponds in lattice theory to a supremum – and this is something we wish to avoid here. Similarly, the definition of a *product* can be given as an instance of a *limit*.

Definition 8.1 (Sum) A *sum* (or *coproduct*) of two objects x and y is an object $x+y$, together with two *injections* $\mathsf{inl}_{x,y} \in x+y \leftarrow x$ and $\mathsf{inr}_{x,y} \in x+y \leftarrow y$ such that for any object z and pair of arrows $f \in z \leftarrow x$ and $g \in z \leftarrow y$ there is a unique arrow $h \in z \leftarrow x+y$ such that (omitting the subscripts)

$$h \circ \mathsf{inl} = f \wedge h \circ \mathsf{inr} = g \quad .$$

□

Definition 8.2 (Product) A *product* of two objects x and y is an object $x \times y$, together with two *projections* $\mathsf{exl}_{x,y} \in x \leftarrow x \times y$ and $\mathsf{exr}_{x,y} \in y \leftarrow x \times y$ such that for any object z and pair of arrows $f \in x \leftarrow z$ and $g \in y \leftarrow z$ there is a unique arrow $h \in x \times y \leftarrow z$ such that (omitting the subscripts)

$$\mathsf{exl} \circ h = f \wedge \mathsf{exr} \circ h = g \quad .$$

□

Sums and products are unique up to isomorphism. A product of two objects in a category is a sum of the two objects in the opposite category. As a result product and sum have similar properties. Without proof we state the following isomorphisms:

$$(a+b)+c \cong a+(b+c)$$

and

$$(a\times b)\times c \cong a\times(b\times c) \quad .$$

In expressions we will assume that $+$ has a lower precedence than $\times$, i.e. an expression like $a\times b+c$ must be parsed as $(a\times b)+c$.

Definition 8.3 (Distributive Category) A category which has an initial object $\emptyset$ and a terminal object $\mathbb{1}$, in which product and sums of pairs of objects exist, and which satisfies the following isomorphisms: for all objects a, b and c

$$a\times(b+c) \cong (a\times b)+(a\times c) \quad ,$$

$$(a+b)\times c \cong (a\times c)+(b\times c) \quad ,$$

$$a\times\emptyset \cong \emptyset$$

and

$$\emptyset\times a \cong \emptyset$$

is called a *distributive* category.

□

We assume all categories in this chapter to be distributive categories.

Having the terminal object $\mathbb{1}$ we can derive the following isomorphisms

$$\mathbb{1}\times a \cong a \quad \text{and} \quad a\times\mathbb{1} \cong a \quad .$$

We repeat the isomorphisms we have so far introduced and furthermore give specific names for the corresponding witnesses. Note, that these isomorphisms hold for all categories in this chapter. Let a, b and c be objects, then:

$$\mathsf{lunit}_a \in \mathbb{1}\times a \cong a \quad ,$$

$$\mathsf{runit}_a \in a\times\mathbb{1} \cong a \quad ,$$

$$\mathsf{sumass}_{a,b,c} \in (a+b)+c \cong a+(b+c) \quad ,$$

$$\mathsf{proass}_{a,b,c} \in (a\times b)\times c \cong a\times(b\times c) \quad ,$$

$$\mathsf{rdist}_{a,b,c} \in (a+b)\times c \cong (a\times c)+(b\times c) \quad ,$$

$$\mathsf{ldist}_{a,b,c} \in a\times(b+c) \cong (a\times b)+(a\times c) \quad .$$

In all cases if η is a witness to an isomorphism then we will denote its inverse by $\eta^{\cup}$.

In a distributive category sums of all objects exist, so we can extend the definition of sum to a binary functor $+$, that maps two objects x and y to the object $x{+}y$. For arrows we argue as follows, suppose $f \in x \leftarrow u$ and $g \in y \leftarrow v$ then we have two arrows $\mathsf{inl}_{x,y} \circ f \in x{+}y \leftarrow u$ and $\mathsf{inr}_{x,y} \circ g \in x{+}y \leftarrow v$ so there is a unique arrow to $x{+}y$ from $u{+}v$. We will denote this arrow by $f{+}g$. Similarly, we can define the binary functor $\times$.

