Domain Operators and Domain Kinds

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Abstract

The notions of domain operator and domain kind are introduced. Several examples are presented. In particular, it is shown that the partial equivalence relations form a domain kind. The proof involves the construction of a Galois connection demonstrating that the partial equivalence relations form a complete lattice under the so-called domain ordering, thus providing another illustration of the importance of the early recognition of Galois connections.

1 Introduction

This paper has been specially prepared for the workshop on Galois connections organised by the (Dutch) Mathematics of Programming group and held on 13th and 14th September, 1993. It is essentially a tutorial on how recognition of a Galois connection aids the construction of a complete lattice of partial equivalence relations. Additionally, the notions domain operator and domain kind are introduced.

A central element of the calculus of datatypes [3, 2, 4, 1] developed by the authors and their students is the notion of a monotype (called a co-reflexive by Freyd and Scedrov [7]). The calculus is a calculus of *total* operators on *untyped* relations but the so-called left and right domain operators in the calculus can always be used to give a minimal typing to the objects of the calculus in terms of a pair of monotypes. Furthermore the monotypes form a complete lattice.

Voermans [10] and Hutton and Voermans [8] have explored a variation on this theory in which the role of the monotypes, and the left and right (monotype) domain operators is replaced by partial equivalence relations (henceforth abbreviated to pers) and the per left and right domain operators. As part of his research Voermans proved that the pers form a complete lattice under a particular ordering. His proof was subsequently improved by van der Woude, exploiting the notion of a Galois connection, and reported by Voermans and van der Woude in outline in [11].

This paper is a tutorial-style presentation of that proof. It also suggests a novel reorganisation of Voermans' work, based on the notion of domain kind introduced here, simultaneously providing the link between his work and the earlier work of Backhouse, Malcolm and van der Woude [5, 6].

2 Domain Operators

Our concern is to formulate a general notion of domain operator using the monotype domain operators as model. The characteristics of the monotypes from which we want to abstract are as follows:

(a) The right (monotype) domain operator satisfies, for all specs R,

$$R{>} = (R{>}){>} \quad ,$$

and

$$R = R{>} \;\equiv\; \mathit{monotype}.R \quad .$$

(b) For all specs R, and all monotypes B,

$$R \circ B = R \;\equiv\; R{>} \sqsubseteq B \quad .$$

(c) Dually, the left (monotype) domain operator satisfies, for all specs R,

$$R{<} = (R{<}){<} \quad ,$$

and

$$R = R{<} \;\equiv\; \mathit{monotype}.R \quad .$$

(d) For all specs R, and all monotypes B,

$$B \circ R = R \;\equiv\; R{<} \sqsubseteq B \quad .$$

(e) The monotypes form a complete lattice (under the usual spec ordering).

Property (a) states that the range of the operator $>$ is equal to its set of fixed points (the monotypes); (c) is just the dual property of the operator $<$. Reading the equation $R \circ B = R$ as "B is a right domain of R", property (b) states that $R{>}$ is the least right domain of R, and (d) is its dual.

Note that the monotypes are uniquely defined by the right domain operator. Let us therefore introduce the following abstract notion of right domain operator.

Definition 1 (Right Domain Operator) A *right domain operator* is a function $\dagger$ from specs to specs such that

(a) $\dagger \;=\; \dagger \cdot \dagger \quad .$

(I.e. the range of $\dagger$ is equal to its set of fixed points.)

(b) The fixed points of $\dagger$ form a complete lattice under some ordering $\lhd_r$.

(c) For all specs R, and all fixed points B of $\dagger$,

$$R \circ B = R \;\equiv\; R\dagger \lhd_r B \quad .$$

□

If $\dagger$ is a right domain operator then we shall say that its set of fixed points is a *right domain kind.*

Note that we do not require that a right domain operator be monotonic with respect to the normal spec ordering. Had we done so then requirement (a) would become equivalent to Scott's [9] notion of a "retraction". The principal example in this paper involves a right domain operator that is not a retraction.

A *left domain operator* is defined dually. Specifically,

Definition 2 (Left Domain Operator) A *left domain operator* is a function $\dagger$ from specs to specs such that

(a) $\dagger \;=\; \dagger \cdot \dagger \quad .$
(b) The fixed points of $\dagger$ form a complete lattice under some ordering $\lhd_l$
(c) For all specs R, and all fixed points B of $\dagger$,
$$B \circ R = R \;\equiv\; R\dagger \lhd_l B \quad .$$

$\Box$

A *left domain kind* is a subset of the set of all specs that is the range of a left domain operator.

Definition 3 (Domain Kind) A subset $\mathcal{K}$ of the specs is said to be a *domain kind* if it is both a left and a right domain kind and the orderings $\lhd_l$ and $\lhd_r$ coincide on its elements.

$\Box$

Note that being both a left domain kind and a right domain kind does not make a subset of the specs a domain kind. We shall exhibit a set of specs (the partial preorders) as an example.

3 Some Examples

There are three well-known examples of domain kinds. The monotypes have already been mentioned and form the basis for the definition. The other two examples are the preorders (the reflexive, transitive specs) and the equivalence relations (the symmetric preorders).

The preorders are a right domain kind because:

(a) The preorders form a complete lattice under $\sqsupseteq$, the converse of the spec ordering.
(b) For all specs R,
$$(R\backslash R)\backslash(R\backslash R) \;=\; R\backslash R$$
and, for all specs R,
$$R = R\backslash R \;\equiv\; R \text{ is a preorder.}$$
(c) For all specs R and all preorders B,
$$R \circ B = R \;\equiv\; R\backslash R \;\sqsupseteq\; B \quad .$$

Thus the preorder right domain operator is the function mapping spec R to $R\backslash R$. Dually, the preorders form a left domain kind (with left domain operator the function mapping spec R to R/R), and, since the domain ordering is in both cases the converse of the spec ordering, they form a domain kind.

The ers (equivalence relations) also form a right domain kind because:

(a) The ers form a complete lattice under $\sqsupseteq$, the converse of the spec ordering.

(b) For all specs R,

$$R\backslash R \sqcap (R\backslash R)\cup \text{ is an er,}$$

and, for all specs R,

$$R = R\backslash R \sqcap (R\backslash R)\cup \equiv R \text{ is an er .}$$

(c) For all specs R and all ers B,

$$R \circ B = R \equiv R\backslash R \sqcap (R\backslash R)\cup \sqsupseteq B \ .$$

(The form of (b) differs from that in the definition of right domain operator but it is easily verified that the two are equivalent.)

4 The Domain Ordering

An immediate consequence of clauses (a) and (c) in definition 1 is that the partial ordering needed to show that a given function is a right domain operator is uniquely defined and, indeed, independent of the function. To see this assume $\dagger$ is a right domain operator with range $\mathcal{K}$. Then, for A and B in $\mathcal{K}$,

$$
\begin{array}{ll}
 & A \lhd_r B \\
\equiv & \{ \quad 1(a),\ A = R\dagger, \text{ for some } R \quad \} \\
 & A\dagger \lhd_r B \\
\equiv & \{ \quad 1(c) \quad \} \\
 & A \circ B = A \ .
\end{array}
$$

Thus if $\mathcal{K}$ is a right domain kind, ordered by $\lhd_r$, and A and B are elements of $\mathcal{K}$,

$$A \lhd_r B \equiv A \circ B = A \ .$$

Now, the binary relation $\lhd_r$ defined on pairs of specs R and S by

(4) $\quad R \lhd_r S \equiv R \circ S = R$

is necessarily transitive (since composition is associative) but is typically neither reflexive nor anti-symmetric. To show that $\dagger$ is a right domain operator it is thus necessary (but, of course, not sufficient) to show that the relation is both reflexive and anti-symmetric when restricted to the range of $\dagger$.

Dually, to show that $\dagger$ is a left domain operator we have to show that, restricted to the range of $\dagger$, the relation $\lhd_l$ defined by

(5) $\quad R \lhd_l S \equiv S \circ R = R$

is reflexive and anti-symmetric.

Suppose a right domain operator $\dagger$ has been constructed. Suppose further that its range is the set $\mathcal{K}$. If all elements of $\mathcal{K}$ are symmetric we may then immediately conclude that $\mathcal{K}$ is a domain kind. For the function mapping spec R to $(R\cup)\dagger$ is clearly a left domain operator with range $\mathcal{K}$. Moreover, the two relations $\lhd_r$ and $\lhd_l$ coincide on $\mathcal{K}$ (since for symmetric R and S, $\;S \circ R \;=\; R \;\equiv\; R \circ S \;=\; R\,$).

Our goal is to show that the pers form a domain kind. Since pers are by definition symmetric it suffices, by the above argument, to construct a right domain operator with range the pers. We therefore have to construct a right per domain operator $\dagger$ such that, with relation $\lhd$ defined by

$$A \lhd B \;\equiv\; A \circ B \;=\; A \quad ,$$

we have:

(a) For all specs R,

$$per\,.\,R\dagger \quad ,$$

and, for all specs R,

$$per.R \;\equiv\; R \;=\; R\dagger \quad .$$

(b) For all specs R and all pers B,

$$R \circ B = R \;\equiv\; R\dagger \lhd B \quad .$$

Furthermore the domain relation $\lhd$ should partially order the pers, and the pers should form a complete lattice under this ordering.

Of these tasks the minimality of the domain operator and the completeness of the lattice structure are the most difficult. Resolution of the other tasks, however, gives the necessary insight into their solution.