With Fun we will denote the category of all functors between the distributive categories under consideration. These distributive categories will be referred to as the base categories. The sum in the category Fun is related to the sum in the base categories as follows. Assume $+$ is a fixed sum functor on the base categories. Let F and G be two functors with the same domain and codomain, then we define $F\dot{+}G$ by $(F\dot{+}G).x = F.x + G.x$ for x an object or an arrow. Then $F\dot{+}G$ is a functor. Taking $\mathsf{inl}_{F,G} \in F\dot{+}G \leftarrow F$ to be

$$(\mathsf{inl}_{F,G})_x = \mathsf{inl}_{F.x,G.x}$$

and $\mathsf{inr}_{F,G}$ similarly, the triple $(F\dot{+}G, \mathsf{inl}_{F,G}, \mathsf{inr}_{F,G})$ is a sum of F and G in the category Fun. The constructor $\dot{+}$ can be turned into a binary functor on Fun by defining

$$(\eta\dot{+}\tau)_x = \eta_x{+}\tau_x \quad .$$

Similarly $\dot{\times}$ can be defined such that it is a binary product functor on Fun. Finally, Fun is also a distributive category; where for instance

$$\mathsf{sumass}_{F,G,H} \in (F\dot{+}G)\dot{+}H \cong F\dot{+}(G\dot{+}H)$$

is given by

$$(\mathsf{sumass}_{F,G,H})_x = \mathsf{sumass}_{F.x,G.x,H.x} \quad .$$

What we frequently need in the forthcoming proofs are isomorphisms like:

$$b \mapsto (a{+}b){+}c \cong b \mapsto a{+}(b{+}c) \quad .$$

Thus, we want to be able to abstract from one of the arguments in the isomorphisms for sum and product as listed before. With the remarks made above about isomorphisms in the functor category we can argue as follows. Suppose a is an object we let $\mathsf{K}.a$ denote the constant functor on a, i.e. let x be an object then $(\mathsf{K}.a).x = a$ and let f be an arrow then $(\mathsf{K}.a).f = \mathsf{id}_a$. Now we can extend the definition of K to arrows: on arrows f we define $\mathsf{K}.f$ to be a natural transformation by letting $\mathsf{K}.f_x$ equal f for all objects x in the category. We leave it to the reader to verify that also K is a functor. With these definitions we can make calculations similar to the one below: let F, G and H be functors and let a, b and c denote objects:

$$(F\dot{+}G)\dot{+}H \cong F\dot{+}(G\dot{+}H)$$

$\Rightarrow$ { instantiate $F,G,H := \mathsf{K}.a,\mathsf{Id},\mathsf{K}.c$ }

$$(\mathsf{K}.a \dot{+} \mathsf{Id}) \dot{+} \mathsf{K}.c \cong \mathsf{K}.a \dot{+} (\mathsf{Id} \dot{+} \mathsf{K}.c)$$

$\equiv$ { definition $\dot{+}$ }

$$b \mapsto ((\mathsf{K}.a).b + \mathsf{Id}.b) + (\mathsf{K}.c).b \;\cong\; b \mapsto (\mathsf{K}.a).b + (\mathsf{Id}.b + (\mathsf{K}.c).b)$$

$\equiv$ { definition $\mathsf{K}.a$ and Id }

$$b \mapsto (a+b)+c \cong b \mapsto a+(b+c) \quad .$$

We let $\mathsf{sumass}_{a,\_,c}$ denote the isomorphism $b \mapsto (a+b)+c \cong b \mapsto a+(b+c)$. Similarly, we let $\mathsf{sumass}_{a,\_,\_}$ denote the isomorphism $b \mapsto (a+b)+b \cong b \mapsto a+(b+b)$. Recalling the Godement notation: for three functors F, G and H:

$$F\bullet\mathsf{sumass}_{a,\_,\_}\bullet(G,H) \in b \mapsto F.((a+G.b)+H.b) \cong b \mapsto F.(a+(G.b+H.b)) \quad .$$

Definition 8.4 (Cartesian Closed Category) A category is *cartesian closed* if for all objects y the functors $y\times$ and $\times y$, i.e. the functors obtained by fixing in turn the arguments of the binary functor $\times$ to y, both have an upper adjoints.

□

We assume all categories in this section to be cartesian closed.

In this chapter we will frequently construct binary functors using the functors $+$ and $\times$ and then fix one of the arguments. The result is a functor $x\oplus$ for some binary functor $\oplus$. We assume that all these functors have an initial algebra. Thus, we can talk about the codomain of an initial $x\oplus$-algebra denoted by $\mu(x\oplus)$.