Note that it is trivial to show that the pers form a complete lattice under the usual spec ordering: a spec R is a per if and only if it is a fixed point of the function $(R \;\mapsto\; R \circ R\cup)$. Since this function is monotonic with respect to the spec ordering the result follows by Tarski's theorem (the fixed points of a monotonic endofunction on a complete lattice form a complete lattice). However, it is not this lattice we are interested in. This is not made clear in Voerman's work.

5 Partial Preorders

Rather than rush straight into a proof that the pers form a domain kind we take a hint from the proof that the ers form a domain kind. Note that the ers are the *symmetric* preorders and the preorders form a domain kind. Knowing the latter and knowing that the converse operator is an isomorphism, it is straightforward to see that the ers also form a domain kind. Guided by this train of thought we postpone consideration of the fact that pers are symmetric until the last possible moment. Instead of considering the partial *equivalence* relations we thus begin by considering the partial *preorders*.

Definition 6 (Partial preorders) Spec R is said to be a partial preorder if and only if

$$R{<} \sqsubseteq R \;\wedge\; R{>} \sqsubseteq R \;\wedge\; R \circ R \sqsubseteq R \;\;.$$

We use $\mathcal{P}$ to denote the set of partial preorders.

□

Partial preorders are "partial" in that they are "partially" reflexive. At the point level the notion of partial reflexivity is expressed by:

$$\forall(x : \exists(y :: xRy) \vee \exists(y :: yRx) : xRx) \;\;.$$

The claim is that the partial preorders form a right domain kind. (They also form a left domain kind but they do not form a domain kind.) We begin a proof of this claim by first assembling some lemmas characterising the partial preorders in alternative ways. This groundwork leads to more efficient proofs later.

5.1 Alternative Characterisations

The first lemma shows that the left and right monotype domains of a partial preorder are equal as well as strengthening the transitivity property.

Lemma 7 $\quad R{\in}\mathcal{P} \quad\equiv\quad R{<} = R{>} \sqsubseteq R \;\;\wedge\;\; R \circ R = R \;\;.$

Proof Follows-from is obvious. Only the implication needs proof. Assume that R is a partial preorder. Then:

$$\begin{array}{ll}
 & R{<} \;=\; R{>} \\
\equiv & \qquad \{ \qquad \text{calculus} \quad \} \\
 & R{<} \sqsubseteq R{>} \;\wedge\; R{<} \sqsupseteq R{>} \\
\equiv & \qquad \{ \qquad (R{<}){>} \;=\; R{<} \;\wedge\; (R{>}){<} \;=\; R{>} \quad \} \\
 & (R{<}){>} \sqsubseteq R{>} \;\wedge\; R{<} \sqsupseteq (R{>}){<} \\
\Leftarrow & \qquad \{ \qquad \text{monotonicity} \quad \} \\
 & R{<} \sqsubseteq R \;\wedge\; R \sqsupseteq R{>} \\
\Leftarrow & \qquad \{ \qquad \text{definition} \quad \} \\
 & R{\in}\mathcal{P} \;\;.
\end{array}$$

Second:

$$\begin{array}{ll}
 & R \circ R \;=\; R \\
\equiv & \qquad \{ \qquad R \text{ is transitive} \quad \} \\
 & R \circ R \sqsupseteq R \\
\equiv & \qquad \{ \qquad \text{monotype domains} \quad \}
\end{array}$$

$$
\begin{array}{ll}
 & R \circ R \sqsupseteq R \circ R{>} \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R \sqsupseteq R{>} \\
\Leftarrow & \{ \quad \text{definition} \quad \} \\
 & R{\in}\mathcal{P} \quad .
\end{array}
$$

□

In the second lemma we examine the individual clauses in the definition of a partial preorder with a view to unifying them. The form of the lemmas is guided by the knowledge that $R\backslash R$ is the right domain operator of the preorders.

Lemma 8

(a) $\quad R{>} \sqsubseteq R \;\equiv\; R{>} \circ R\backslash R \sqsubseteq R$,

(b) $\quad R{<} \sqsubseteq R \;\equiv\; R{<} \circ R\backslash R \sqsubseteq R$,

(c) $\quad R \circ R \sqsubseteq R \;\equiv\; R \sqsubseteq R{<} \circ R\backslash R$,

(d) $\quad (R{>} \circ R\backslash R){<} \;=\; R{>} \;=\; (R{>} \circ R\backslash R){>}$.

Proof Parts (a) and (b) are both instances of the factor cancellation property:

$$R \;=\; R/(R\backslash R) \quad ,$$

which is established by the following argument: for all specs R and S,

$$
\begin{array}{ll}
 & S \sqsubseteq R \\
\Leftarrow & \{ \quad I \sqsubseteq R\backslash R \quad \} \\
 & S \circ R\backslash R \sqsubseteq R \\
\Leftarrow & \{ \quad \text{factor cancellation} \quad \} \\
 & S \sqsubseteq R \quad .
\end{array}
$$

For (c) we have:

$$
\begin{array}{ll}
 & R \circ R \sqsubseteq R \\
\equiv & \{ \quad \text{factors} \quad \} \\
 & R \sqsubseteq R\backslash R \\
\equiv & \{ \quad (\Rightarrow)\ \text{monotonicity and } R{<} \circ R \;=\; R\ , \quad (\Leftarrow)\ R{<} \sqsubseteq I \quad \} \\
 & R \sqsubseteq R{<} \circ R\backslash R \quad .
\end{array}
$$

Finally for (d) we have:

$$
\begin{array}{ll}
 & (R{>} \circ R\backslash R){<} \\
= & \{ \quad \text{monotype domains} \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & (R{>} \circ (R\backslash R){<}){<} \\
= & \quad \{ \quad (R\backslash R){<} \;=\; I \quad \} \\
 & R{>} \\
= & \quad \{ \quad \text{factor cancellation} \quad \} \\
 & (R \circ R\backslash R){>} \\
= & \quad \{ \quad \text{monotype domains} \quad \} \\
 & (R{>} \circ R\backslash R){>} \quad .
\end{array}
$$

□

5.2 The Right Domain Operator

Lemma 8 gives us a candidate for the partial preorder right domain operator, namely the function mapping spec R to $R{>} \circ R\backslash R$. We therefore propose the definition:

(9) $\quad R{\succ} \;=\; R{>} \circ R\backslash R$.

Note that $R \;=\; R \circ R{\succ}$ since $R \;=\; R \circ R{>}$ and $R \;=\; R \circ R\backslash R$. So $R{\succ}$ is a "domain" of R.

The function $\succ$ was introduced and studied in [6] where the goal was to deduce a general characterisation of a congruence relation on a datatype. Indeed, the strategy we are now following of postponing consideration of symmetry is motivated by its successful use in that application.

In the following lemma we observe that the fixed points of $\succ$ are exactly the partial preorders. Recall that $\mathcal{P}$ denotes the set of partial preorders.

Lemma 10 For all specs R,

(a) $\quad R{\succ} \in \mathcal{P}$,
(b) $\quad R{\in}\mathcal{P} \;\equiv\; R \;=\; R{\succ}$.

Proof We begin with (a). By the definition of partial preorder we have to verify three properties:

$$
\begin{array}{l}
(R{>} \circ R\backslash R){>} \;\sqsubseteq\; R{>} \circ R\backslash R \;\; , \\
(R{>} \circ R\backslash R){<} \;\sqsubseteq\; R{>} \circ R\backslash R \;\; , \\
R{>} \circ R\backslash R \quad \text{is transitive.}
\end{array}
$$

The first two are consequences of 8 (d) and $I \sqsubseteq R\backslash R$. The third requires a short calculation:

$$
\begin{array}{ll}
 & R{>} \circ R\backslash R \circ R{>} \circ R\backslash R \\
= & \quad \{ \quad \text{lemma 8(d), monotype domains} \quad \} \\
 & R{>} \circ R\backslash R \circ R\backslash R \\
= & \quad \{ \quad \text{factor cancellation} \quad \} \\
 & R{>} \circ R\backslash R \quad .
\end{array}
$$

Thus (a) has been established. Now, immediate from (a) is the follows-from part of (b). It suffices thus to establish the implication. Assume $R{\in}\mathcal{P}$. Then:

$$
\begin{array}{ll}
 & \mathit{true} \\
\Rightarrow & \quad \{ \quad \text{definition of partial preorder and 8(b) and (c)} \quad \} \\
 & R \;=\; R{<} \circ R\backslash R \\
\equiv & \quad \{ \quad \text{by lemma 7,} \quad R{>} \;=\; R{<} \quad \} \\
 & R \;=\; R{>} \circ R\backslash R \quad .
\end{array}
$$

□

5.3 Poset Structure

As discussed earlier, for it to be at all possible that the partial preorders form a right domain kind it is necessary that the relation $\lhd_r$ is both reflexive and anti-symmetric. This is soon verified. For reflexivity we have, for all partial preorders A:

$$
\begin{array}{ll}
 & A \;\lhd_r\; A \\
\equiv & \quad \{ \quad \text{definition} \quad \} \\
 & A \circ A \;=\; A \\
\equiv & \quad \{ \quad \text{lemma 7} \quad \} \\
 & \mathit{true} \quad .
\end{array}
$$