Lists can be defined in various (isomorphic) ways. In the examples that follow we use "cons" lists. That is, we assume the functor $*$ is defined by: let a be an object

$$*a = \mu(y \mapsto \mathbb{1} + (a \times y)) \quad .$$

In contrast to normal functor applications we will skip the dot when the functor $*$ is applied to an object. Note, that $*$ is a functor follows immediately from the fact that $*$ is a map operator as defined in section 5.4.

We have seen that we could 'lift' the binary sum and product functor $+$ and $\times$ defined on base categories to a binary sum and product functor $\dot{+}$ and $\dot{\times}$ defined on a functor category. We want to have a similar result for $*$.

With the abstraction theorem for algebras we have proven that $\varpi\bullet\mathsf{F}$ is the carrier of a particular initial $\mathsf{F}\dot{\oplus}$-algebra. Thus we may choose $\mathsf{mu}(\mathsf{F}\dot{\oplus})$ to have $\varpi\bullet\mathsf{F}$ as carrier i.e.

$$(a \mapsto \mu(\mathsf{F}.a\oplus) =)\varpi\bullet\mathsf{F} = \mu(\mathsf{F}\dot{\oplus}) \quad .$$

With the definition of $*$ and the remarks made about abstraction we can make the following calculation:

$$\begin{array}{ll}
& *\bullet\mathsf{F} \\
= & \quad\{ \quad \text{definition } * \quad \} \\
& (a \mapsto \mu(y \mapsto \mathbb{1}+(a\times y)))\bullet\mathsf{F} \\
= & \quad\{ \quad \text{composition} \quad \} \\
& a \mapsto \mu(y \ \mapsto \ \mathbb{1} + \mathsf{F}.a \times y) \\
= & \quad\{ \quad \text{heading for abstraction define } a\oplus y \ = \ \mathbb{1} + \mathsf{F}.a \times y \quad \} \\
& a \mapsto \mu(a\oplus) \\
= & \quad\{ \quad \text{abstraction theorem} \quad \} \\
& \mu(\mathsf{G} \mapsto \mathsf{Id}\dot{\oplus}\mathsf{G}) \\
= & \quad\{ \quad (\mathsf{Id}\dot{\oplus}\mathsf{G}).x = x \oplus \mathsf{G}.x = \mathbb{1}+(\mathsf{F}.x \times \mathsf{G}.x) \\
& \qquad\qquad = (\mathsf{K}.\mathbb{1} \dot{+} \mathsf{F}\dot{\times}\mathsf{G}).x\,,\ \text{extensionality} \quad \} \\
& \mu(\mathsf{G} \mapsto \mathsf{K}.\mathbb{1} \dot{+} \mathsf{F}\dot{\times}\mathsf{G}) \quad .
\end{array}$$

If, we now define the functor $\dot{*}$ on objects F by $\mu(\mathsf{G} \mapsto \mathsf{K}.\mathbb{1} \dot{+} \mathsf{F}\dot{\times}\mathsf{G})$ then we can write

(8.5) $\quad *\bullet\mathsf{F} = \dot{*}\mathsf{F} \quad .$

We recall from the previous chapter that for the functors F, G and H all of appropriate type we have the following isomorphisms

(8.6) $\quad \mathsf{roll}_{\mathsf{F},\mathsf{G}} \ \in \ \mathsf{F}.\mu(\mathsf{G}\bullet\mathsf{F}) \cong \mu(\mathsf{F}\bullet\mathsf{G}) \quad ,$

(8.7) $\quad \mathsf{fuse}_{\mathsf{F},\mathsf{G},\mathsf{H}}.\mathsf{swap} \ \in \ \mathsf{F}.\mu\mathsf{G} \cong \mu\mathsf{H} \quad ,$

where $\mathsf{swap} \in \mathsf{F}\bullet\mathsf{G} \cong \mathsf{H}\bullet\mathsf{F}$ and F has an upper adjoint,

(8.8) $\quad \mathsf{diag}_{\oplus} \in \mu(x \mapsto \mu(y \mapsto x\oplus y)) \cong \mu(x \mapsto x\oplus x) \quad ,$

(8.9) $\quad \mathsf{exch}_{\mathsf{F},\mathsf{G},\mathsf{H}}.\mathsf{mirror} \in \mu(\mathsf{F}\bullet\mathsf{G}) \cong \mu(\mathsf{F}\bullet\mathsf{H}) \quad ,$

where both G and H have an upper adjoint and $\mathsf{mirror} \in \mathsf{H}\bullet\mathsf{F}\bullet\mathsf{G} \cong \mathsf{G}\bullet\mathsf{F}\bullet\mathsf{H}$. Again the inverse of a witness η will be denoted by $\eta^{\cup}$.