Now we check anti-symmetry. Assume A and B are partial preorders. Then, observing from lemma 7 that $A \;\sqsupseteq\; A{>} \;=\; A{<}$ and likewise for B, we have:

$$
\begin{array}{ll}
 & A \;\lhd_r\; B \quad \wedge \quad B \;\lhd_r\; A \\
\equiv & \quad \{ \quad \text{definition} \quad \} \\
 & A \circ B \;=\; A \;\wedge\; B \circ A \;=\; B \\
\Rightarrow & \quad \{ \quad \text{monotype domains, } A \sqsupseteq A{>},\, B \sqsupseteq B{>} \quad \} \\
 & B{>} \sqsupseteq A{>} \quad \wedge \quad A{>} \sqsupseteq B{>} \quad \wedge \quad A{>} \circ B \;\sqsubseteq\; A \quad \wedge \quad B{>} \circ A \;\sqsubseteq\; B \\
\equiv & \quad \{ \quad \text{anti-symmetry and substitution} \quad \} \\
 & B{>} \;=\; A{>} \quad \wedge \quad B{>} \circ B \;\sqsubseteq\; A \quad \wedge \quad A{>} \circ A \;\sqsubseteq\; B \\
\Rightarrow & \quad \{ \quad A{>} \;=\; A{<} \text{ and } B{>} \;=\; B{<} \text{ ; monotype domains} \quad \} \\
 & A \;=\; B \quad .
\end{array}
$$

We conclude:

Lemma 11 The binary relation $\lhd_r$ is a partial ordering when restricted to partial preorders.

□

5.4 Complete Lattice

We now show that the partial preorders form a complete lattice. We almost have all the ingredients of a Galois connection that will allow us to draw this conclusion. Consider the function τ defined on all specs by

$$\tau.R \;=\; (R\!>\,,\, R\backslash R) \quad .$$

The function τ maps specs into the (complete) lattice $(\mathcal{M}\times\mathcal{T}\;,\;\sqsubseteq\times\sqsupseteq)$ where $\mathcal{M}$ denotes the monotype domain kind and $\mathcal{T}$ the preorder domain kind. Also, considering the function λ defined by

$$\lambda.(R{,}S) \;=\; R\circ S$$

we have, by lemma 10(b) that

$$(12) \qquad R{\in}\mathcal{P} \;\;\equiv\;\; R \;=\; \lambda.(\tau.R) \quad .$$

The functions λ and τ, suitably restricted in their domains, will form the two adjoint functions in our Galois connection.

Let $\mathcal{Q}$ denote some, as yet unspecified, subset of $\mathcal{M}\times\mathcal{T}$. Let $\tau_{\mathcal{P}}$ and $\lambda_{\mathcal{Q}}$ denote the restriction of the functions τ and λ to the domains $\mathcal{P}$ and $\mathcal{Q}$, respectively. In order to conclude that $\mathcal{P}$ is a complete lattice we construct $\mathcal{Q}$ such that

(a) $\tau_{\mathcal{P}}\in\mathcal{Q}{\leftarrow}\mathcal{P}$, $\lambda_{\mathcal{Q}}\in\mathcal{P}{\leftarrow}\mathcal{Q}$ and $\lambda_{\mathcal{Q}}$ is surjective .

(b) For all $p{\in}\mathcal{P}$ and $q{\in}\mathcal{Q}$,

$$\tau_{\mathcal{P}}.p \;\;\sqsubseteq\times\sqsupseteq\;\; q \quad\equiv\quad p \;\lhd_r\; \lambda_{\mathcal{Q}}.q \quad .$$

(c) $\mathcal{Q}$ is a complete lattice.

We construct $\mathcal{Q}$ by guaranteeing that it is the largest subset of $\mathcal{M}\times\mathcal{T}$ satisfying (a).

Lemma 13 Let m be a monotype and let T be a preorder. Then

$$\lambda.(m{,}T)\in\mathcal{P} \quad\equiv\quad (m\circ T)\!>\,\sqsubseteq m \quad .$$

Hence with $\mathcal{Q}$ defined by

$$\mathcal{Q} \;=\; \{(m{,}T)|\; m{\in}\mathcal{M} \;\wedge\; T{\in}\mathcal{T} \;\wedge\; (m\circ T)\!>\,\sqsubseteq m\}$$

we have

$$\tau_{\mathcal{P}}\in\mathcal{Q}{\leftarrow}\mathcal{P} \;\;,\quad \lambda_{\mathcal{Q}}\in\mathcal{P}{\leftarrow}\mathcal{Q} \quad\text{and}\quad \lambda_{\mathcal{Q}} \text{ is surjective onto } \mathcal{P}.$$

Proof

$$
\begin{array}{ll}
 & \lambda.(m,T) \in \mathcal{P} \\
\equiv & \{ \quad \text{definition of } \lambda \text{ and } \mathcal{P} \quad \} \\
 & (m \circ T)> \;=\; (m \circ T)< \;\sqsubseteq\; m \circ T \quad \wedge \quad m \circ T \circ m \circ T \sqsubseteq m \circ T \\
\equiv & \{ \quad (m \circ T)< \;=\; m \;\sqsubseteq\; m \circ T \quad \text{since } T< \;=\; I \text{ and } I \sqsubseteq T. \\
 & \quad\;\; m \circ T \circ m \circ T \sqsubseteq m \circ T \quad \text{since} \quad m \sqsubseteq I \quad \text{and} \quad T \circ T \sqsubseteq T \\
 & \quad\;\; (m \circ T)> \sqsupseteq m \quad \text{since} \quad I \sqsubseteq T. \quad \} \\
 & (m \circ T)> \sqsubseteq m \quad .
\end{array}
$$

Thus $\lambda_{\mathcal{Q}} \in \mathcal{P} \leftarrow \mathcal{Q}$.