If we instantiate $\mathsf{F} := \mathsf{Id}$ in (8.7) we get (Id has an upper adjoint):

$$\mathsf{fuse}_{\mathsf{Id},\mathsf{G},\mathsf{H}}.\mathsf{swap} \in \mu\mathsf{G} \cong \mu\mathsf{H} \quad ,$$

where $\mathsf{swap} \in \mathsf{G} \cong \mathsf{H}$. The functors G and H can be inferred from the typing of the natural isomorphism swap, so in the case that F equals the identity functor we will not explicitly mention the functors anymore, i.e.

$$\mathsf{fuse}.\mathsf{swap} \in \mu\mathsf{G} \cong \mu\mathsf{H} \quad .$$

In this chapter we employ a notation whereby the witnesses to isomorphisms are included in the hints marked by a bullet (“$\bullet$”). Specifically a proof step of the form

$$\begin{array}{ll} & \mathsf{F} \\ \cong & \{ \quad \text{hint}, \bullet \quad \alpha \quad \} \\ & \mathsf{G} \end{array}$$

is short for

$$\alpha \in \mathsf{F} \cong \mathsf{G} \Leftarrow \text{ hint} \quad .$$

8.2 Theorems

In this section, we prove isomorphisms between certain list structures and simultaneously construct its witnesses. The first example is chosen for its simplicity.

Theorem 8.10 (List Fusion)

$$*a \times b \cong \mu(y \mapsto b + a\times y) \quad .$$

Proof We construct a witness α as follows

$$\begin{array}{ll} & \alpha \in *a \times b \cong \mu(y \mapsto b + a\times y) \\ \equiv & \{ \quad \text{definition of } * \quad \} \\ & \alpha \in \mu(y \mapsto \mathbb{1} + a\times y) \times b \cong \mu(y \mapsto b + a\times y) \\ \Leftarrow & \{ \quad \times b \text{ is a lower adjoint for all } b\text{, fusion (8.7)} \\ & \qquad \bullet \quad \alpha = \mathsf{fuse}_{\mathsf{F},\mathsf{G},\mathsf{H}}.\beta\,, \\ & \qquad \text{where } \mathsf{F},\mathsf{G},\mathsf{H} := \times b\,, (y \mapsto \mathbb{1} + a\times y)\,, (y \mapsto b + a\times y) \quad \} \\ & \beta \in y \mapsto (\mathbb{1} + a\times y)\times b \cong y \mapsto b + a\times(y\times b) \\ \Leftarrow & \{ \quad \text{above isomorphisms}, \bullet \quad \beta = \mathsf{rdist}_{\mathbb{1},\_,b}\bullet(a\times)\circ\gamma \quad \} \end{array}$$

$$
\begin{array}{ll}
 & \gamma \;\in\; y \mapsto (\mathbb{1}\times b) + (a\times y)\times b \;\cong\; y \mapsto b + a\times(y\times b) \\
\equiv & \qquad \{ \qquad \text{definition } \dot{+} \quad \} \\
 & \gamma \in (y\mapsto \mathbb{1}\times b)\dot{+}(y\mapsto (a\times y)\times b) \cong (y\mapsto b)\dot{+}(y\mapsto a\times(y\times b)) \\
\Leftarrow & \qquad \{ \qquad \text{definition } \dot{+} \quad \} \\
 & \gamma = \mathsf{K}.(\mathsf{lunit}_b) \dot{+} \mathsf{proass}_{a,\_,b} \quad .
\end{array}
$$

We have thus constructed

$$
\begin{array}{ll}
 & \mathsf{fuse}_{F,G,H}.(\mathsf{rdist}_{\mathbb{1},\_,b}\bullet(a\times)\circ(\mathsf{K}.(\mathsf{lunit}_b)\dot{+}\mathsf{proass}_{a,\_,b})) \\
\in & *a\times b \cong \mu(y \mapsto b + a\times y) \quad ,
\end{array}
$$

where $F,G,H := \times b\,,(y \mapsto \mathbb{1} + a\times y)\,,(y \mapsto b + a\times y)\,.$

□

Corollary 8.11 $\quad (a\times)^{\star} \cong (*a)\times$

Proof Straightforward combination of the abstraction theorem, theorem 7.19 and theorem 8.10.