Now, for all specs R, $\tau.R \;=\; (R> \,,\, R\backslash R)$ and $(R> \circ R\backslash R)> \;=\; R>$. Thus $\tau.R \in \mathcal{Q}$. In particular, $\tau_{\mathcal{P}} \in \mathcal{Q} \leftarrow \mathcal{P}$.

Finally, by lemma 8(b),

$$
p{\in}\mathcal{P} \quad \equiv \quad p \;=\; \lambda.(\tau.p) \quad .
$$

Hence $\lambda_{\mathcal{Q}} \circ \tau_{\mathcal{P}}$ is the identity function on $\mathcal{P}$ from which it is immediate that $\lambda_{\mathcal{Q}}$ is surjective onto $\mathcal{P}$.
□

Now we verify (b).

Lemma 14 For all specs R and $(m,T) \in \mathcal{Q}$,

(15) $\quad \tau.R \;\; \sqsubseteq \times \sqsupseteq \;\; (m,T) \quad \equiv \quad R \;=\; R \circ m \circ T \quad .$

In particular for all $p{\in}\mathcal{P}$ and $(m,T) \in \mathcal{Q}$,

(16) $\quad \tau.p \;\; \sqsubseteq \times \sqsupseteq \;\; (m,T) \quad \equiv \quad p \;\lhd_r\; \lambda.(m,T) \quad .$

Proof For all specs R and $(m,T) \in \mathcal{Q}$,

$$
\begin{array}{ll}
 & \tau.R \;\sqsubseteq \times \sqsupseteq\; (m,T) \\
\equiv & \{ \quad \text{definition of } \tau \quad \} \\
 & R> \sqsubseteq m \;\wedge\; R\backslash R \sqsupseteq T \\
\equiv & \{ \quad \text{monotype domains; factors and } I \sqsubseteq T \quad \} \\
 & R \;=\; R \circ m \;\wedge\; R \;=\; R \circ T \\
\equiv & \{ \quad (\Rightarrow) \quad \text{calculus;} \\
 & \quad\;\; (\Leftarrow) \;\; (m,T) \in \mathcal{Q} \text{ . I.e.} \quad (m \circ T)> \;=\; m \text{ and } T \;=\; T \circ T \quad \} \\
 & R \;=\; R \circ m \circ T \quad .
\end{array}
$$

Property 16 follows by definition of λ and the relation $\lhd_r$.
□

The final task is to prove (c).

Lemma 17 $\mathcal{Q}$ is a complete lattice.

Proof $(\mathcal{M} \times \mathcal{T}\ ,\ \sqsubseteq\ \times\ \sqsupseteq)$ is clearly a complete lattice. The supremum of a bag of pairs $\{(m_i,T_i)|\ i{\in}I\}$ is the pair $(\sqcup.(i\ ::\ \ m_i)\ ,\sqcap.(i\ ::\ \ T_i))$. It suffices thus to show that if $\{(m_i,T_i)|\ i{\in}I\} \subseteq \mathcal{Q}$ then $(\sqcup.(i\ ::\ m_i)\ ,\sqcap.(i\ ::\ T_i)) \in \mathcal{Q}$. So, assume for all i, $(m_i,T_i) \in \mathcal{Q}$. Then,

$$
\begin{array}{ll}
 & (\sqcup.(i :: \ m_i) \circ \sqcap.(i :: \ T_i))\!> \\
= & \qquad \{ \qquad \text{distributivity of composition and } > \quad \} \\
 & \sqcup.(i :: \ (m_i \circ \sqcap.(j :: \ T_j))\!>) \\
\sqsubseteq & \qquad \{ \qquad \sqcap.(j :: \ T_j) \sqsubseteq T_i \quad \} \\
 & \sqcup.(i :: \ (m_i \circ T_i)\!>) \\
\sqsubseteq & \qquad \{ \qquad (m_i,T_i) \in \mathcal{Q} \quad \} \\
 & \sqcup.(i :: \ m_i) \quad .
\end{array}
$$

□

We conclude:

Theorem 18 The partial preorders form a complete lattice.

Proof By lemmas 13, 14 and 17 we have constructed a Galois connection between a complete lattice and the set of partial preorders whereby the adjoint function to the partial preorders is surjective. It thus follows that the partial preorders form a complete lattice.
□

5.5 Least Right Domain

Our next task is to verify that $R\succ$ is the least partial-preorder right domain of R.

Theorem 19 For all partial preorders p,

$$R \circ p \ = \ R \quad \equiv \quad R\succ \ \lhd_r \ p \quad .$$

Proof

$$
\begin{array}{ll}
 & R\succ \ \lhd_r \ p \\
\equiv & \quad \{ \quad \text{definition of } \lhd_r \quad \} \\
 & R\succ \circ p \ = \ R\succ \\
\Rightarrow & \quad \{ \quad R \circ R\succ \ = \ R \quad \} \\
 & R \circ p \ = \ R \\
\equiv & \quad \{ \quad \lambda.(\tau.p) \ = \ p \quad \} \\
 & R \circ \lambda.(\tau.p) \ = \ R \\
\equiv & \quad \{ \quad (15),\ \tau.p \in \mathcal{Q} \quad \} \\
 & \tau.R \ \ \sqsubseteq \times \sqsupseteq \ \ \tau.p \\
\Rightarrow & \quad \{ \quad \tau.R \in \mathcal{Q} \quad \text{and} \quad \lambda_{\mathcal{Q}} \quad \text{is monotonic} \quad \} \\
 & \lambda.(\tau.R) \ \lhd_r \ \lambda.(\tau.p) \\
\equiv & \quad \{ \quad \lambda.(\tau.p) \ = \ p \ ; \text{def. } \succ \quad \} \\
 & R\succ \ \lhd_r \ p \quad .
\end{array}
$$

□

6 Pers

The pers are the symmetric partial preorders and they form a subset of the partial preorders on which the left and the right domain orders coincide. More precisely, defining the function sym by

$$sym.R \ = \ R \sqcap R\cup \quad ,$$

we have:

Lemma 20 For all specs R,

$$per.R \ \equiv \ \exists(p: \ p \text{ is a partial preorder: } \ R \ = \ sym.p)$$

Proof Follows-from is well-known: the partial preorders are transitive, and transitivity is preserved under converse ($\cup$) and intersection ($\sqcap$). Moreover all specs in the range of sym are clearly symmetric. Thus if p is a partial preorder then $sym.p$ is symmetric and transitive, i.e. a per.

Conversely, if R is a per then $R \ = \ sym.R$, since R is symmetric. Moreover, R is a partial preorder by definition.
□

With this characterisation of the pers the task of showing that the pers form a complete lattice under the domain ordering can be resolved by showing that sym is a lower adjoint of the embedding of the pers in the partial preorders:

Lemma 21 For all partial preorders p and all pers A,

$$p \sqcap p^{\cup} \lhd A \quad \equiv \quad p \lhd_r A \ .$$

(Note: we use the symbol $\lhd$ for the domain ordering when restricted to pers since, on pers, the two relations $\lhd_r$ and $\lhd_l$ are equal.)

Proof First note that $p \lhd_r p \sqcap p^{\cup}$ since

$$\begin{array}{ll} & p \circ (p \sqcap p^{\cup}) \\ \sqsupseteq & \{ \quad p \sqcap p^{\cup} \sqsupseteq p> \quad \} \\ & p \\ = & \{ \quad p \lhd_r p \quad \} \\ & p \circ p \\ \sqsupseteq & \{ \quad \text{monotonicity} \quad \} \\ & p \circ (p \sqcap p^{\cup}) \ . \end{array}$$

Hence,

$$\begin{array}{ll} & p \sqcap p^{\cup} \lhd A \\ \equiv & \{ \quad \text{definition} \quad \} \\ & (p \sqcap p^{\cup}) \circ A = p \sqcap p^{\cup} \\ \Rightarrow & \{ \quad \text{Leibniz} \quad \} \\ & p \circ (p \sqcap p^{\cup}) \circ A = p \circ (p \sqcap p^{\cup}) \\ \equiv & \{ \quad p \lhd_r p \sqcap p^{\cup} \text{, definition of } \lhd_r \quad \} \\ & p \lhd_r A \ . \end{array}$$

This proves one half of the equivalence. For the other half a more complicated argument is needed. Assume $p \lhd_r A$. I.e. assume $p \circ A = p$.

We split the equality $(p \sqcap p^{\cup}) \circ A = p \sqcap p^{\cup}$ into two inclusions. For the case

$$(p \sqcap p^{\cup}) \circ A \sqsupseteq p \sqcap p^{\cup}$$

we argue that it suffices to prove that $A> \sqsupseteq (p \sqcap p^{\cup})>$ since, for all specs R and S,

$$\begin{array}{ll} & R \circ S \sqsupseteq R \\ \Rightarrow & \{ \quad \text{monotype domains} \quad \} \\ & S> \sqsupseteq R> \\ \Rightarrow & \{ \quad \text{monotonicity, monotype domains} \quad \} \end{array}$$

$$
\begin{array}{ll}
 & R \circ S{>} \;\sqsupseteq\; R \\
\Rightarrow & \quad \{ \quad \bullet \quad S \sqsupseteq S{>}\ , \quad \} \\
 & R \circ S \sqsupseteq R \ .
\end{array}
$$