□

In the remainder of this section we will denote the witness to list fusion by $\mathsf{listfuse}_{a,b}$, so

$$
\mathsf{listfuse}_{a,b} \;\in\; *a\times b \cong \mu(y \mapsto b + a\times y) \quad .
$$

The next example has been chosen for its relative difficulty, and its practical relevance. An instance is the so-called "lines-unlines" problem: given a sequence of two types of characters, delimiters and non-delimiters, write a program to divide the sequence into (possibly empty) "lines" of non-delimiters separated by single delimiters. Construct in addition the inverse of the program.

Expressed as an isomorphism between datatypes, the problem boils down to showing that the star decomposition theorem of regular languages is constructively valid.

Theorem 8.12 (List Decomposition)

$$
*a \times *(b\times *a) \cong *(b{+}a) \quad .
$$

Proof

$$
\begin{array}{ll}
 & *a\times *(b\times *a) \\
= & \qquad \{ \qquad \text{definition of } * \quad \} \\
 & *a\times \mu(y \mapsto \mathbb{1} + (b\times *a)\times y)
\end{array}
$$

$$
\begin{array}{ll}
\cong & \{ \quad \text{Godement's rules,} \bullet \quad \mathsf{id}_{*a} \times \mathsf{fuse}.((\mathbb{1}+)\bullet(\mathsf{proass}_{b,*a,\_})) \quad \} \\
& *a \times \mu(y \mapsto \mathbb{1} + b{\times}(*a \times y)) \\
\cong & \{ \quad \text{rolling rule,} \bullet \quad \mathsf{roll}_{*a\times,F} \\
& \quad\quad \text{we let } F \text{ denote } y \mapsto \mathbb{1} + b{\times}y \quad \} \\
& \mu(y \mapsto *a \times (\mathbb{1} + b{\times}y)) \\
\cong & \{ \quad \text{fusion rule,} \bullet \quad \mathsf{fuse}.((\mathsf{listfuse}_{a,\_})\bullet F) \quad \} \\
& \mu(y \mapsto \mu(z \mapsto (\mathbb{1} + b{\times}y) + a{\times}z)) \\
\cong & \{ \quad \text{diagonal rule,} \bullet \quad \mathsf{diag}_{\oplus} \text{, where } y{\oplus}z = (\mathbb{1} + b{\times}y) + a{\times}z \quad \} \\
& \mu(y \mapsto (\mathbb{1} + b{\times}y) + a{\times}y) \\
\cong & \{ \quad \text{fusion rule,} \bullet \quad \mathsf{fuse}.((\mathsf{sumass}_{\mathbb{1},\_,\_})\bullet(b\times\,,\,a\times)\circ(\mathbb{1}+)\bullet(\mathsf{rdist}^{\cup}_{b,a,\_})) \quad \} \\
& \mu(y \mapsto \mathbb{1} + (b{+}a){\times}y) \\
= & \{ \quad \text{definition of } * \quad \} \\
& *(b{+}a) \quad .
\end{array}
$$

We conclude that,

$$
\begin{array}{ll}
& \mathsf{id}_{*a} \times \mathsf{fuse}.((\mathbb{1}+)\bullet(\mathsf{proass}_{b,*a,\_})) \\
\circ & \mathsf{roll}_{*a\times,F} \\
\circ & \mathsf{fuse}.((\mathsf{listfuse}_{a,\_})\bullet F) \\
\circ & \mathsf{diag}_{\oplus} \\
\circ & \mathsf{fuse}.((\mathsf{sumass}_{\mathbb{1},\_,\_})\bullet(b\times\,,\,a\times)\circ((\mathbb{1}+)\bullet(\mathsf{rdist}^{\cup}_{b,a,\_}))) \\
\in & *a \times *(b \times *a) \cong *(b{+}a) \quad .
\end{array}
$$

where $F = y \mapsto \mathbb{1} + b{\times}y$ and $y{\oplus}z = (\mathbb{1} + b{\times}y) + a{\times}z$.

□

(An alternative "higher level" proof is to use the monad decomposition theorem together with corollary 8.11. This instructive exercise is left to the reader.)