I.e. $(R \circ S \sqsupseteq R \;\equiv\; S{>} \sqsupseteq R{>}) \;\Leftarrow\; S \sqsupseteq S{>}$. But for all pers A it is indeed the case that $A \sqsupseteq A{>}$. Hence,

$$
\begin{array}{ll}
 & (p \sqcap p^{\cup}) \circ A \;\sqsupseteq\; p \sqcap p^{\cup} \\
\equiv & \quad \{ \quad \text{above argument} \quad \} \\
 & A{>} \sqsupseteq (p \sqcap p^{\cup}){>} \\
\equiv & \quad \{ \quad \text{assumption: } p \circ A \;=\; p\text{. Hence } A{>} \sqsupseteq p{>}. \quad \} \\
 & p{>} \sqsupseteq (p \sqcap p^{\cup}){>} \\
\equiv & \quad \{ \quad \text{monotonicity} \quad \} \\
 & \mathit{true} \ .
\end{array}
$$

Now we turn to the other inclusion.

$$
\begin{array}{ll}
 & (p \sqcap p^{\cup}) \circ A \;\sqsubseteq\; p \sqcap p^{\cup} \\
\Leftarrow & \quad \{ \quad \text{infimum calculus} \quad \} \\
 & (p \circ A) \sqcap (p^{\cup} \circ A) \;\sqsubseteq\; p \sqcap p^{\cup} \\
\Leftarrow & \quad \{ \quad \text{assumption: } \; p \circ A \;=\; p \quad \} \\
 & p \sqcap (p^{\cup} \circ A) \sqsubseteq p^{\cup} \\
\Leftarrow & \quad \{ \quad \text{dedekind and } \; p \circ p \;=\; p \quad \} \\
 & p \sqcap A \;\sqsubseteq\; p^{\cup} \\
\Leftarrow & \quad \{ \quad \text{monotype domains: } \; A \circ p{>} \;=\; (\top\!\!\top \circ p) \sqcap A \;\sqsupseteq\; p \sqcap A \quad \} \\
 & A \circ p{>} \;\sqsubseteq\; p^{\cup} \\
\Leftarrow & \quad \{ \quad p{>} \;\sqsubseteq\; p^{\cup} \quad \text{and the assumption} \quad \} \\
 & \mathit{true} \ .
\end{array}
$$

□

Corollary 22 The pers form a complete lattice under the domain ordering.

□

For the pers to be a domain kind we define the right per domain operator $\dagger$ to be the composition $\mathit{sym} \cdot \succ$. That is we define, for all specs R,

$$R\dagger \;=\; \mathit{sym}.R{\succ}$$

and we prove that the pers are just the fixpoints of $\dagger$:

$$
\begin{array}{ll}
& R \;=\; R\dagger \\
\Rightarrow & \quad \{ \quad R\dagger \text{ is a per } \} \\
& per.R \\
\equiv & \quad \{ \quad \text{a per is a symmetric partial preorder } \} \\
& R \;=\; sym.R \;\wedge\; R \;=\; R\succ \\
\Rightarrow & \quad \{ \quad \text{substitution, definition of } \dagger \} \\
& R \;=\; R\dagger \;\; .
\end{array}
$$

In addition the least domain property holds:

Lemma 23 For all specs R and all pers A,

$$R \circ A \;=\; R \quad \equiv \quad sym.R\succ \;\lhd\; A \;\; .$$

Proof

$$
\begin{array}{ll}
& R \circ A \;=\; R \\
\equiv & \quad \{ \quad \text{theorem 19 } \} \\
& R\succ \;\lhd_r\; A \\
\equiv & \quad \{ \quad \text{lemma 21 } \} \\
& sym.R\succ \;\lhd\; A \;\; .
\end{array}
$$

□

In conclusion:

Theorem 24 The pers form a domain kind with right domain operator the function mapping R to $sym.R\succ$ and left domain operator the function mapping R to $sym.R\prec$, where $R\prec \;=\; (R\cup)\succ$.

□

7 Conclusion

We have introduced the notion of domain kind and shown that the pers are an instance. By splitting the proof into a proof that the partial preorders form a right domain kind followed by the consideration of symmetry we have tried to condense the calculations into small and manageable fragments. It is nevertheless frustrating that the establishment of the Galois connection between the pers and the partial preorders is so much more difficult than the remaining parts of the proof. There may be a simplification possible but we have not been able to find it.

In [6] it was shown that, for relator F, the function mapping spec R to $sym.(\!|F;\ R|\!)\succ$ is idempotent and has fixed points the F-congruences (i.e. F-substitutive pers where, by definition, R is F-substitutive equivales $R \sqsupseteq F.R$). This is the first step on the way to showing that the function is a right domain operator. The obvious question raised by the current paper — which we have as yet had insufficient time to tackle — is whether it is indeed a right domain operator and whether the F-congruences form a domain kind.

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