Suppose we denote the above witness by $\mathsf{decomp}_{a,b}$. We can use abstraction to lift the above isomorphism to the level of functors.

$$
\begin{array}{ll}
& a,b \mapsto *a \times *(b \times *a) \\
= & \{ \quad \text{Introducing the functors } \mathit{Exl} \text{ and } \mathit{Exr}\text{,} \\
& \quad\quad \mathit{Exl}.(a,b) = a \text{ and } \mathit{Exr}.(a,b) = b \quad \} \\
& a,b \mapsto *(\mathit{Exl}.(a,b)) \times *(\mathit{Exr}.(a,b) \times *(\mathit{Exl}.(a,b))) \\
= & \{ \quad \text{abstraction} \quad \} \\
& (*\bullet\mathit{Exl}) \dot{\times} (*\bullet(\mathit{Exr} \dot{\times} (*\bullet\mathit{Exl})))
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad \text{abstraction: (8.5)} \quad \} \\
& \dot{*}Exl\dot{\times}\dot{*}(Exr\dot{\times}\dot{*}Exl) \\
\cong & \{ \quad \text{theorem 8.12} \\
& \quad \bullet \quad \mathsf{decomp}_{Exl,Exr} \quad \} \\
& \dot{*}(Exr\dot{+}Exl) \\
= & \{ \quad \text{abstraction} \quad \} \\
& \mathsf{a},\mathsf{b} \mapsto *(\mathsf{b}+\mathsf{a}) \quad .
\end{array}
$$

If full details of the definition of $\mathsf{decomp}_{Exl,Exr}$ are required then we would have to instantiate the above witness in the following way:

$$\mathsf{a},\mathsf{b} \;:=\; Exl,Exr \quad ,$$

$$+,\times,* \;:=\; \dot{+},\dot{\times},\dot{*}$$

and

$$\mathbb{1} \;:=\; \mathsf{K}.\mathbb{1} \quad .$$

Theorem 8.13 (Leapfrog Preservation) Suppose F is a functor and $\otimes$ is a binary functor such that $\otimes\mathsf{a}$ is a lower adjoint for every object a. Define the object part of the functor $\hat{\mathsf{F}}$ by

$$\hat{\mathsf{F}} = \mathsf{a} \mapsto \mu(\mathsf{F}\bullet\mathsf{a}\otimes) \quad .$$

Suppose we have two natural isomorphisms

$$\mathsf{ass}_{\mathsf{a},\mathsf{b},\mathsf{c}} \in (\mathsf{a}\otimes\mathsf{b})\otimes\mathsf{c} \cong \mathsf{a}\otimes(\mathsf{b}\otimes\mathsf{c})$$

and

$$\mathsf{leapF}_{\mathsf{a},\mathsf{b}} \in \mathsf{F}.(\mathsf{a}\otimes\mathsf{b})\otimes\mathsf{a} \cong \mathsf{a}\otimes\mathsf{F}.(\mathsf{b}\otimes\mathsf{a}) \quad .$$

Then

$$\mathsf{a}\otimes\hat{\mathsf{F}}.(\mathsf{b}\otimes\mathsf{a}) \cong \hat{\mathsf{F}}.(\mathsf{a}\otimes\mathsf{b})\otimes\mathsf{a}$$

Proof We start with the left hand side:

$$
\begin{array}{ll}
& \mathsf{a}\otimes\hat{\mathsf{F}}.(\mathsf{b}\otimes\mathsf{a}) \\
= & \{ \quad \text{definition of } \hat{\mathsf{F}} \quad \} \\
& \mathsf{a}\otimes\mu(\mathsf{y}\mapsto\mathsf{F}.((\mathsf{b}\otimes\mathsf{a})\otimes\mathsf{y})) \\
\cong & \{ \quad \bullet \quad \mathsf{id}_{\mathsf{a}}\otimes\mathsf{fuse}.(\mathsf{F}\bullet\mathsf{ass}_{\mathsf{b},\mathsf{a},\_}) \quad \} \\
& \mathsf{a}\otimes\mu(\mathsf{y}\mapsto\mathsf{F}.(\mathsf{b}\otimes(\mathsf{a}\otimes\mathsf{y}))) \\
\cong & \{ \quad \bullet \quad \mathsf{roll}_{\mathsf{a}\otimes,\mathsf{F}\bullet\mathsf{b}\otimes} \quad \} \\
& \mu(\mathsf{y}\mapsto\mathsf{a}\otimes\mathsf{F}.(\mathsf{b}\otimes\mathsf{y})) \quad .
\end{array}
$$

Thus,

$$
\begin{array}{ll}
& \alpha \ \in\ a\otimes\hat{F}.(b\otimes a) \cong \hat{F}.(a\otimes b)\otimes a \\
\Leftarrow & \quad\{\quad \text{above calculation} \\
& \qquad \bullet \quad \alpha \ =\ \mathsf{id}_a \otimes \mathsf{fuse}.(F\bullet\mathsf{ass}_{b,a,\_}) \circ \mathsf{roll}_{a\otimes,F\bullet b\otimes} \circ \beta \quad \} \\
& \beta \ \in\ \mu(y\mapsto a\otimes F.(b\otimes y)) \cong \hat{F}.(a\otimes b)\otimes a \\
= & \quad\{\quad \text{definition of } \hat{F} \quad \} \\
& \beta \ \in\ \mu(y\mapsto a\otimes F.(b\otimes y)) \cong \mu(y\mapsto F.((a\otimes b)\otimes y))\otimes a \\
\Leftarrow & \quad\{\quad \otimes a \text{ is a lower adjoint,} \\
& \qquad \bullet \quad \beta = (\mathsf{fuse}_{\otimes a,G,H}.\gamma)^{\cup} \\
& \qquad \text{where } G,H := y\mapsto F.((a\otimes b)\otimes y), y\mapsto a\otimes F.(b\otimes y) \quad \} \\
& \gamma \ \in\ y\mapsto F.((a\otimes b)\otimes y)\otimes a \ \cong\ y\mapsto a\otimes F.(b\otimes(y\otimes a)) \\
\Leftarrow & \quad\{\quad \bullet \quad \gamma = ((\otimes a)\bullet F\bullet\mathsf{ass}_{a,b,\_})\circ\delta\circ((a\otimes)\bullet F\bullet\mathsf{ass}_{b,\_,a}) \quad \} \\
& \delta \ \in\ y\mapsto F.(a\otimes(b\otimes y))\otimes a \ \cong\ y\mapsto a\otimes F.((b\otimes y)\otimes a) \\
\Leftarrow & \quad\{\quad \text{Godement's rules, assumption} \quad \} \\
& \delta = (\mathsf{leapF}_{a,\_})\bullet(b\otimes) \quad .
\end{array}
$$

We conclude that

$$
\begin{array}{ll}
& \mathsf{id}_a\otimes\mathsf{fuse}.(F\bullet\mathsf{ass}_{b,a,\_}) \circ \mathsf{roll}_{a\otimes,F\bullet b\otimes} \\
\circ & (\mathsf{fuse}_{\otimes a,G,H}.(((\otimes a)\bullet F\bullet\mathsf{ass}_{a,b,\_})\circ(\mathsf{leapF}_{a,\_})\bullet(b\otimes)\circ((a\otimes)\bullet F\bullet\mathsf{ass}_{b,\_,a})))^{\cup} \\
\in & a\otimes\hat{F}.(b\otimes a) \cong \hat{F}.(a\otimes b)\otimes a \quad .
\end{array}
$$

where $G,H := (y\mapsto F.((a\otimes b)\otimes y)), (y\mapsto a\otimes F.(b\otimes y))$.
□

Corollary 8.14 (List Leapfrog)

$$a\times *(b\times a) \cong *(a\times b)\times a \quad .$$

Proof Instantiate in theorem 8.13:

$$F \ := \ \mathbb{1}+ \quad ,$$

$$\otimes \ := \ \times$$

and

$$\mathsf{leapF}_{a,b} \ := \ \mathsf{rdist}_{\mathbb{1},a\times b,a}\circ(\mathsf{lunit}_a\circ\mathsf{runit}_a^{\cup})+\mathsf{proass}_{a,b,a}\circ\mathsf{ldist}_{a,\mathbb{1},b\times a}^{\cup} \quad .$$

To these must be added the substitution

$$\mathsf{ass} := \mathsf{proass} \quad .$$

□

Theorem 8.15 Suppose $\mathsf{Id}\dot{\otimes}$ and $\dot{\otimes}\mathsf{Id}$ are lower adjoints and we have the following isomorphism

$$\mathsf{leapF}_{a,b} \in \mathsf{F}.(a{\otimes}b) \otimes a \cong a \otimes \mathsf{F}.(b{\otimes}a) \quad .$$

Define the map functors $\hat{\mathsf{F}}$ and $\check{\mathsf{F}}$ by

$$\hat{\mathsf{F}} = a \mapsto \mu(\mathsf{F}{\bullet}a{\otimes}) \quad ,$$

and

$$\check{\mathsf{F}} = a \mapsto \mu(\mathsf{F}{\bullet}{\otimes}a) \quad .$$

Then

$$\hat{\mathsf{F}} \cong \check{\mathsf{F}} \quad .$$

Proof We first use the abstraction theorem:

$$\begin{array}{ll}
 & \hat{\mathsf{F}} \\
= & \quad \{ \quad \text{definition} \quad \} \\
 & a \mapsto \mu(\mathsf{F}{\bullet}a{\otimes}) \\
= & \quad \{ \quad \text{define } a{\oplus}y = \mathsf{F}.(a{\otimes}y) \quad \} \\
 & a \mapsto \mu(a{\oplus}) \\
= & \quad \{ \quad \text{abstraction} \quad \} \\
 & \mu(\mathsf{Id}\dot{\oplus}) \\
= & \quad \{ \quad ((\mathsf{Id}\dot{\oplus}).\mathsf{G}).x = x \oplus \mathsf{G}.x = \mathsf{F}.(x \otimes \mathsf{G}.x) \\
 & \qquad\qquad = (((\mathsf{F}{\bullet}){\bullet}(\mathsf{Id}\dot{\otimes})).\mathsf{G}).x \quad \} \\
 & \mu((\mathsf{F}{\bullet}){\bullet}(\mathsf{Id}\dot{\otimes})) \quad .
\end{array}$$

Thus,

$$\begin{array}{ll}
 & \alpha \in \hat{\mathsf{F}} \cong \check{\mathsf{F}} \\
\equiv & \quad \{ \quad \text{above} \quad \} \\
 & \alpha \in \mu((\mathsf{F}{\bullet}){\bullet}(\mathsf{Id}\dot{\otimes})) \cong \mu((\mathsf{F}{\bullet}){\bullet}(\dot{\otimes}\mathsf{Id})) \\
\Leftarrow & \quad \{ \quad \text{exchange rule, } \mathsf{Id}\dot{\otimes} \text{ and } \dot{\otimes}\mathsf{Id} \text{ are lower adjoints}
\end{array}$$

$$\bullet \quad \alpha = \mathsf{exch}_{\mathsf{F}\bullet,\mathsf{Id}\dot{\otimes},\dot{\otimes}\mathsf{Id}}.\beta \quad \}$$

$$\beta \in (\dot{\otimes}\mathsf{Id})\bullet(\mathsf{F}\bullet)\bullet(\mathsf{Id}\dot{\otimes}) \cong (\mathsf{Id}\dot{\otimes})\bullet(\mathsf{F}\bullet)\bullet(\dot{\otimes}\mathsf{Id})$$

$$\equiv \qquad \{ \quad \text{assumption} \quad \}$$

$$\beta = \mathsf{leapF}_{\mathsf{Id},\_} \quad .$$

Thus,

$$\mathsf{exch}_{\mathsf{F}\bullet,\mathsf{Id}\dot{\otimes},\dot{\otimes}\mathsf{Id}}.(\mathsf{leapF}_{\mathsf{Id},\_}) \in \hat{\mathsf{F}} \cong \check{\mathsf{F}} \quad .$$

□

Corollary 8.16 (Cons, Snoc List Isomorphism) Defining "cons" lists by the equation

$$\mathsf{Clist} = a \mapsto \mu(y \mapsto \mathbb{1} + a \times y)$$

and "snoc" lists by the equation

$$\mathsf{Slist} = a \mapsto \mu(y \mapsto \mathbb{1} + y \times a) \quad ,$$

we have

$$\mathsf{Clist} \cong \mathsf{Slist} \quad .$$

Proof Instantiate in theorem 8.15:

$$\mathsf{F} := \mathbb{1}+ \quad ,$$

$$\otimes := \times$$

and

$$\mathsf{leapF}_{a,b} := \mathsf{rdist}_{\mathbb{1},a\times b,a} \circ (\mathsf{lunit}_a \circ \mathsf{runit}_a^{\cup}) + \mathsf{proass}_{a,b,a} \circ \mathsf{ldist}_{a,\mathbb{1},b\times a}^{\cup} \quad .$$

□

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